NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Tuesday, 13th March 1984

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

CANDIDATES ARE NOT EXPECTED TO ATTEMPT ALL SEVEN QUESTIONS.

1. P, Q, R are arbitrary points on the sides BC, CA, AB respectively of triangle ABC. Prove that the triangle whose vertices are the centres of the circles AQR, BRP, CPQ is similar to triangle ABC.

2. Let a_n be the number of binomial coefficients $\binom{n}{r}$ $(0 \leq r \leq n)$ which leave remainder 1 on division by 3 and let b_n be the number which leave remainder 2. Prove that $a_n > b_n$ for all positive integers n.

3. (i) Prove that, for all positive integers m,

$$\left(2 - \frac{1}{m}\right)\left(2 - \frac{3}{m}\right)\left(2 - \frac{5}{m}\right) \ldots \left(2 - \frac{2m-1}{m}\right) \leq m!$$

(ii) Prove that if a, b, c, d, e are positive real numbers then

$$\left(\frac{a}{b}\right)^4 + \left(\frac{b}{c}\right)^4 + \left(\frac{c}{d}\right)^4 + \left(\frac{d}{e}\right)^4 + \left(\frac{e}{a}\right)^4$$

$$\geq \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d} + \frac{a}{e} .$$

4. Let N be a positive integer. Determine with proof the number of solutions of the equation

$$x^2 - [x^2] = (x - [x])^2$$

lying in the interval $1 \leq x \leq N$.

(For a real number x the "integer part" [x] is the largest integer which is $\leqslant x$.)

5. A plane cuts a right circular cone with vertex V in an ellipse E and meets the axis of the cone at C ; A is an extremity of the major axis of E. Prove that the area of the curved surface of the slant cone with V as vertex and E as base is

$$\frac{VA}{AC} \times (\text{area of } E).$$

6. Let a, m be positive integers. Prove that if there exists an integer x such that $a^2x - a$ is divisible by m then there exists an integer y such that both $a^2y - a$ and $ay^2 - y$ are divisible by m.

7. ABCD is a quadrilateral which has an inscribed circle. With the side AB is associated

$$u_{AB} = p_1 \sin \widehat{DAB} + p_2 \sin \widehat{ABC}$$

where p_1, p_2 are the perpendiculars from A, B respectively to the opposite side CD. Define u_{BC}, u_{CD}, u_{DA} likewise, using in each case perpendiculars to the opposite side. Show that

$$u_{AB} = u_{BC} = u_{CD} = u_{DA}.$$