Introduction to plethysm

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Outline

§1 Motivation: the Wronskian isomorphism

§2 Polynomial representations and plethysms of Schur functions

§3 Decomposition numbers for S_{2n} from $\mathrm{Sym}^n \mathrm{Sym}^2 E$

§4 Maximal summands in plethysms

§5 Foulkes' Conjecture and plethysm stability

§1 Motivation: the Wronskian isomorphism

Let V be a vector space.

- $$\begin{aligned}\mathrm{Sym}^2 V &= V^{\otimes 2}/\langle v \otimes w - w \otimes v : v, w \in V\rangle \\ &= \langle vw : v \in V, w \in V\rangle\end{aligned}$$
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Observation. $\mathrm{Sym}^2\, \mathbb{C}^d$ and $\bigwedge^2 \mathbb{C}^{d+1}$ both have dimension $\binom{d+1}{2}$.

- Proof. If $e_1, \ldots, e_d$ is a basis for $\mathbb{C}^d$ then $\mathrm{Sym}^2 \mathbb{C}^d$ has basis $e_1^2, \ldots, e_d^2, e_1 e_2, \ldots, e_{d-1} e_d$, of size $d + \binom{d}{2}$.

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Question. Asked by მამუკა ჯიბლაძე on MathOverflow: Is there a natural isomorphism between these vector spaces?

Answer. Yes! Let E be the 2-dimensional natural representation of $\mathrm{SL}_2(\mathbb{C})$. Then $\mathrm{Sym}^{d-1}E$ is d-dimensional and

$$\mathrm{Sym}^2\mathrm{Sym}^{d-1}E \cong_{\mathrm{SL}_2(\mathbb{C})} \bigwedge^2 \mathrm{Sym}^d E.$$

§1 Motivation: the Wronskian isomorphism

Are there nice isomorphisms $\mathrm{S}^2(k^n) \cong \Lambda^2(k^{n+1})$?

Asked 1 year, 1 month ago Active 1 year, 1 month ago Viewed 349 times

12

This might be forced to migrate to math.SE but let me still risk it.

The spaces $\mathrm{S}^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

⋮

share cite edit close flag edited Jan 15 '19 at 10:52

asked Jan 15 '19 at 9:45
მამუკა ჯიბლაძე
13.9k ● 3 ● 50 ● 125

19

Let E be a 2-dimensional k-vector space. The Wronksian isomorphism is an isomorphism of $\mathrm{SL}(E)$-modules $\bigwedge^m \mathrm{S}^{m+r-1}(E) \cong \mathrm{S}^m\mathrm{S}^r(E)$. It is easiest to deduce it from the corresponding identity in symmetric functions (specialized to 1 and q), but it can also be defined explicitly: see for example Section 2.5 of this paper of Abdesselam and Chipalkatti.

In particular, identifying $\mathrm{S}^n(E)$ with the homogeneous polynomial functions on E of degree n, their definition becomes the map $\wedge^2\mathrm{S}^n(E) \to \mathrm{S}^2\mathrm{S}^{n-1}(E)$ defined by

$$f \wedge g \mapsto \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Now $\mathrm{S}^n(E) \cong k^{n+1}$ and $\mathrm{S}^{n-1}(E) \cong k^n$, so we have the required isomorphism $\mathrm{S}^2 k^n \cong \wedge^2 k^{n+1}$.

share cite edit delete flag edited Jan 15 '19 at 11:49

answered Jan 15 '19 at 11:09
Mark Wildon
8,018 ● 1 ● 32 ● 51

Action of $\mathrm{SL}_2(F)$ on $\bigwedge^2 \mathrm{Sym}^2 E$ where $E = \langle v, w \rangle$

$$\begin{array}{c} \begin{array}{cc} v & w \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array} \longmapsto \begin{array}{c} \begin{array}{ccc} v^2 \wedge vw & w^2 \wedge vw & v^2 \wedge w^2 \end{array} \\ \begin{pmatrix} \alpha^3\delta - \alpha^2\beta\gamma & \alpha\beta^2\delta - \alpha\beta^2\gamma & 2\alpha^2\beta\delta - 2\alpha\beta^2\gamma \\ -\alpha\gamma^2\delta + \beta\gamma^3 & \alpha\delta^3 - \beta\gamma\delta^2 & 2\beta\gamma^2\delta - 2\alpha\gamma\delta^2 \\ \alpha^2\gamma\delta - \alpha\gamma^2\beta & \beta^2\gamma\delta - \alpha\beta\delta^2 & \alpha^2\delta^2 - \beta^2\gamma^2 \end{pmatrix} \end{array}$$

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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$. The matrices are not even conjugate if $\mathrm{char}\, F = 2$! Instead it is $\mathrm{SL}_2(F)$ acting on $\mathrm{Sym}_2 E = \langle v \otimes v, w \otimes w, v \otimes w + w \otimes v \rangle$.
- Thus $(\mathrm{Sym}^2 E)^\star \cong_{\mathrm{SL}_2(F)} \bigwedge^2 \mathrm{Sym}^2 E$ and the duality is critical.

