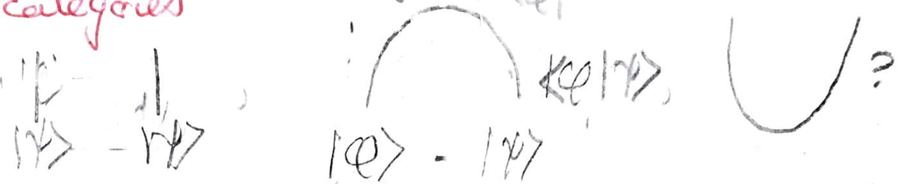
Untangling the ZX-calculus

Based on: Coecke, Horsman, Kissinger, Wang ①
Kindergarden quantum mechanics graduates
& John van de Wetering, ZX-Calculus for the working computer scientist

§0 Monoidal categories

Example Hilb



what is dual to $H \otimes H \to \mathbb{C}$ $|\varphi\rangle \otimes |\psi\rangle \to \langle\varphi|\psi\rangle$? Answer

$\langle v_1 \cdot v_n \rangle$, $1 \to \sum_{i=1}^{n} v_i \otimes v_i$ (NB $\alpha|\varphi\rangle = \bar{\alpha}\langle\varphi|$)

$H = \langle |0\rangle, |1\rangle \rangle$, ∪ is Bell state. Symmetric!

$|00\rangle + |11\rangle = |{+}{+}\rangle + |{-}{-}\rangle = |uu\rangle + |vv\rangle$

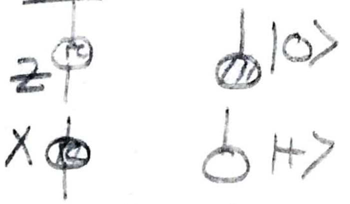
ADD EXp4 later

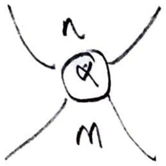
Comparison with other categories: a state is a morphism from the unit object. In Set no entangled states (states are elements); in Rel pure states in A tensor B = A times B are subsets A' times B'.

§1 ZX-spiders

Z , green , white $|0\rangle$ $|1\rangle$

X , red , grey $|+\rangle$ $|-\rangle$



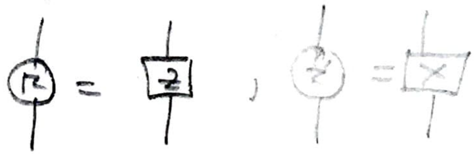


$\leftrightarrow |0 \cdot\cdot 0\rangle_n \langle 0 \cdot\cdot 0|_m + e^{i\alpha} |1 \cdot\cdot 1\rangle_n \langle 1 \cdot\cdot 1|_m$

$\leftrightarrow |+ \cdot\cdot +\rangle\langle + \cdot\cdot +| + e^{i\alpha} |- \cdot\cdot -\rangle\langle - \cdot\cdot -|$

- if $m = n = 1$:

$= \begin{pmatrix} 1 & \\ & e^{i\alpha} \end{pmatrix}$ with columns $|0\rangle$ $|1\rangle$ so

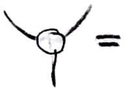


- if $m = 0$ $n = 1$:

$= \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}$ $\mathbb{C}$ so

$= |0\rangle + |1\rangle \propto |+\rangle$

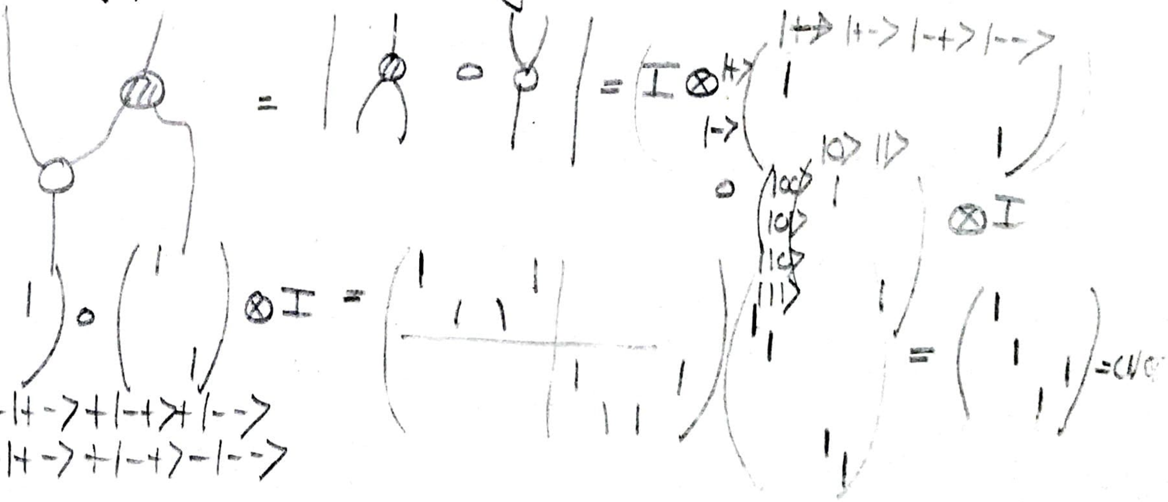
$= |+\rangle + |-\rangle \propto |0\rangle$

