Mathematisches Forschungsinstitut Oberwolfach

Report No. 9/2020

DOI: 10.4171/OWR/2020/9

Mini-Workshop: Kronecker, Plethysm, and Sylow Branching Coefficients and their Applications to Complexity Theory

Organized by
Christine Bessenrodt, Hannover
Chris Bowman, Kent
Eugenio Giannelli, Florence

23 February – 29 February 2020

An introduction to plethysm

Mark Wildon
(joint work with Melanie de Boeck and Rowena Paget)

Let Λ be the ring of symmetric functions and let $s_\lambda = \sum_{t \in \mathrm{SSYT}(\lambda)} x^t$ be the Schur function labelled by the partition λ, defined combinatorially as the generating function enumerating semistandard tableaux of shape λ. For example,

$$(1) \qquad \begin{aligned} s_{(2,1)}(x_1, x_2, x_3) &= x^{\begin{smallmatrix}1&1\\2\end{smallmatrix}} + x^{\begin{smallmatrix}1&1\\3\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\2\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\3\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\2\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\3\end{smallmatrix}} + x^{\begin{smallmatrix}2&2\\3\end{smallmatrix}} + x^{\begin{smallmatrix}2&3\\3\end{smallmatrix}} \\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3. \end{aligned}$$

Informally, the *plethysm* $f \circ g$ of $f, g \in \Lambda$ is defined by substituting the monomials in g for the variables in f. This definition is unambiguous and easy to work with when g is a sum of distinct monomials. We give an example using $s_{(2)}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$ and $s_{(2)}(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_1y_3 + y_2y_3$. Substituting monomials we find

$$(2) \qquad (f \circ g)(x_1, x_2) = f(x_1^2, x_2^2, x_1x_2) = x_1^4 + x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 + x_2^4.$$

Note that since f is a symmetric function, it does not matter how we order the monomials of g; for instance,

$$f(x_1^2, x_2^2, x_1x_2) = f(x_1x_2, x_1^2, x_2^2) \tag{3}$$

Moreover, since g is symmetric, $f \circ g$ is symmetric. If g has a repeated monomial then it is substituted in f according to its multiplicity: for instance if $g = (x_1+x_2)^2$ then $s_{(2)} \circ g = s_{(2)}(x_1^2, x_2^2, x_1x_2, x_1x_2)$. As this may indicate, there are subtleties in extending the plethysm product to arbitrary g: see [9] for the general definition and an excellent introduction to plethysm.

A fundamental open problem in algebraic combinatorics is to find the coefficients $\langle s_\nu \circ s_\mu, s_\lambda \rangle$ in the decomposition of the plethysm $s_\nu \circ s_\mu$ as a linear combination of Schur functions. This problem can be attacked using representations of general linear and symmetric groups, invariant theory, and ideas from combinatorial enumeration, such as the cycle index and the plethystic semistandard tableaux defined below. In my talk I surveyed some of these connections and gave some of the more useful rules for computing plethysms. I ended with a summary of the state of the art on Foulkes' Conjecture.

A combinatorial model. Let $\mathrm{PSSYT}(\mu^\nu)$ be the set of semistandard ν-tableaux whose entries are themselves semistandard μ-tableaux. (This requires the semistandard μ-tableaux to be ordered in some way: as seen in (3), the choice of order is irrelevant.) We define the *weight* of a plethystic semistandard tableau to be the sum of the weights of its μ-tableau entries. The 'substitute monomials' rule implies that $s_\nu \circ s_\mu = \sum_{T \in \mathrm{PSSYT}(\mu^\nu)} x^T$. This definition appears in [7, Definition 3.1], where it is used to find the maximal constituent of $s_\nu \circ s_\mu$ in the reverse lexicographic order on partitions. To give a small example, $\boxed{\boxed{1\,1}\;\boxed{1\,2}}$ has weight $(3,1)$ and, using the same formalism as (1),