Duality and the modular Wronskian isomorphism

Theorem (McDowell–W 2020)

Let F be any field. Let E be the 2-dimensional natural representation of $\mathrm{SL}_2(F)$. There is an explicit isomorphism

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(F)} \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

Here $\mathrm{Sym}_n V$ is the invariant subspace of $V^{\otimes n}$ under the permutation action of S_r on tensors and $\mathrm{Sym}^n V$ is the usual quotient of $V^{\otimes n}$.

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As a corollary we obtain a modular version of Hermite reciprocity.

Corollary (Hermite 1854 over $\mathbb{C}$, McDowell–W 2020)

Let F be any field. Let m, $\ell \in \mathbb{N}$ and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(F)$. Then

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by an explicit map.

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Question. What other classical $\mathrm{SL}_2(\mathbb{C})$-isomorphisms have modular analogues?

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- Polynomial representations of $\mathrm{GL}(E)$ with $E = \langle e_1, e_2, e_3 \rangle \cong \mathbb{C}^3$.

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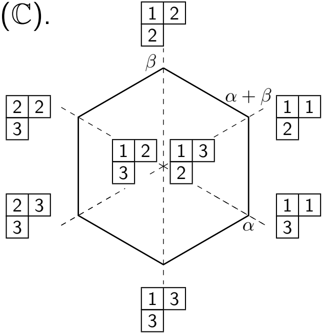
  Here $\nabla^{(2,1)}(E)$ has basis all $F(t)$ for t a semistandard tableaux of shape $(2,1)$ with entries from $\{1,2,3\}$:

  $$F\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = e_a e_b \otimes e_c - e_c e_b \otimes e_a \in \mathrm{Sym}^2 E \otimes E.$$

  You might also know it as the adjoint representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.



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  Now take $E = \langle e_1, e_2\rangle \cong \mathbb{C}^2$
  - Tensor product: $\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E$
  - Symmetric power of symmetric power: $\mathrm{Sym}^2\mathrm{Sym}^2 E$ with basis $(e_1^2)(e_1^2)$, $(e_1^2)(e_2^2)$, $(e_1^2)(e_1 e_2)$, $(e_2^2)(e_2^2)$, $(e_2^2)(e_1 e_2)$, $(e_1 e_2)(e_1 e_2)$

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- Symmetric functions
  - $s_{(2)}(y_1, y_2, y_3) = y_1^2 + y_1 y_2 + y_1 y_3 + y_2^2 + y_2 y_3 + y_3^3$
  - $$s_{(2,1)}(x_1, x_2, x_3) = x^{\begin{smallmatrix}1&1\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&1\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} + \cdots + x^{\begin{smallmatrix}2&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&3\\3&\end{smallmatrix}}$$
    $$= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + \cdots + x_2^2 x_3 + x_2 x_3^2$$

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  - Multiplication: $s_{(2)}(x_1, x_2)^2 = (x_1^2 + x_2^2 + x_1 x_2)^2$
  - Evaluate $s_{(2)}(y_1, y_2, y_3)$ at monomials in $s_{(2)}(x_1, x_2)$ to get

$$s_{(2)}(x_1^2, x_2^2, x_1 x_2) = (x_1^2)(x_1^2) + (x_1^2)(x_2^2) + (x_1^2)(x_1 x_2) + \cdots + (x_1 x_2)(x_1 x_2).$$

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  - Evaluate $s_{(2)}(y_1, y_2, y_3)$ at monomials in $s_{(2)}(x_1, x_2)$ to get

$$s_{(2)}(x_1^2, x_2^2, x_1 x_2) = (x_1^2)(x_1^2) + (x_1^2)(x_2^2) + (x_1^2)(x_1 x_2) + \cdots + (x_1 x_2)(x_1 x_2).$$

This is the plethysm $(s_{(2)} \circ s_{(2)})(x_1, x_2)$, obtained by evaluating $s_{(2)}$ at the monomials x_1^2, x_2^2, $x_1 x_2$ in $s_{(2)}(x_1, x_2)$.

Combinatorial definition of plethysm

Given a tableau t let $x^t = x_1^{a_1} x_2^{a_2} \ldots$ where a_i is the number of entries of t equal to i. We say t has *weight* $(a_1, a_2, \ldots)$.