- 

$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ (not a clone)

Note not unitary: just hackin' with building blocks

Example



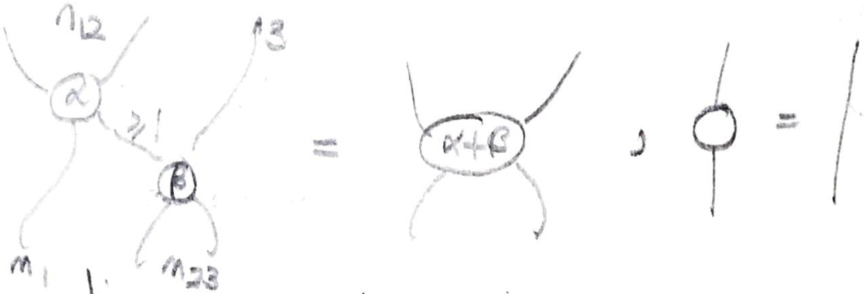
$|00\rangle = |{+}{+}\rangle + |{+}{-}\rangle + |{-}{+}\rangle + |{-}{-}\rangle$

$|01\rangle = |{+}{+}\rangle - |{+}{-}\rangle + |{-}{+}\rangle - |{-}{-}\rangle$

etc.

§2 Basic ZX rules

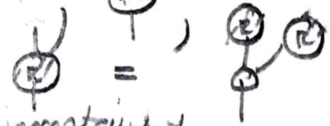
- spider fusion



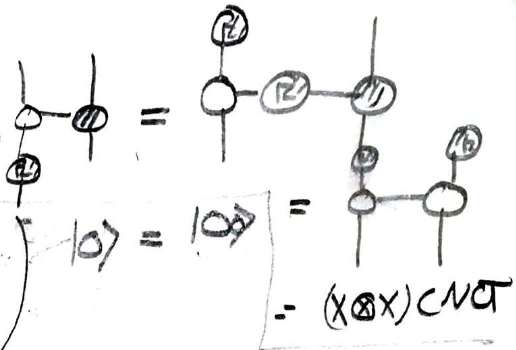
- duality

where □ = H

- π-copy



Example
CNOT(X⊗I)



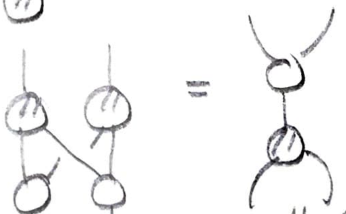
= (X⊗X)CNOT

- strong complementarity

(a)

