From representations of general linear groups to plethysms of symmetric functions and back

§1 GL_n MathOverflow: is there a nice isomorphism $\mathrm{Sym}^2 \mathbb{F}^{n} \cong \Lambda^2 \mathbb{F}^{n+1}$

Answer: yes! (19 pts) In fact if $E = \mathbb{C}^2$ then

$$\mathrm{Sym}^r \mathrm{Sym}^{n-1} E \cong \Lambda^r \mathrm{Sym}^{n+r-2} E$$

as representations of $SL_2(\mathbb{C})$ (Wronskian iso). This categorifies

$$\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r}$$

multiset

(but didn't answer qn)

If $E = \langle X, Y \rangle_{\mathbb{F}}$ then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \xrightarrow{\mathrm{Sym}^2} \begin{array}{c} \begin{array}{ccc} X^2 & Y^2 & XY \end{array} \\ \begin{pmatrix} \alpha^2 & \beta^2 & \alpha\beta \\ \gamma^2 & \delta^2 & \gamma\delta \\ 2\alpha\gamma & 2\beta\delta & \alpha\delta+\beta\gamma \end{pmatrix} \end{array} \qquad \mathrm{Sym}^2 E \cong \frac{\det}{\nabla^{(2)} E} \quad \text{char } \mathbb{F} = 2$$

$$\xrightarrow{\mathrm{Sym}_2} \begin{array}{ccc} X\otimes X & Y\otimes Y & X\otimes Y + Y\otimes X \\ \alpha^2 & \beta^2 & 2\alpha\beta \\ \gamma^2 & \delta^2 & 2\gamma\delta \\ \alpha\gamma & \beta\delta & \alpha\delta+\beta\gamma \end{array} \qquad \mathrm{Sym}_2 E \cong \frac{L^{(2)} E}{\det} \cong (\mathrm{Sym}^2 E)^*$$

Theorem 1 (McDowell – W 2020)

$$\mathrm{Sym}_m \mathrm{Sym}^{\ell} E \cong_{SL_2(\mathbb{F})} \Lambda^r \mathrm{Sym}^{r+\ell-1} E$$

Proof • Give the right map. E.g. if $m = X^{\ell-a}Y^a$

$$m \otimes m \otimes \cdots \otimes m \to mY^{r-1} \otimes mY^{r-2}X \otimes \cdots \otimes mX^{r-1}$$

• Prove injective (not obvious)

• Prove $SL_2(\mathbb{F})$ equivariant: $SL_2(\mathbb{F}_p) \leftarrow SL_2(\mathbb{Z}_p) \subseteq SL_2(\overline{\mathbb{Q}}_p) \cong SL_2(\mathbb{C})$ using Lie algebras.

§2 To plethysms Take $\mathbb{F} = \mathbb{C}$, $E = \langle e_1, \ldots, e_d \rangle_{\mathbb{C}}$

$$E \otimes E \cong \mathrm{Sym}^2 E \oplus \Lambda^2 E$$

$$E \otimes E \otimes E \cong \mathrm{Sym}^3 E \oplus \Lambda^3 E \oplus \nabla^{(2,1)} E \oplus \nabla^{(2,1)} E$$

(2 dofs)

Construct $\nabla^{(2,1)} E \subseteq \mathrm{Sym}^2 E \otimes \mathrm{Sym}\, E$ as $\langle F\begin{pmatrix} a & b \\ c & \end{pmatrix} : a, b, c \in [1,d] \rangle$

where $F\begin{pmatrix} a & b \\ c & \end{pmatrix} = e_a e_b \otimes e_c - e_c e_b \otimes e_a$ Basis $F\left(\begin{smallmatrix} a & b \\ c & \end{smallmatrix}\right)$

Whitehead constructed all polreps of $GL_d(\mathbb{C})$ eg

$$\mathrm{Sym}^\ell E = \langle F(\boxed{a_1 \; \ldots \; a_r}) \rangle, \quad \Lambda^r E = \langle F\left(\begin{smallmatrix} a_1 \\ \vdots \\ a_r \end{smallmatrix}\right) \rangle = e_{b_1} \wedge \cdots \wedge e_{b_r}$$

Generally $\nabla^\lambda E = \langle F(t) : t$ semistandard λ-tableaux$\rangle$

A rep $\rho : GL_d(\mathbb{C}) \to GL(V)$ has a <u>character</u>

$$\chi_\rho(\alpha_1 \cdots \alpha_d) = \operatorname{tr} \rho \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_d \end{pmatrix} = f_\rho(x_1 \cdots x_n)\big|_{x_i = \alpha_i}$$