$$\begin{aligned}(s_{(2)} \circ s_{(2)})(x_1,x_2) &= x^{\boxed{\boxed{1\,1}\;\boxed{1\,1}}} + x^{\boxed{\boxed{1\,1}\;\boxed{1\,2}}} + x^{\boxed{\boxed{1\,1}\;\boxed{2\,2}}} \\ &\quad + x^{\boxed{\boxed{1\,2}\;\boxed{1\,2}}} + x^{\boxed{\boxed{1\,2}\;\boxed{2\,2}}} + x^{\boxed{\boxed{2\,2}\;\boxed{2\,2}}} \end{aligned} \tag{4}$$

$$= x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4$$

$$= s_{(4)}(x_1,x_2) + s_{(2,2)}(x_1,x_2). \tag{5}$$

This agrees with (2). Working with further variables gives nothing new: in fact $s_{(2)} \circ s_{(2)} = s_{(4)} + s_{(2,2)}$.

General linear groups and invariant theory. Given $\lambda \in \mathrm{Par}(r)$, let ∇^λ denote the corresponding Schur functor: thus if V is a polynomial representation of $\mathrm{GL}_d(\mathbf{C})$ of degree s then $\nabla^\lambda(V)$ is a polynomial representation of degree rs. For example, $\nabla^{(r)}$ and $\nabla^{(1^r)}$ are the rth symmetric power and rth exterior power functors, respectively. Let Φ_W denote the formal character of a representation W; for instance, if E is the natural representation of $\mathrm{GL}_d(\mathbf{C})$ then

$$\Phi_{\nabla^\lambda(E)} = s_{(r)}(x_1,\ldots,x_d)$$

and correspondingly, $\nabla^\lambda(E)$ has a canonical basis of weight vectors indexed by $\mathrm{SSYT}(\lambda)$. The fundamental bridge between plethysm and Schur functors is the relation

$$\Phi_{\nabla^\nu\left(\nabla^\mu(E)\right)} = (s_\nu \circ s_\mu)(x_1, \ldots, x_d). \tag{6}$$

For example, if $E = \langle e_1, e_2 \rangle$ then $\mathrm{Sym}^2 E = \langle e_1^2, e_1e_2, e_2^2 \rangle$ and

$$\mathrm{Sym}^2(\mathrm{Sym}^2 E) = \big\langle (e_1^2)(e_1^2), (e_1^2)(e_1e_2), (e_1^2)(e_2^2), (e_1e_2)(e_1e_2), (e_1e_2)(e_2^2), (e_2^2)(e_2^2) \big\rangle$$

where the basis vectors are ordered to correspond with (4). Using this we may verify (6) and see the decomposition in (4) algebraically: the 'multiply out' map $\mathrm{Sym}^2(\mathrm{Sym}^2 E) \to \mathrm{Sym}^4 E$ has kernel spanned by $(e_1^2)(e_2^2) - (e_1e_2)(e_1e_2)$, which is a highest weight vector in $\nabla^{(2,2)}E$. This is interpreted geometrically in a very instructive example in [6, §11.3]; in my talk I sketched a proof using the related invariant theory of $\mathrm{SL}_2(\mathbf{C})$ that

$$\mathrm{Sym}^2 \mathrm{Sym}^n E \cong \sum_{0 \le s \le n/2} \nabla^{(2m-2s,2s)} E. \tag{7}$$

Symmetric groups. Now suppose that $E = \langle e_1, \ldots, e_d \rangle$ where $d \ge r$. Let λ be a partition of r. The polynomial representation $\nabla^\lambda(E)$ of $\mathrm{GL}_d(\mathbf{C})$ has a (1^r)-weight space, denoted $\nabla^\lambda(E)_{(1^r)}$, in which the diagonal matrix $\mathrm{diag}(\alpha_1, \ldots, \alpha_d)$ acts by multiplication by $x_1 \ldots x_d$. This weight space is invariant under the permutation matrices in $\mathrm{GL}_d(\mathbf{C})$ that permute $e_1, \ldots, e_d$ amongst themselves. The fundamental bridge between representations of general linear and symmetric groups is that $\nabla^\lambda(E)_{(1^r)} \cong S^\lambda$, where S^λ is the Specht module canonically labelled by λ.

