Mathematical Olympiad for Girls

Tuesday 11th October 2016

Organised by the United Kingdom Mathematics Trust

Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks. Full marks require clearly written solutions — not just answers — including complete proofs of any assertions you may make.

   Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt.
4. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.

   However, one complete solution will gain more credit than several unfinished attempts.
5. Earlier questions tend to be easier. Some questions have two parts. Part (a) introduces results or ideas useful in solving part (b).
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Start each question on a fresh sheet of paper. Write on one side of the paper only.

   On each sheet of working write the number of the question in the top left-hand corner and your name, initials and school in the top right-hand corner.
8. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
9. Staple all the pages neatly together in the top left hand corner.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 08:00 BST on Wednesday 12th October.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

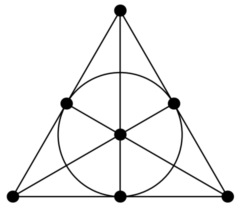
MOG, UK Mathematics Trust, School of Mathematics Satellite,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1. The diagram shows a figure consisting of six line segments and a circle, each containing three points.

Each point is labelled with a real number. The sum of the three numbers on each line segment or circle is T.

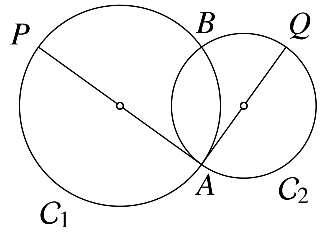
Prove that each number is equal to $\frac{1}{3}T$.



2. The diagram shows two circles C_1 and C_2 with diameters PA and AQ. The circles meet at the points A and B, and the line PA is a tangent to C_2 at A.

Prove that

$$\frac{PB}{BQ} = \frac{\text{area}\, C_1}{\text{area}\, C_2}.$$



3. Punam puts counters onto some of the cells of a 5×5 board. She can put more than one counter on each cell, and she can leave some cells empty. She tells Quinn how many counters there are in each row and column. These ten numbers are all different.

Can Quinn always work out which cells, if any, are empty?

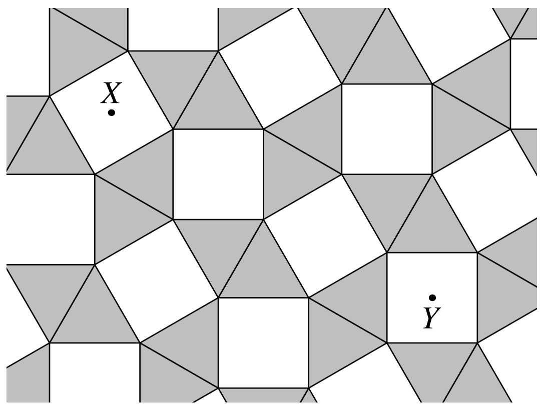
4. (a) In the trapezium $ABCD$, the edges AB and DC are parallel. The point M is the midpoint of BC, and N is the midpoint of DA.

Prove that $2MN = AB + CD$.

(b) The diagram shows part of a tiling of the plane by squares and equilateral triangles.

Each tile has edges of length 2. The points X and Y are at the centres of square tiles.

What is the distance XY?



5. Alia, Bella and Catherine are multiplying fractions, aiming to obtain integers. Each of them can multiply as many fractions as she likes (including just one), and can use the same fraction more than once.

Alia's fractions are of the form $\dfrac{n+1}{n}$, where n is a positive integer.

Bella's fractions are of the form $\dfrac{6p-5}{3p+6}$, where p is a positive integer.

Catherine's fractions are of the form $\dfrac{4q-1}{2q+1}$, where q is a positive integer.

Which integers can each of them obtain?