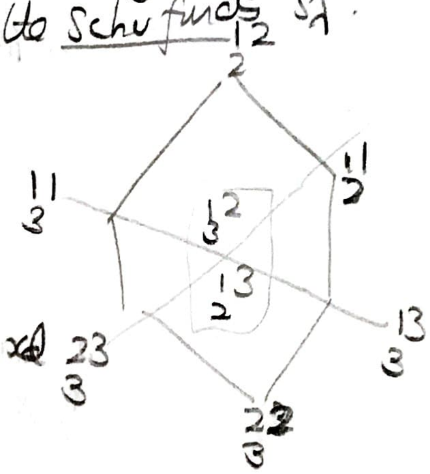
for a symmetric polynomial f_ρ. Eg

$$\chi_{Sym^2}(\alpha_1, \alpha_2) = \begin{pmatrix} \alpha_1^2 & & \\ & \alpha_2^2 & \\ & & \alpha_1\alpha_2 \end{pmatrix} \quad \text{so } f_{Sym^2 E}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$$

Generally each $F(t)$ is a simultaneous eigenvector for the diagonal matrices ($\boxed{1|1|1}$, $\boxed{2|2}$, $\boxed{1|2}$) and $f_{\nabla^\lambda E}(x_1 \cdots x_d) = \sum_{t \in SSYT(\lambda)} x^t$ is the <u>Schur function</u> s_λ.

<u>Example</u>

$$s_{(2,1)}(x_1, x_2, x_3) = x^{\substack{12\\2}} + \cdots + x^{\substack{11\\3}}$$
$$= x_1x_2^2 + \cdots + x_1^2x_3$$
$$= s_{(2,1)}(x_1, x_2, x_3)$$



$$s_{(1^\ell)}(x_1 \cdots x_d) = x_1 \cdots x_\ell + \cdots + x_{d-\ell+1} \cdots x_d$$
$$= e_\ell(x_1 \cdots x_d)$$

<u>Translations</u>

	GL_n-reps	Sym funcs	Symmetric gp
	$\nabla^\lambda E$	s_λ	
Geometric want theory	$U \otimes V$	$f_U f_V$	

$Sym^\bullet(Sym^2 E) \supseteq Sym^2(Sym^2 E)$ $f_{(2)}(x_1^2, x_1x_2, x_2^2)$

$= \mathbb{C}[U, V, W] / (UV - W^2)$ $= Sym^2\langle e_1^2, e_1e_2, e_2^2 \rangle$ $= (x_1^2)(x_1^2) + (x_1^2)(x_1x_2) + \cdots$

$= \mathcal{O}(\ldots)$ $= \langle (e_1^2)(e_1^2), (e_1^2)(e_1e_2), \ldots \rangle$

$\nabla^\lambda(\nabla^\mu E)$ $(s_\lambda \circ s_\mu)(x_1 \cdots x_d)$
$$= s_\lambda(x^t : t \in SSYT_{[1..d]}(\mu))$$

where $\circ$ is the plethysm product: $f \circ g$ = evaluate f at the monomials in g

§3 Plethysms for $\nabla^\lambda Sym^\ell E$, $\dim E = 2$

<u>Prop</u>: The following are equivalent:

(1) $\nabla^\lambda Sym^\ell E \cong_{SL_2(\mathbb{C})} \nabla^\mu Sym^m E$ } same character

(2) $s_\lambda \circ s_{(\ell)}(\xi, \xi^{-1}) = s_\mu \circ s_{(m)}(\xi, \xi^{-1})$ } plethysm defn

(3) $s_\lambda(\xi^\ell, \xi^{\ell-2}, \ldots, \xi^{-\ell}) = s_\mu(\xi^m, \ldots, \xi^{-m})$ } Stanley HCF: $s_\lambda(1, \ldots, \xi^\ell) = \prod_{\square \in [\lambda]} \frac{q^{c(\square)+\ell+1} - 1}{q^{h(\square)} - 1}$

(4) $C(\lambda) + \ell + 1 / H(\lambda) = C(\mu) + m + 1 / H(\mu)$

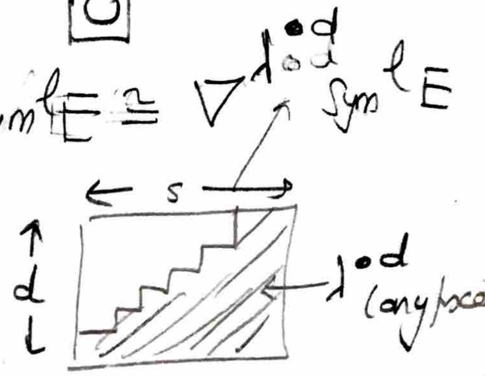
where $C(\lambda) = \{ j - i : (i, j) \in [\lambda] \}$