Definition (Schur function)

Let μ be a partition. The *Schur function* s_μ is the generating function enumerating semistandard tableaux of shape μ by weight:

$$s_\mu = \sum_{t \in \mathrm{SSYT}(\mu)} x^t.$$

For instance

$$\begin{aligned} s_{(2)}(x_1, x_2, \ldots) &= x^{\boxed{1}\boxed{1}} + x^{\boxed{1}\boxed{2}} + x^{\boxed{2}\boxed{2}} + x^{\boxed{1}\boxed{3}} + \cdots \\ &= x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots \end{aligned}$$

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$$\begin{aligned} s_{(2)}(x_1, x_2, \ldots) &= x^{\boxed{1\,|\,1}} + x^{\boxed{1\,|\,2}} + x^{\boxed{2\,|\,2}} + x^{\boxed{1\,|\,3}} + \cdots \\ &= x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots \end{aligned}$$

Equivalently, $s_\mu(x_1, \ldots, x_d)$ is the trace of $\mathrm{diag}(x_1, \ldots, x_n)$ acting on $\nabla^\mu(E)$. For instance $s_{(n)}(x_1, \ldots, x_d)$ is the character of $\mathrm{Sym}^n E$.

Definition (Plethysm of Schur functions)

Let μ and ν be partitions. Let $\mathrm{SSYT}(\mu) = \{t(1), t(2), \ldots\}$. The *plethystic product* of s_ν and s_μ is $s_\nu \circ s_\mu = s_\nu(x^{t(1)}, x^{t(2)}, \ldots)$.

By definition of the Hall inner product, $\langle f, s_\lambda \rangle$ is the multiplicity of s_λ as a summand of the symmetric function f.

Problem (Stanley's Problem 9, 2000)

Find a combinatorial interpretation of the plethysm coefficients $\langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle$ *that makes it clear they are non-negative.*

Equivalently, find a combinatorial interpretation for the multiplicity of the irreducible $\mathrm{GL}_d(\mathbb{C})$-module $\nabla^\lambda(E)$ in $\mathrm{Sym}^n\mathrm{Sym}^m E$.

§3 Decomposition numbers for S_{2n} from $\mathrm{Sym}^n\mathrm{Sym}^2 E$

Problem (Decomposition numbers)

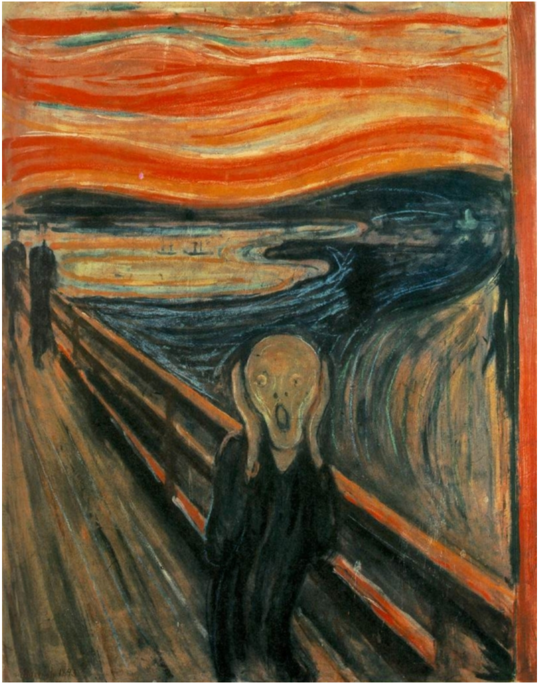
Determine the composition factors of Specht modules over fields of prime characteristic.

For instance in characteristic 3 the Specht module $\mathrm{Sp}^{(3,3)}$ has composition factors labelled by $(5,1)$ and $(3,3)$.

	(6)	(5,1)	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
$(5,1)$	1	1					
$(4,2)$	·	·	1				
$(3,3)$	·	1	·	1			
$(4,1,1)$	·	1	·	·	1		
$(3,2,1)$	1	1	·	1	1	1	
$(2,2,1,1)$	·	·	·	·	·	·	1
$(2,2,2)$	1	·	·	·	·	1	·
$(3,1,1,1)$	·	·	·	·	1	1	·
$(2,1,1,1,1)$	·	·	·	1	·	1	·
$(1,1,1,1,1,1)$	·	·	·	1	·	·	·

Decomposition matrix of principal block of $\mathbb{F}_2 S_{10}$

	(10)	(9,1)	(8,2)	(7,3)	(6,4)	(6,3,1)	(5,3,2)
(10)	1						
$(9,1)$	1	1					
$(8,2)$	1	1	1				
$(7,3)$	1	·	1	1			
$(6,4)$	·	·	1	1	1		
$(6,3,1)$	1	·	2	1	1	1	
$(5,3,2)$	2	1	1	·	1	1	1
$(5,5)$	·	·	1	·	1	·	·
$(8,1,1)$	2	1	1	·	·	·	·
$(6,2,2)$	1	·	1	·	·	1	·
$(4,4,2)$	2	1	1	·	1	·	1
$(4,3,3)$	2	1	·	·	·	·	1
$(7,1,1,1)$	2	1	1	1	·	·	·
$(6,2,1)$	2	1	3	1	1	1	·
$(5,3,1,1)$	3	1	3	1	2	1	1
$(4,4,1,1)$	2	1	1	1	1	·	1
$(5,2,2,1)$	3	1	2	1	1	1	1
$(6,1,1,1,1)$	2	1	2	1	1	·	·