To see how composition of polynomial representations is reflected in weight spaces, an example is helpful. Observe that $\mathrm{Sym}^r E_{(1^r)} = \langle e_1e_2 \ldots e_r \rangle$ is the trivial module and, more generally,

$$\big(\mathrm{Sym}^m E\big)^{\otimes n}_{(1^{mn})} = \langle e_{i_1} \ldots e_{i_m} \otimes \cdots \otimes e_{j_1} \ldots e_{j_m} \rangle$$

where (in slightly informal notation), $\big(\{i_1, \ldots, i_m\}, \ldots, \{j_1, \ldots, j_m\}\big)$ is an ordered partition of $\{1, \ldots, r\}$. Hence the weight space is isomorphic to the permutation module of S_{mn} acting on the cosets of the Young subgroup $S_m \times \cdots \times S_m$ of S_{mn}. Suppose we replace $\otimes^n$ with the Schur functor Sym^n. The basis for the weight space then becomes

$$(e_{i_1 \ldots i_m}) \ldots (e_{j_1} \ldots e_{j_m}) \in \mathrm{Sym}^n \mathrm{Sym}^m E$$

where concatenation shows the product for Sym^n. The order of sets in the partition is now irrelevant, and so the weight space is isomorphic to the permutation module of S_{mn} acting on the cosets of the wreath product $S_m \wr S_n$ containing the Young subgroup $S_m \times \cdots \times S_m$ as its base group. This is the *Foulkes module* $H^{(m^n)}$.

More generally, one can show that

$$\nabla^\nu\big(\nabla^\mu(E)\big)_{(1^{nm})} \cong \big(\widetilde{(S^\mu)}^{\otimes n} \otimes \mathrm{Inf}_{S_n}^{S_m \wr S_n} S^\nu\big)\big\uparrow_{S_m \wr S_n}^{S_{mn}}. \tag{8}$$

Here the tilde denotes that the action of $S_m \times \cdots \times S_m$ on $(S^\mu)^{\otimes n}$ is extended to a top group S_n in the wreath product $S_m \wr S_n$ by permuting factors; in the example above, the representation we induce is the trivial representation of $S_m \wr S_n$.

Rules for computing plethysm. Generalizing the result of Iijima [7] mentioned above, de Boeck, Paget and the author [4, Theorem 1.5] proved the following theorem.

Theorem 1. *The maximal constituents of $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of shape μ^ν.*

This strengthened an earlier result proved by Paget and Wildon in [11] using (8). Also in [4], the authors gave a simpler proof of a result originally due to Brion [1], strengthened with an explicit combinatorial bound on the stable multiplicity.

Theorem 2. *Let $\nu \in \mathrm{Par}(n)$ and let μ be a partition. If $r \in \mathbf{N}$ then*

$$\langle s_\nu \circ s_{\mu+(1^r)}, s_{\lambda+(n^r)} \rangle \geq \langle s_\nu \circ s_\mu, s_\lambda \rangle$$

for all partitions λ. Moreover $\langle s_\nu \circ s_{\mu+N(1^r)}, s_{\lambda+N(n^r)} \rangle$ is constant for $N \geq n(\mu_1 + \cdots + \mu_{r-1}) + (n-1)\mu_r + \mu_{r+1} - (\lambda_1 + \cdots + \lambda_r)$.

Still in [4], the authors proved the following two theorems, generalizing results due to Newell, Conca and Varbaro [2], and Ikenmeyer [8, Theorem 4.3.4] respectively.

