United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 30 November 2007

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

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2007/8 British Mathematical Olympiad
Round 1: Friday, 30 November 2007

1. Find the value of
$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}.$$

2. Find all solutions in positive integers x, y, z to the simultaneous equations
$$x + y - z = 12$$
$$x^2 + y^2 - z^2 = 12.$$

3. Let ABC be a triangle, with an obtuse angle at A. Let Q be a point (other than A, B or C) on the circumcircle of the triangle, on the same side of chord BC as A, and let P be the other end of the diameter through Q. Let V and W be the feet of the perpendiculars from Q onto CA and AB respectively. Prove that the triangles PBC and AWV are similar. [*Note: the circumcircle of the triangle ABC is the circle which passes through the vertices A, B and C.*]

4. Let S be a subset of the set of numbers $\{1, 2, 3, ..., 2008\}$ which consists of 756 distinct numbers. Show that there are two distinct elements a, b of S such that $a + b$ is divisible by 8.

5. Let P be an internal point of triangle ABC. The line through P parallel to AB meets BC at L, the line through P parallel to BC meets CA at M, and the line through P parallel to CA meets AB at N. Prove that
$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} \leq \frac{1}{8}$$
and locate the position of P in triangle ABC when equality holds.

6. The function f is defined on the set of positive integers by $f(1) = 1$, $f(2n) = 2f(n)$, and $nf(2n + 1) = (2n + 1)(f(n) + n)$ for all $n \geq 1$.

   i) Prove that $f(n)$ is always an integer.

   ii) For how many positive integers less than 2007 is $f(n) = 2n$?