$\mathrm{Sym}^n\mathrm{Sym}^2 E$ and even partitions

As usual, let $E = \langle e_1, \ldots, e_d \rangle$ be the d-dimensional natural representation of $\mathrm{GL}_n(\mathbb{C})$. For $n \in \mathbb{N}$,

$$\mathrm{Sym}^n\,\mathrm{Sym}^2 E = \sum_{\substack{\lambda \in \mathrm{Par}(n) \\ \ell(\lambda) \le d}} \nabla^{2\lambda}(E)$$

where 2λ is the even partition obtained by doubling each part of λ and $\nabla^{2\lambda}(E)$ is an irreducible $\mathrm{GL}_n(\mathbb{C})$-representation. Equivalently

$$\mathbb{C}\big\uparrow_{S_2 \wr S_n}^{S_{2n}} = \bigoplus_{\lambda \in \mathrm{Par}(n)} \mathrm{Sp}^{2\lambda}.$$

Example. Take $d = 4$. Let $\mathcal{F}(V)$ be the $(1,1,1,1)$-weight space of V.

$$\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E \xrightarrow{\mathcal{F}} \left\langle \begin{matrix} e_1 e_2 \otimes e_3 e_4 & e_3 e_4 \otimes e_1 e_2 \\ e_1 e_3 \otimes e_2 e_4 & e_2 e_4 \otimes e_1 e_3 \\ e_1 e_4 \otimes e_2 e_3 & e_2 e_3 \otimes e_1 e_4 \end{matrix} \right\rangle \xrightarrow{\cong} \underset{\mathrm{Sp}^{(4)} \oplus \mathrm{Sp}^{(3,1)} \oplus \mathrm{Sp}^{(2,2)}}{\mathbb{C}\big\uparrow_{S_2 \times S_2}^{S_4}}$$

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Example. Take $d = 4$. Let $\mathcal{F}(V)$ be the $(1,1,1,1)$-weight space of V.

$$\begin{tikzcd}
\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E \arrow[r, "\mathcal{F}"] \arrow[d, two heads] & \left\langle \begin{matrix} e_1 e_2 \otimes e_3 e_4 & e_3 e_4 \otimes e_1 e_2 \\ e_1 e_3 \otimes e_2 e_4 & e_2 e_4 \otimes e_1 e_3 \\ e_1 e_4 \otimes e_2 e_3 & e_2 e_3 \otimes e_1 e_4 \end{matrix} \right\rangle \arrow[r, "\cong"] \arrow[d, two heads] & \begin{matrix} \mathbb{C}\big\uparrow_{S_2 \times S_2}^{S_4} \\ \mathrm{Sp}^{(4)} \oplus \mathrm{Sp}^{(3,1)} \oplus \mathrm{Sp}^{(2,2)} \end{matrix} \arrow[d, two heads] \\
\mathrm{Sym}^2 \, \mathrm{Sym}^2 E \arrow[r, "\mathcal{F}"] & \left\langle \begin{matrix} (e_1 e_2)(e_3 e_4) \\ (e_1 e_3)(e_2 e_4) \\ (e_1 e_4)(e_2 e_3) \end{matrix} \right\rangle \arrow[r, "\cong"] & \begin{matrix} \mathbb{C}\big\uparrow_{S_2 \wr S_2}^{S_4} \\ \mathrm{Sp}^{(4)} \oplus \mathrm{Sp}^{(2,2)} \end{matrix}
\end{tikzcd}$$

From $\mathrm{Sym}^n\mathrm{Sym}^2 E = \bigoplus \nabla^{2\lambda}(E)$ to decomposition numbers

Given a p-core γ, let $\mathcal{E}(\gamma)$ be the set of even partitions obtained from γ by adding the least possible number of disjoint p-hooks.

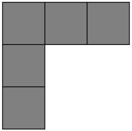
- For example if $p = 3$ then $\mathcal{E}\left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right) = \Big\{(6,2),(4,4),(4,2,2)\Big\}$

Theorem (Giannelli–W 2014)

Let p be an odd prime and let γ be a p-core. Let $\lambda \in \mathcal{E}(\gamma)$ be greatest in the lexicographic order. The column of the decomposition matrix labelled by λ has entries 0 *and* 1. *Moreover its non-zero entries are in rows labelled by $E(\gamma)$.*