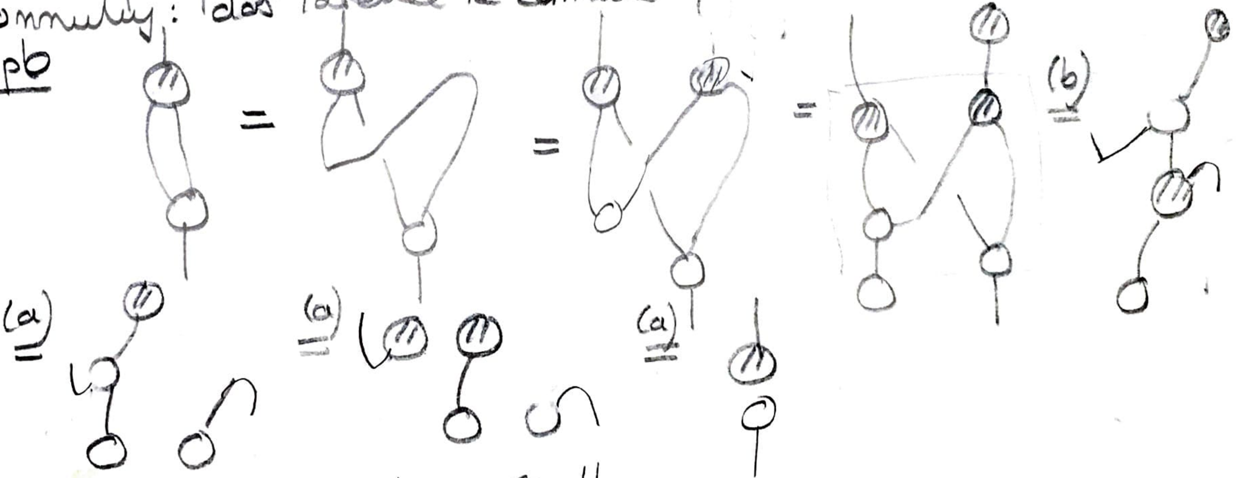
Pf $\begin{pmatrix}1&0\\0&0\\0&0\\0&1\end{pmatrix}|0\rangle = |00\rangle$

(b)



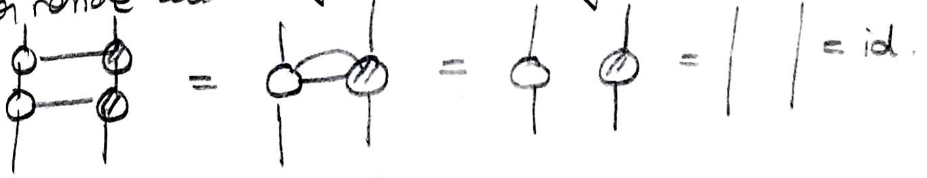
- commutativity: dots labelled π commute

Example



So can remove all 2-cycles. Corollary

CNOT^2 =



§3 Clifford group (Gottesman)

$P_n = \langle X_i, Z_i \rangle \leq U_{2^n}(\mathbb{C})$

$Cl_n = N_{U_{2^n}}(\mathbb{C}) \cong \dfrac{Sp_{2n}(\mathbb{F}_2)}{P_n}$

Rough idea why: $g \in Cl_n$ preserves commuting/anticommuting on Paulis which is described by symplectic form $\begin{array}{c|cc} & X_1 & Z_1 \\ X_1 & 0 & 1 \\ Z_1 & 1 & 0 \\ & & & 0 & 1 \\ & & & 1 & 0 \end{array}$. So

$Cl_n \longrightarrow Sp_{2n}(\mathbb{F}_2)$

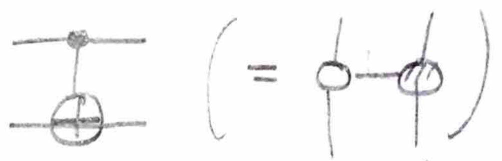
Kernel is P_n since P_n rep is irreducible, not obvious is onto but can show $Cl_n = \langle P_n, H_i, CNOT_{ij} \rangle$ Gottesman-Knill Thm

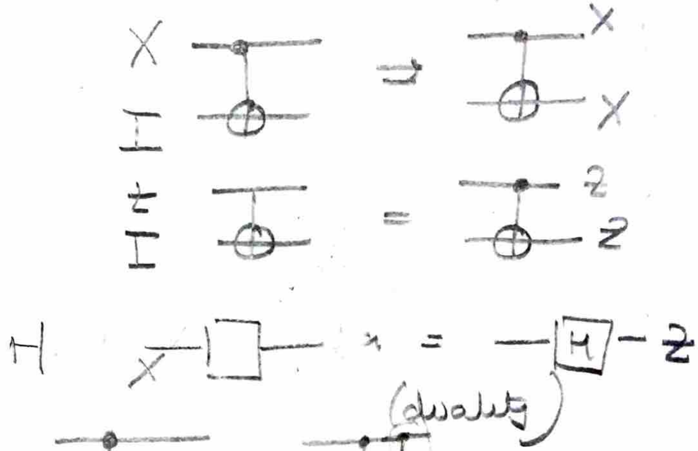
The rules so far prove all identities in the Clifford gp.

