11-20 – 12.10

50 mins

Plethysms of symmetric functions Thank Christine Thorsten w/ Rowena Paget

§1 Polynomial reps of $GL_d(\mathbb{C})$

Let $E = \langle e_1 \dots e_d \rangle_{\mathbb{C}} \circlearrowleft GL(E)$

$E \otimes E \cong Sym^2 E \oplus \Lambda^2 E$

$$d=2 \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} e_1^2 & e_1e_2 & e_2^2 \end{array} \\ \begin{pmatrix} \alpha^2 & \alpha\gamma & \gamma^2 \\ 2\alpha\beta & \alpha\delta+\beta\gamma & 2\gamma\delta \\ \beta^2 & \beta\delta & \delta^2 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longrightarrow (\alpha\delta - \beta\gamma)$$

both polydeg 2

$\boxed{\cancel{\text{formal char } x_1^2 + x_1x_2 + x_2^2 \quad x_1x_2}}$

Defn Let $V \in GL(E)$-mod deg. r. Let $\varepsilon \in \mathbb{N}_0^d$, $\varepsilon_1 + \dots + \varepsilon_d = r$.

(i) $v \in V$ has weight ε if $\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_d \end{pmatrix} v = \alpha_1^{\varepsilon_1} \dots \alpha_d^{\varepsilon_d} v$ for all diag

(ii) Let $V_\varepsilon = \{ v \in V : \text{weight } v = \varepsilon \}$

(iii) Say $v \in V_\varepsilon \setminus \{0\}$ is highest weight if $\begin{pmatrix} 0 & * \\ & 0 \end{pmatrix} \cdot v = 0$ for all strict UT mats (Lie alg action)

$\boxed{\cancel{\text{Rec}}}$

Fact If $v \in V$ is hw weight λ then v generates an irreducible submodule, isomorphic to $\nabla^\lambda E$. (defn postponed)

Example $E^{\otimes 3} = Sym^3 E \oplus \Lambda^3 E \oplus \nabla^{(2,1)} E^{\oplus 2}$

$\phantom{E^{\otimes 3}} = \nabla^{(3)}E \qquad = \nabla^{(1^3)}E$

$\searrow Sym^2 E \otimes E$

$\cup$

$v = e_1e_1 \otimes e_2 - e_1e_2 \otimes e_1$ weight $(2,1)$

highest weight: $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $X \cdot e_1 = 0$ $X \cdot e_2 = e_1$

$X \cdot v = e_1^2 \otimes X \cdot e_2 - e_1 X \cdot e_2 \otimes e_1 = 0$.

(By defn) v generates $\nabla^{(2,1)}E = \left\langle F\begin{pmatrix} a & c \\ b & \end{pmatrix} : a \le c \right\rangle \in GL(E)\text{-mod}_3$

$F\begin{pmatrix} a & c \\ b & \end{pmatrix} = e_ae_c \otimes e_b - e_be_c \otimes e_a$

§2 Schur functors

Let $\mathrm{Par}_d(r) = \{\lambda \in \mathbb{N}_0^d : \lambda_1 \geq \dots \geq \lambda_d,\ \lambda_1 + \dots + \lambda_d = r\}$.

For $\lambda \in \mathrm{Par}_d(r)$ have Schur functor

$$\nabla^\lambda : GL(E)\text{-mod deg } s \to GL(E)\text{-mod deg } rs$$

Suppose V has basis $\mathcal{B}$, totally ordered by $<$. Then $\nabla^\lambda V$ has basis

$$\{F(T) : T \in SSYT_{\mathcal{B}}(\lambda)\}$$

where $SSYT_{\mathcal{B}}(\lambda) = \left\{ \text{tableaux } \begin{smallmatrix} a \leq \leq \\ \hat{b} \end{smallmatrix} \text{ shape } \nu \right\}$ and $F(T) \in \prod \mathrm{Sym}^{\lambda_j'} V$ defined by column symmetry