Example: 3-block of S_{12} with core $(3,1,1)$

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



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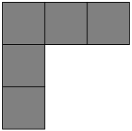
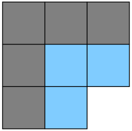
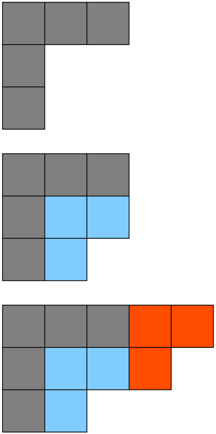
	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2) = \langle 2\rangle$	1									
$(9,4,1) = \langle 2,2\rangle$	1	1								
$(9,3,2) = \langle 2,1\rangle$	2	1	1							
$(8,4,2) = \langle 1\rangle$	1	1	1	1						
$(6^2,2) = \langle 1,2\rangle$				1	1					
$(6,4^4) = \langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2) = \langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1) = \langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1) = \langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2) = \langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5) = \langle 2,3\rangle$			1							
$(6,4,1^4) = \langle 2,2,3\rangle$							1			
$(6,3,2,1^3) = \langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2) = \langle 3,2\rangle$								1		
$(6,1^8) = \langle 2,3,3\rangle$						1				
$(5,4,2,1^3) = \langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2) = \langle 3,1\rangle$	1		1			1				1
$(3^2,2^4) = \langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4) = \langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6) = \langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5) = \langle 3,3\rangle$					1				1	
$(3,1^{11}) = \langle 3,3,3\rangle$					1					





Example: 3-block of S_{12} with core $(3,1,1)$

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



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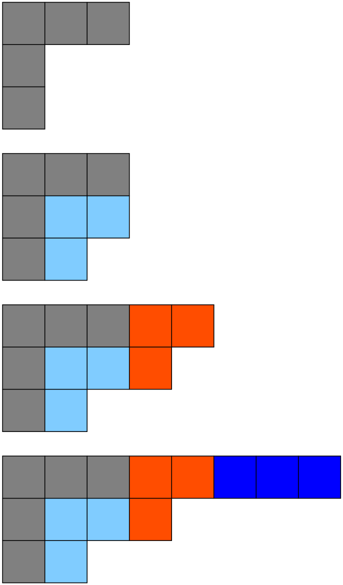
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



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$(12,1^2)=\langle 2\rangle$	1									
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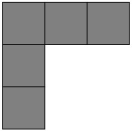
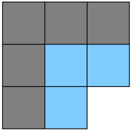
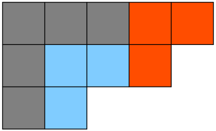
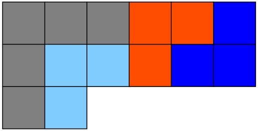
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					









From $\mathrm{Sym}^n\mathrm{Sym}^2 E = \bigoplus \nabla^{2\lambda}(E)$ to decomposition numbers

Given a p-core γ, let $\mathcal{E}(\gamma)$ be the set of even partitions obtained from γ by adding the least possible number of disjoint p-hooks.

- For example if $p = 3$ then $\mathcal{E}\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = \left\{(6,2),(4,4),(4,2,2)\right\}$

Theorem (Giannelli–W 2014)

Let p be an odd prime and let γ be a p-core. Let $\lambda \in \mathcal{E}(\gamma)$ be greatest in the lexicographic order. The column of the decomposition matrix labelled by λ has entries 0 *and* 1. *Moreover its non-zero entries are in rows labelled by $E(\gamma)$*

Idea of proof. Study the reduction modulo p of the symmetric group module $\mathbb{C}\big\uparrow_{S_2\wr S_n}^{S_{2n}}$, corresponding to $\mathrm{Sym}^n\mathrm{Sym}^2 E$.

From $\mathrm{Sym}^n\mathrm{Sym}^2 E = \bigoplus \nabla^{2\lambda}(E)$ to decomposition numbers

Given a p-core γ, let $\mathcal{E}(\gamma)$ be the set of even partitions obtained from γ by adding the least possible number of disjoint p-hooks.

- For example if $p = 3$ then $\mathcal{E}\left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right) = \Big\{(6,2), (4,4), (4,2,2)\Big\}$

Theorem (Giannelli–W 2014)

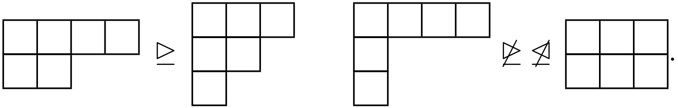
Let p be an odd prime and let γ be a p-core. Let $\lambda \in \mathcal{E}(\gamma)$ be greatest in the lexicographic order. The column of the decomposition matrix labelled by λ has entries 0 *and* 1. *Moreover its non-zero entries are in rows labelled by $E(\gamma)$*

Idea of proof. Study the reduction modulo p of the symmetric group module $\mathbb{C}\big\uparrow_{S_2 \wr S_n}^{S_{2n}}$, corresponding to $\mathrm{Sym}^n\mathrm{Sym}^2 E$.

- Main step: show that the only summands of $\mathbb{F}_p\big\uparrow_{S_2 \wr S_n}^{S_{2n}}$ in the block of S_{2n} with p-core γ are projective.
- From the decomposition of $\mathrm{Sym}^n\,\mathrm{Sym}^2 E$, each projective lifts to a direct sum of Specht modules over $\mathbb{C}$ labelled by even partitions.
- By Brauer reciprocity we get information about columns of decomposition matrix.