§4 Stabiliser formalism (skipped)

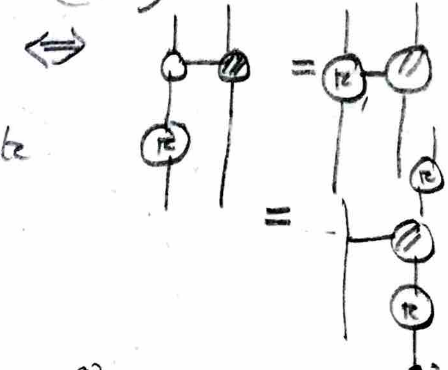
$g \in Cl_n$ is determined by its action on Paulis

Example CNOT

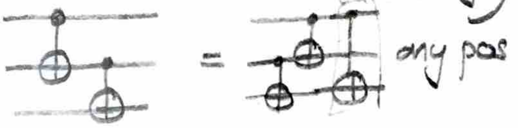




$\Leftrightarrow (X\otimes I)^{CNOT} = X\otimes X$

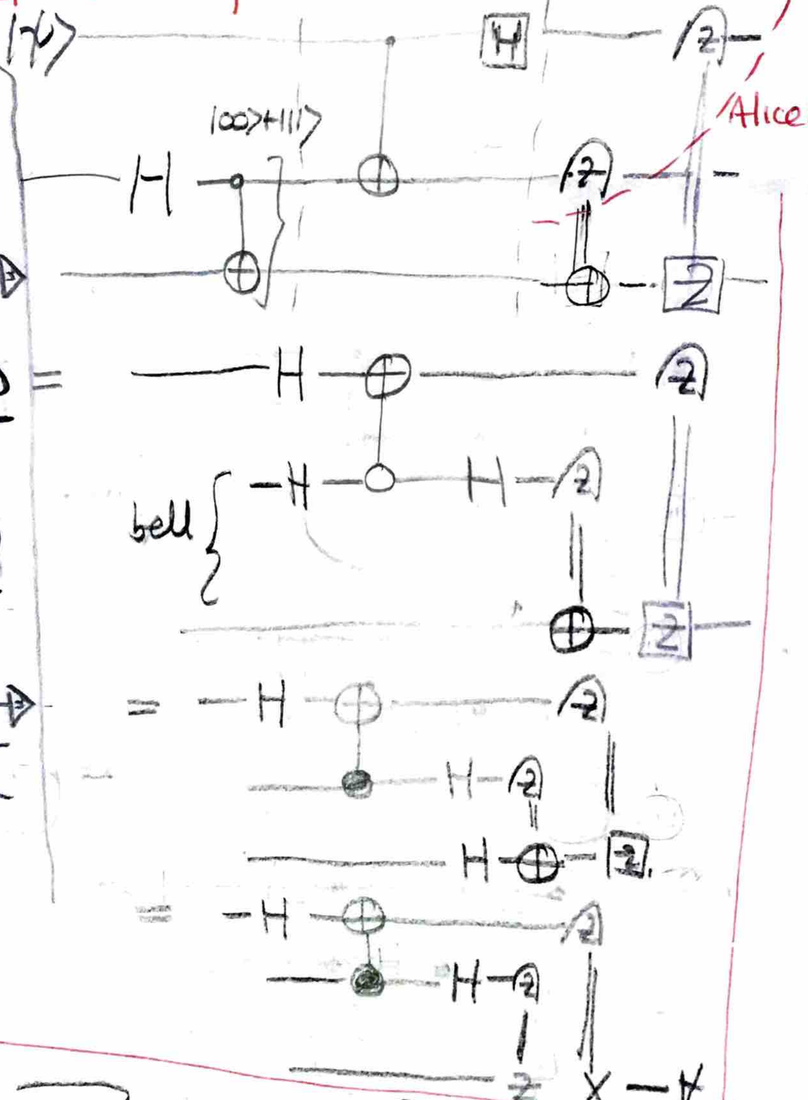


Exercise



Exercise $H^{\otimes 2}CNOT_{12} = CNOT_{21}H^{\otimes 2}$

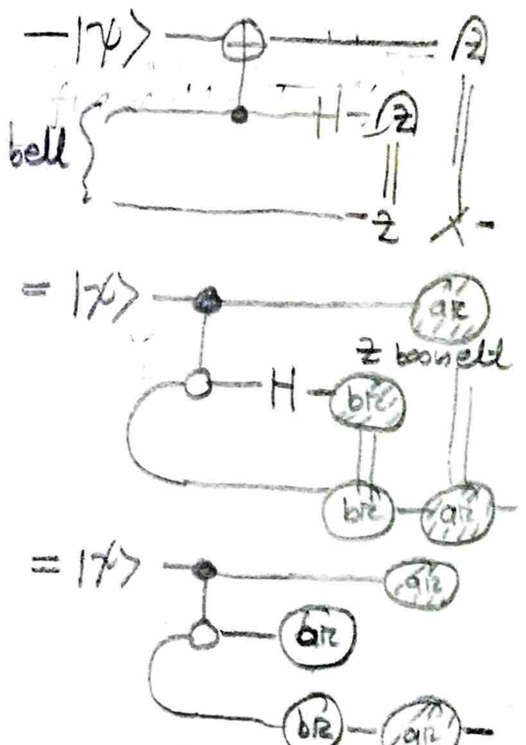
§5 Quantum teleportation

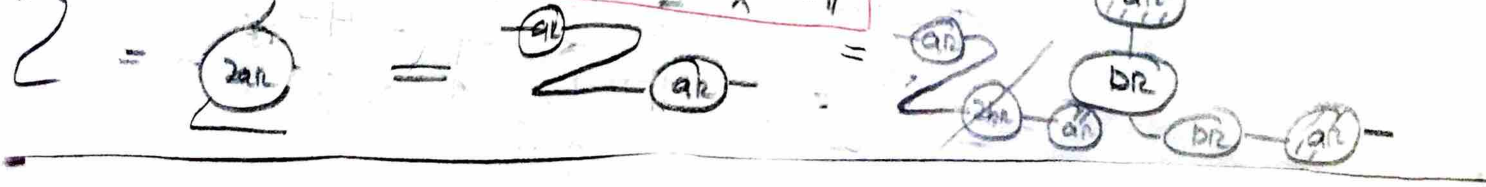


Basic calculation

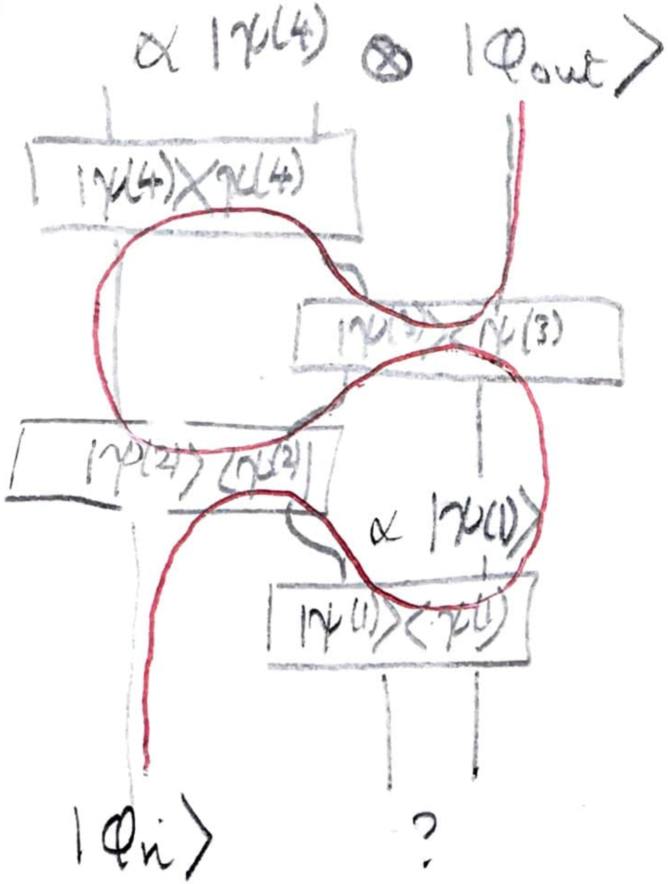
$(\langle 00| + \langle 11|) \otimes \langle u|)(|v\rangle \otimes |00\rangle + |11\rangle) = \langle u|v\rangle$

So up to conjugating |ψ⟩ by H is equal to





§0 Projectors in monoidal categories moved.



Qn: what is $|\varphi_{out}\rangle$ in terms of $|\psi(1)\rangle \, .. \, |\psi(4)\rangle$? (up to scalar).

Ans Let $|\psi(a)\rangle = \sum_{ij} w^{(a)}_{ij} |ij\rangle$

Then

$$|\varphi_{out}\rangle = w(3)^t \, \overline{w(4)}^t \, w(2) \cdot \overline{w(3)} \, w(1)^t \, \overline{w(2)}^t \, |\varphi_{in}\rangle$$