Theorem 3. *Let $\nu \in \mathrm{Par}(n)$ and let μ be a partition. If r is at least the greatest part of μ then $\langle s_\nu \circ s_{(r)\sqcup\mu}, s_{(nr)\sqcup\lambda} \rangle = \langle s_\nu \circ s_\mu, s_\lambda \rangle$ for all partitions λ.*

Theorem 4. *Let μ be a partition. If $\langle s_{(n^\star)} \circ s_\mu, s_{\lambda^\star} \rangle \geq 1$ then $\langle s_{(n+n^\star)} \circ s_\mu, s_{\lambda+\lambda^\star} \rangle \geq \langle s_{(n)} \circ s_\mu, s_\lambda \rangle$.*

Many further results on plethysm are known and it will be clear that the selection above is biased to the author's work.

Foulkes' Conjecture. In the language of symmetric functions, Foulkes' Conjecture states that if $n \geq m$ then $\langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle \geq \langle s_{(m)} \circ s_{(n)}, s_\lambda \rangle$ for all partitions λ of mn. Equivalently, using the symmetric group, $H^{(n^m)}$ is isomorphic to a submodule of $H^{(m^n)}$. Foulkes' Conjecture is proved only when $n \leq 5$ (see [3] for the case $n = 5$), when $m+n \leq 19$ (computationally in [5] for $m+n \leq 19$, extending [10]) and when n is very large compared to m (see [1]). The full decomposition of $s_{(n)} \circ s_{(m)}$ is known for all m only when $n = 2$, when we have (7) and $s_{(n)} \circ s_{(2)} = \sum_{\lambda \in \mathrm{Par}(n)} s_{2\lambda}$. Problem 9 in Stanley's influential survey article [12] is to find a combinatorial interpretation of the multiplicity $\langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle$. Even a solution in the special case $s_{(n)} \circ s_{(3)}$ would be of considerable interest.

References

[1] Michel Brion, *Stable properties of plethysm: on two conjectures of Foulkes*, Manuscripta Math. **80** (1993), no. 4, 347–371.

[2] Winfried Bruns, Aldo Conca, and Matteo Varbaro, *Relations between the minors of a generic matrix*, Adv. Math. **244** (2013), 171–206.

[3] Man-Wai Cheung, Christian Ikenmeyer, and Sevak Mkrtchyan, *Symmetrizing tableaux and the 5th case of the Foulkes conjecture*, J. Symbolic Comput. **80** (2017), no. part 3, 833–843.

[4] Melanie de Boeck, Rowena Paget, and Mark Wildon, *Plethysms of symmetric functions and highest weight representations*, Submitted. ArXiv:1810.03448 (September 2018), 35 pages.

[5] Anton Evseev, Rowena Paget, and Mark Wildon, *Character deflations and a generalization of the Murnaghan-Nakayama rule*, J. Group Theory **17** (2014), no. 6, 1035–1070.

[6] William Fulton and Joe Harris, *Representation theory, a first course*, Graduate Texts in Mathematics (Readings in Mathematics), vol. 129, Springer, 1991.

[7] Kazuto Iijima, *The first term of plethysms*, arXiv:1108.4915 (2011), 11.

[8] Christian Ikenmeyer, *Geometric complexity theory, tensor rank, and littlewood–richardson coefficients*, Ph.D. thesis, Universität Paderborn, 2012.

[9] Nicholas A. Loehr and Jeffrey B. Remmel, *A computational and combinatorial exposé of plethystic calculus*, J. Algebraic Combin. **33** (2011), no. 2, 163–198.

[10] Jurgen Müller and Max Neunhöffer, *Some computations regarding Foulkes' conjecture*, Experiment. Math. **14** (2005), no. 3, 277–283.

[11] Rowena Paget and Mark Wildon, *Generalized Foulkes modules and maximal and minimal constituents of plethysms of Schur functions*, Proc. Lond. Math. Soc. (3) **118** (2019), no. 5, 1153–1187.

[12] Richard P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 295–319.