§4: Maximal summands in plethysms

A partition λ *dominates* a partition κ if the Young diagram of κ can be obtained from the Young diagram of λ by repeatedly moving boxes downwards. For instance



Quiz. Choose partitions κ and λ of n (a very large number) uniformly at random. What, roughly, is the chance that κ and λ are comparable in the dominance order?

§4: Maximal summands in plethysms

A partition λ *dominates* a partition κ if the Young diagram of κ can be obtained from the Young diagram of λ by repeatedly moving boxes downwards. For instance

$$\ydiagram{4,2} \trianglerighteq \ydiagram{3,2,1} \qquad \ydiagram{4,1,1} \ntrianglerighteq \ntrianglelefteq \ydiagram{3,3}.$$

Quiz. Choose partitions κ and λ of n (a very large number) uniformly at random. What, roughly, is the chance that κ and λ are comparable in the dominance order?

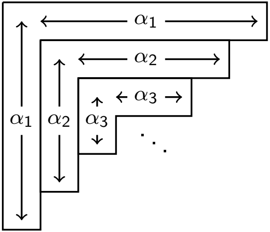
Answer. Asymptotically 0, by a theorem of Pittel (1997).

n	5	6	10	20	30	35
$p_{\text{comparable}}$	1	0.967	0.904	0.782	0.716	0.694

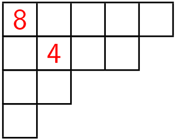
But no problem if you guessed something else: the convergence is very slow, and the small cases are misleading.

Most plethysms have many different maximal summands.

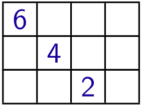
Extreme example: $s_{(1^n)} \circ s_{(2)}$. Let $n \in \mathbb{N}$. Given a partition α of n with distinct parts, let $2[\alpha]$ be the partition of $2n$ with leading diagonal hook lengths $2\alpha_1, 2\alpha_2, \ldots$



$2[4,2]$



$2[3,2,1]$



The plethysm $s_{(1^n)} \circ s_{(2)}$ corresponding to $\bigwedge^n \mathrm{Sym}^2 E$ is

$$s_{(1^n)} \circ s_2 = \sum_{\alpha \in \mathrm{Par}_{\mathrm{distinct}}(n)} s_{2[\alpha]}$$

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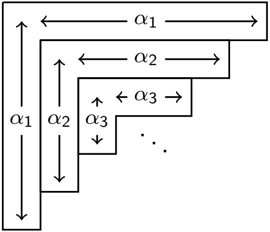

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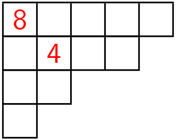

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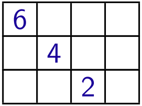

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For instance, if $n = 6$ then

$$s_{(1^6)} \circ s_2 = s_{(7,1^5)} + s_{(6,3,1,1,1)} + s_{(5,4,2,1)} + s_{(4,4,4)}$$

and $(7, 1^5), (6, 3, 1, 1, 1), (5, 4, 2, 1), (4, 4, 4)$ are all incomparable.

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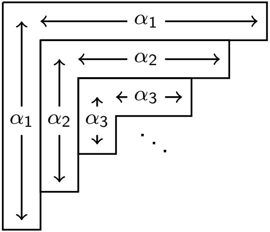

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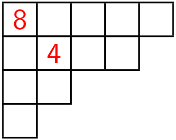

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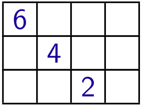

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in which $2[\alpha]$ and $2[\beta]$ are incomparable for all distinct α and β.

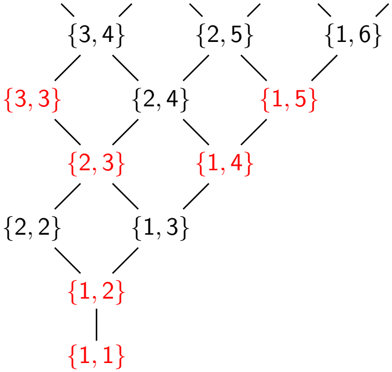
Thus every constituent of $s_{(1^n)} \circ s_{(2)}$ is both maximal *and* minimal. All of them are determined by our theorem.

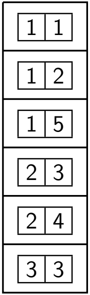
Theorem (Paget–W 2018)

The maximal constituents of the plethysm $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of outer shape ν and inner shape μ.

A plethystic semistandard tableaux of outer shape (1^n) and inner shape (m) is the same as a set of n distinct m-multisets of $\mathbb{N}$, ordered by the majorization order.