Example (1) $\nabla^{(2,1)} \nabla^{(3,1)} E$ has basis labelled by $SSYT_{SSYT_{\leq d}(3,1)}(2,1)$

e.g. $T = \begin{array}{|c|c|} \hline \begin{smallmatrix}111\\2\end{smallmatrix} & \begin{smallmatrix}112\\2\end{smallmatrix} \\ \hline \begin{smallmatrix}122\\2\end{smallmatrix} & \\ \hline \end{array} \leftrightarrow F(T) \in \nabla^{(2,1)}(\nabla^{(3,1)} E)$

$$= F\left(\begin{smallmatrix}111\\2\end{smallmatrix}\right) F\left(\begin{smallmatrix}112\\2\end{smallmatrix}\right) \otimes F\left(\begin{smallmatrix}122\\2\end{smallmatrix}\right) - F\left(\begin{smallmatrix}122\\2\end{smallmatrix}\right) F\left(\begin{smallmatrix}112\\2\end{smallmatrix}\right) \otimes F\left(\begin{smallmatrix}112\\2\end{smallmatrix}\right) \quad (6,6)$$

(2) Let $V = \mathrm{Sym}^2\langle e_1, e_2\rangle = \langle e_1^2, e_1e_2, e_2^2\rangle$

$$\mathrm{Sym}^2 V = \langle F(\boxed{11}\,\boxed{11}),\ F(\boxed{11}\,\boxed{12}) \ \dots\ F(\boxed{22}\,\boxed{22})\rangle$$

$$= \langle (e_1^2)(e_1^2),\ (e_1^2)(e_1e_2),\ \dots\ (e_2^2)(e_2^2)\rangle$$

$$\downarrow \pi$$

$\mathrm{Sym}^4 V$: $\quad e_1^4, \quad e_1^3 e_2, \quad \dots \quad e_2^4$

Omitted geometry

$\mathrm{Sym}^2(\mathrm{Sym}^2 E^*) \subseteq \mathcal{O}(\mathrm{Sym}^2 E) = \mathbb{C}[e_1^2, e_1e_2, e_2^2]$

$\downarrow \pi$ (pull back / restrict)

$\bigoplus \mathrm{Sym}^{2n} E^* \cong \mathbb{C}[e_1, e_2] \subseteq \mathcal{O}(E)$

induced by $C = \{\alpha^2 e_1^2 + 2\alpha\beta e_1e_2 + \beta^2 e_2^2\}$

$\uparrow$

$E = \{\alpha e_1 + \beta e_2\}$

$\ker \pi \ni (e_1^2)(e_2^2) - (e_1e_2)^2$ generated by, highest weight $(2,2)$

$$X \cdot (e_1^2)(e_2^2) = (e_1^2)(Xe_2)e_2 + (e_1^2)(e_2(Xe_2)) = 2(e_1^2)(e_1e_2)$$

$$X \cdot (e_1e_2)^2 = 2(e_1e_2)X(e_1e_2) = 2(e_1e_2)(e_1^2)$$

$\operatorname{im} \pi \ni e_1^4$ highest weight (4)

$$\therefore\ \mathrm{Sym}^2\mathrm{Sym}^2 E \cong \nabla^{(2,2)}E \oplus \mathrm{Sym}^4 E.$$

More generally $\left((e_1^2)(e_2^2) - (e_1e_2)^2\right)^s e_1^{2s'} \in \mathrm{Sym}^{s+s'}\mathrm{Sym}^2 E$ has highest weight $(2,2)s + 2s' = 2(s+s', s)$ and

$$\mathrm{Sym}^n \mathrm{Sym}^2 E \cong \bigoplus_{2s+s'=n} \nabla^{2(s+s',s)} E.$$

~~geometry~~ on back

§3 πληθυσμοσ and Schur polynomials

<u>Defn</u> Let $V \in GL(E)$-mod deg r. The formal character of V is
$$f_V(x_1 \cdots x_d) = \sum_{\varepsilon \in \mathbb{N}_0^d} (\dim V_\varepsilon)\, x_1^{\varepsilon_1} \cdots x_d^{\varepsilon_d}$$

<u>Rmk</u> $\mathrm{tr}_V \begin{pmatrix} \alpha_1 & \\ & \alpha_d \end{pmatrix} = f_V(\alpha_1 \cdots \alpha_d)$

f_V is symmetric : Weyl gp act on wts. — rearrange le getting to S_n

<u>Defn</u> Say a tableau T has weight ε iff T has ε_i ents of i for each i. Define Sch poly s_λ by

$$s_\lambda(x_1 \cdots x_d) = \sum_{T \in SSYT_{\leq d}(\lambda)} x^T$$

(Soichi Okada gave an lisym 2 vo defn 1st talk)

<u>Rmk</u> Since $(\nabla^\lambda E)_\varepsilon = \langle F(T) : T \in SSYT_{\leq d}(\lambda) : \mathrm{wt}\, T = \varepsilon \rangle$

Hence $f_{\nabla^\lambda E} = s_\lambda(x_1 \cdots x_d)$. In ptic s_λ is symmetric.

