Vertices of Specht Modules

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Outline

(1) Vertices and the Brauer Correspondence for Modules
(2) Vertices of Specht Modules
(3) Complexity of Modules and Two Results of K. J. Lim

§1: Vertices and the Brauer Correspondence for Modules

Let G be a finite group. Let F be a field of prime characteristic p. Let V be an indecomposable FG-module.

A subgroup $P \leq G$ is said to be a vertex of V is there is an FG-module U such that $V \mid U\uparrow_P^G$, and P is minimal with this property.

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Green showed in 1959 that

(i) Vertices are p-subgroups of G;

(ii) If P, $Q \leq G$ are vertices of V then $P^x = Q$ for some $x \in G$.

Brauer Correspondence for Modules

Let $V^Q = \{v \in V : vg = v \text{ for all } g \in Q\}$. Given $R \leq Q \leq G$ define the trace map $\mathrm{Tr}_R^Q : V^R \to V^Q$ by

$$\mathrm{Tr}_R^Q(v) = \sum_{i=1}^{m} vg_i$$

where $Q = Rg_1 \cup \ldots \cup Rg_m$.

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Theorem (Broué 1985)

If $V(Q) \neq 0$ then Q is contained in a vertex of V.

Brauer Correspondence for p-Permutation Modules

Let $P_{\max}$ be a Sylow p-subgroup of G.

An FG-module V is p-permutation if it has a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ such that $v_i g \in \mathcal{B}$ for all $g \in P_{\max}$.

Remark: V is an indecomposable p-permutation module if and only if $V \mid F\!\uparrow_P^G$ for some $P \leq G$.

Lemma

Suppose that V is p-permutation with respect to the basis $\mathcal{B}$. If $Q \leq P_{\max}$ then $V(Q) = \left\langle \mathcal{B}^Q \right\rangle_F$.

Theorem (Broué 1985)

Let V be an indecomposable p-permutation FG-module. Then $V(Q) \neq 0 \iff Q$ is contained in a vertex of V. If V has vertex P then $V(P)$ is the Green correspondent of V.

§2 Vertices of Specht Modules

Theorem

Let $n \in \mathbf{N}$ and let p be a prime such that $p \nmid n$. The vertex of $S^{(n-r,1^r)}$, defined over a field of characteristic p, is a Sylow p-subgroup of $S_{n-r-1} \times S_r$.

The proof uses a p-permutation basis for $S^{(n-r-1,1^r)}$.

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Application: In characteristic 2 Specht modules may be decomposable. I used this theorem to give a short proof of a theorem of Murphy (1980): if n is odd and $2^{\ell-1} \leq r < 2^{\ell}$ then $S^{(n-r,1^r)}$ is decomposable, unless $n \equiv 2r+1 \bmod 2^{\ell}$.

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Remark: Suppose that S^λ, defined over a field of characteristic p, is indecomposable with vertex Q. It follows from a theorem of Green (1960) that if g is a p-element such that $\chi^\lambda(g) \neq 0$ then there exists $x \in G$ such that $g \in Q^x$.

Open Problems

Problem

Find vertices of hook Specht modules $S^{(n-r,1^r)}$ over fields of characteristic $p \geq 3$ where $p \mid n$.

Solved when $p = 2$ by Murphy and Peel (1984).

Work is in progress with Susanne Danz and Karin Erdmann on $S^{(n-3,1,1,1)}$ in characteristic 3.

Problem

Clarify the relationship between character values on p-elements and vertices in characteristic p.

A Subgroup Bound on Vertices

Let λ be a partition and let t be a λ-tableau. Let $H(t)$ be the subgroup of the row stabilising group of t that permutes, as blocks for its action, the entries of columns of t of equal length.

For example if $\lambda = (8, 4, 1)$ and

$$t = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ \hline \mathbf{9} & \mathbf{10} & \mathbf{11} & \mathbf{12} \\ \cline{1-4} \mathbf{13} \\ \cline{1-1} \end{array}$$

then $H(t)$ is generated by

$$(2, 3, 4)(10, 11, 12), (2, 3)(10, 11), (5, 6, 7, 8), (5, 6).$$

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Theorem

If S^λ is indecomposable then it has a vertex containing a Sylow p-subgroup of $H(t)$.

Outline Proof

We assume w.l.o.g. t is the greatest tableau under $\trianglerighteq$. Let Q be a Sylow p-subgroup of $H(t)$.

(1) Show that e_t is fixed by every permutation in Q. So for instance, we need

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For (2) it suffices to show that if u is a λ-tableau and $g \in H(t)$ is a p-element then, when $e_u + e_u g + \cdots + e_u g^{p-1}$ is written as a linear combination of standard polytabloids, the coefficient of e_t is 0.

§3 Complexity of Modules and Two Results of K. J. Lim

Definition

Let G be a finite group and let V and an FG-module. Let

$$\to P_2 \to P_1 \to P_0 \to V$$

be a minimal projective resolution of V. The complexity of V is the least non-negative integer c such that

$$\lim_{n\to\infty} \frac{\dim_F P_n}{n^c} = 0.$$

Theorem (Lim 2011, Theorem 3.2)

Suppose that the Specht module S^μ, defined over a field of odd characteristic, has an abelian vertex. Let m be the p-rank of Q. If c is the complexity of S^μ and w is the weight of μ then $c = w = m$ and Q is conjugate to the elementary abelian subgroup

$$\langle(1,\ldots,p)\rangle \times \cdots \times (wp-p+1,\ldots,wp)\rangle \leq S_{wp}.$$

Abelian Vertices

In 2003 I proved:

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Theorem (Lim 2011, Corollary 5.1)
Let p be an odd prime and let $1 \leq m \leq p-1$. The Specht module S^λ, defined over a field of characteristic p has an abelian vertex of p-rank m if and only if the p-weight of μ is m.

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Problem

Classify all Specht modules with abelian vertex.