Taking $m = 2$ we get the decomposition of $s_{(1^n)} \circ s_{(2)}$. For $n = 6$:





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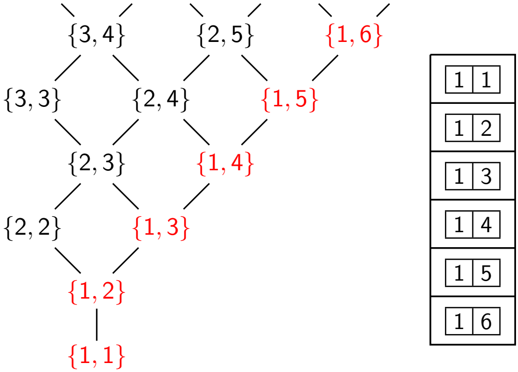

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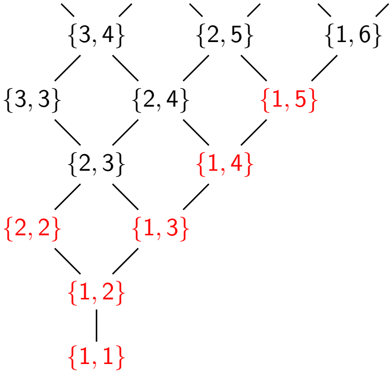

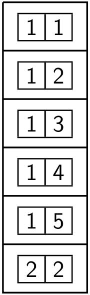

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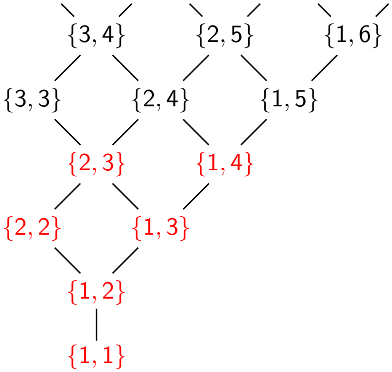

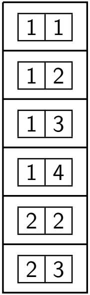

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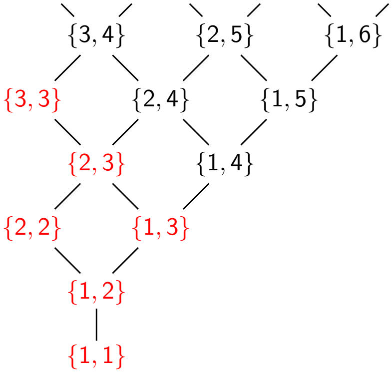

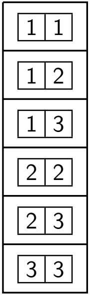

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- The 2018 proof uses the symmetric group.
- In 2020 with Melanie de Boeck we gave a shorter proof using polynomial representations of $\mathrm{GL}_n(\mathbb{C})$.
- Our recent work in 2022–23 gives a still shorter combinatorial proof, with an explicit 'gap' result on the separation between maximal and minimal summands.

§5: Foulkes' Conjecture and plethysm stability

Conjecture (Foulkes 1950)

If $m \le n$ then $s_{(m)} \circ s_{(n)}$ is contained in $s_{(n)} \circ s_{(m)}$.

Equivalently

- There is an injective homomorphism of $\mathrm{GL}(E)$-modules $\mathrm{Sym}^m \mathrm{Sym}^n E \to \mathrm{Sym}^n \mathrm{Sym}^m E$ when $\dim E = mn$.
- There is an injective homomorphism of $\mathbb{C}S_{mn}$-modules $\mathbb{C}\big\uparrow_{S_n \wr S_m}^{S_{mn}} \to \mathbb{C}\big\uparrow_{S_m \wr S_n}^{S_{mn}}$
- Let S_{mn} act on set partitions of $\{1, \ldots, mn\}$. The permutation character for the action on m sets of size n contains the permutation character for the action on n sets of size m.

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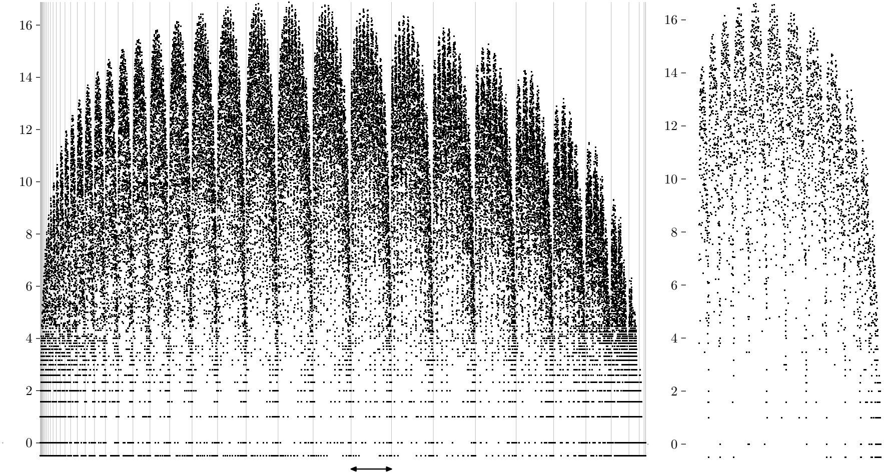
Proved when

- $m = 2$ Thrall (1942)
- $m = 3$ Thrall (1942), Dent and Siemons (2000)
- $m = 4$ McKay (2008),
- $m = 5$ Cheung, Ikenmeyer and Mkrychyan (2015)

and when $m + n \leq 20$, Evseev, Paget and Wildon (2008).