<u>Example</u> • $f_{\mathrm{Sym}^2 E}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 = s_{(2)}(x_1, x_2)$

• $f_{\mathrm{Sym}^2(\mathrm{Sym}^2 E)}(x_1, x_2) = x_1^2x_1^2 + x_1^2(x_1x_2) + \cdots + x_2^2x_2^2$
$$= s_{(2)}(x_1^2, x_1x_2, x_2^2)$$
$$= (s_{(2)} \circ s_{(2)})(x_1, x_2)$$

Generally define plethysm $(s_\lambda \circ s_\mu)(x_1 \cdots x_d) =$
s_λ (each monomial in $s_\mu(x_1 \cdots x_d)$).

<u>Prop</u> $f_{\nabla^\lambda(\nabla^\mu E)} = (s_\lambda \circ s_\mu)(x_1 \cdots x_d)$

<u>Problem</u> (Stanley 2000) Find a combinatorial interprets for the plethysm coeffs $s_\nu \circ s_\mu = \sum_\lambda \langle s_\nu \circ s_\mu, s_\lambda \rangle s_\lambda$.

wir müssen wissen / wir werden wissen

§4 Some results   Let $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$

Theorem [Paget–W 16, new proof 18] The maximal $\lambda \in \mathrm{Par}(mn)$ s.t. $\langle s_\nu \circ s_\mu, s_\lambda \rangle \neq 0$ are precisely the maximal weights of the 'plethystic' SSYT $T \in \mathrm{SSYT}_{\mathcal{B}}(\nu)$ where $\mathcal{B} = \mathrm{SSYT}_{\{1..d\}}(\mu)$.

Corollary (1) Proofs of 3 conjectures of Agaoka (Hiroshima Univ) (1998) on lex min/max constituents. (One already proved by Iijima (2011))

52 mins

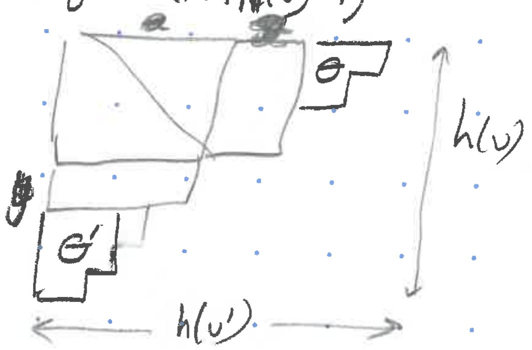
(2) New invariants from rectangle maximals: if m is odd, $n = |\mathrm{SSYT}_{\leq d}(\mu)|$, ~~then~~ $\frac{mn}{d} \in \mathbb{N}$ and ~~then~~ $\nabla^{\left(\frac{mn}{d}\right)^d} E \cong \det^{mn/d}$ is unique minimal in $\mathrm{Sym}^n(\nabla^\mu E)$

(3) Quick pf of deep $s_{(1^n)} \circ s_{(2)}$: all constits min & max

Theorem [Manivel 98] $s_{(a^b)} \circ s_{(c)} = s_{(b^a)} \circ s_{(c+a-b)}$

Theorem [Paget–W 18] $s_\nu \circ s_{(m+h(\nu)-1)} = s_{\nu'} \circ s_{(m+h(\nu')-1)}$

$\Leftrightarrow$ ν has the form



Theorem [Paget–W '19] Let ν be a partition.

$\langle s_{(n)} \circ s_{\mu + (c)}, s_{((n+c)m - |\nu|, \nu)} \rangle$

is constant for n, c suff large.

Generalises earlier result of Brion (93) Manivel (98).

THANKS