Foulkes Module $\mathrm{Sym}^7\mathrm{Sym}^8 E$

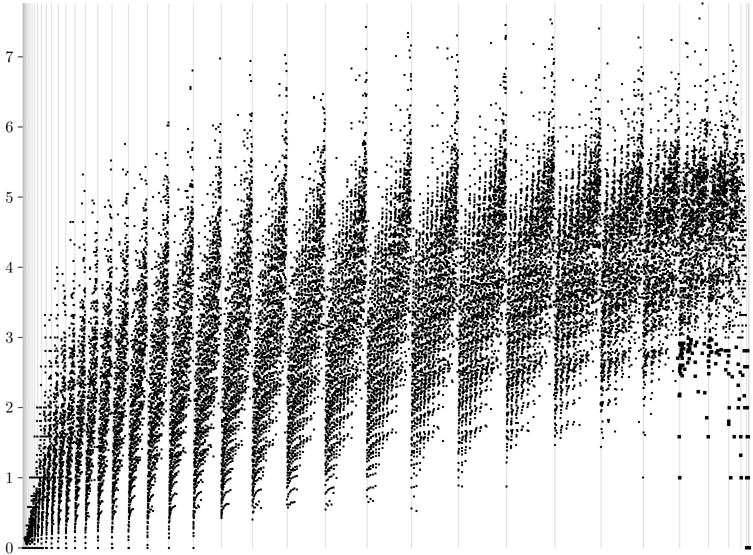
Logarithms of multiplicities of irreducibles $\nabla^\lambda(E)$



The marked interval, enlarged on right, is all partitions of 56 with first part 19

Foulkes Module $\mathrm{Sym}^7\mathrm{Sym}^8 E$

Logarithmic differences in multiplicities: for big dots, smaller multiplicity is 0.



Theorem (Stability for the Foulkes plethysm)

Let γ *be a partition, and let* $(mn - |\gamma|; \gamma)$ *denote the partition* $(mn - |\gamma|, \gamma_1, \ldots, \gamma_\ell)$*. The plethysm coefficient*

$$\left\langle s_{(n)} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \right\rangle$$

is constant for all m *and* n *sufficiently large.*

Hence stable Foulkes Conjecture holds, with equality. Proved by

- Weintraub (1988): recurrence relation on Schur functions
- Carré, Thibon (1992): vertex operators
- Brion (1993): dominant maps of algebraic varieties
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The BP and BPW proofs are notable as the only ones to give an explicit (if intricate) formula for the multiplicity that is clearly non-negative. This is a significant step towards the solution of Stanley's Problem 9.

Using combinatorial arguments with signed plethystic semistandard tableaux Paget and I have given unified proofs of every stability result in the literature.

Here are two representative examples.

Theorem (Brion 1993)

Let $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$, $\lambda \in \mathrm{Par}(mn)$. Let $r \in \mathbb{N}$. The plethysm coefficient

$$\left\langle s_\nu \circ s_{\mu + N(1^r)}, s_{\lambda + N(n^r)} \right\rangle$$

is constant for all N sufficiently large, with an explicit bound.

Theorem (Paget–W 2023)

Let $\nu \in \mathrm{Par}(n)$, $\mu/\mu^\star \in \mathrm{SkewPar}(m)$, $\lambda \in \mathrm{Par}(mn)$. Let $r \in \mathbb{N}$. The plethysm coefficient

$$\left\langle s_\nu \circ s_{\mu + N(1^r)/\mu^\star}, s_{\lambda + N(n^r)} \right\rangle$$

is constant for all N sufficiently large, with an explicit bound.

Theorem (Law–Okitani 2021: Proposition 5.3)

Let $\nu \in \mathrm{Par}(n)$ and $\lambda \in \mathrm{Par}(mn)$. The plethysm coefficient

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(2)}, s_{\lambda + (N) \sqcup (1^N)} \right\rangle$$

is constant for N sufficiently large.

The generalization replacing 2 with an arbitrary $m \in \mathbb{N}$ and $\lambda + (N) \sqcup (1^N)$ with $\lambda + (m-1)N \sqcup (1^N)$ was announced by Law at Oberwolfach in September 2022.

Our methods generalize the Law–Okitani result further, from (m) to an arbitrary rectangular partition. The proof requires signed plethystic semistandard tableaux with negative entries.

Theorem (Paget–W 2023)

Let $\nu \in \mathrm{Par}(n)$, let $a, b \in \mathbb{N}$ and let $\lambda \in \mathrm{Par}(abn)$. The plethysm coefficient

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(a^b)}, s_{\lambda + N(a^{b-1}, a-1) \sqcup (1^N)} \right\rangle$$

is constant for N sufficiently large, with an explicit bound on N.

Thank you! Any questions?