$H(\lambda) = \{$ hook lengths in $[\lambda] \}$ eg $\begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & & \\ \hline \end{array}$

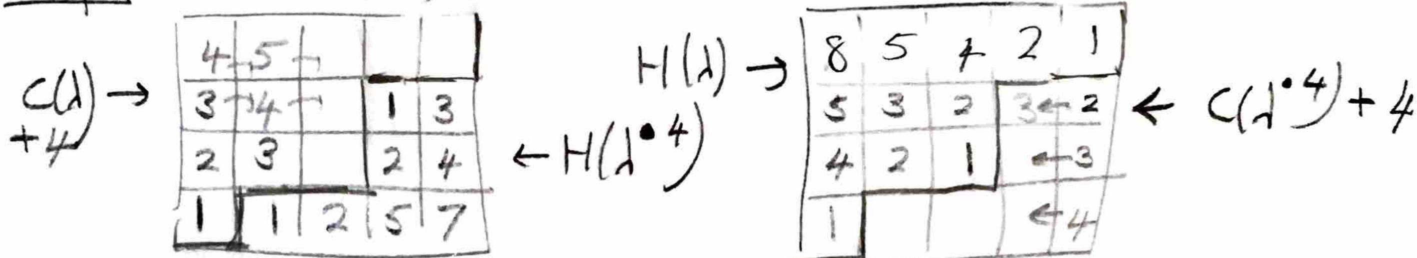
Example (Wronskian)

$$\boxed{l+1\ \ l+2 \cdots l+r} \,\Big/\, \boxed{0\ \ 1 \cdots r-1} \ \doteq\ \begin{array}{|c|} l+r \\ l+r-1 \\ \vdots \\ l+1 \end{array} \,\Big/\, \begin{array}{|c|} r-1 \\ \vdots \\ 0 \end{array} \qquad \square$$

Theorem 2 (King 85 (cf. Paget W 19)) Let ... $\nabla^\lambda \mathrm{Sym}^l E \cong \nabla^{\lambda^{\circ d}} \mathrm{Sym}^l E$
$\iff \lambda = \lambda^{\circ d}$ or $l = d-1$.



Example $\lambda = (4,3,3,1)$



$\{1^3, 2^4, 3^3, 4^3, 5^4, 6^2, 7, 8\}$ So $C(\lambda)+4 \cup H(\lambda^{\circ 4})$
$= C(\lambda^\circ)+4 \cup H(\lambda)$
$\Rightarrow$ (using (4)) $\nabla^\lambda \mathrm{Sym}^4 E \cong \nabla^{\lambda^{\circ 4}} \mathrm{Sym}^4 E$

Theorem 3 [McDowell W 21] Let G be a group, let V be a d-dim'l (finite or infinite) rep of G over an arbitrary field. Let $\lambda^{\circ d}$ be complement in $d \times s$ box. Then

$$\nabla^\lambda V \cong \nabla^{\lambda^{\circ d}} V^* \otimes (\det V)^s$$

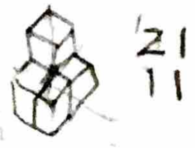
Generalises Theorem 1 to arbitrary fields and groups.

§4 Modular plethysms for $SL_2(\mathbb{F})$

King showed $\nabla^\lambda \mathrm{Sym}^l E \cong_{SL_2(\mathbb{C})} \nabla^{\lambda'} \mathrm{Sym}^{l+\ell(\lambda')-\ell(\lambda)} E$ for a wide class of partitions, including all hooks. The special case for $\lambda = (1^n)$ is the Wronskian iso, and for a rectangle

$$q^{-a\binom{b}{2}} s_{(a^b)}(1, q, \dots, q^{b+c-1}) =_{\text{up to } q} \sum_{\pi \in PP(a,b,c)} q^{|\pi|}$$

3D Young diagrams



Theorem 4 [Paget W 19] King's Thm gives all plethystic isos relating ∇^λ and $\nabla^{\lambda'}$

<u>Theorem 5</u> (McDowell–W 20) Let char $F = p$. There are infinitely many a, b such that if $e \gg 1$, the eight reps of $SL_2(\mathbb{F})$ obtained from

$$\nabla^{(a+1, 1^b)} \, \mathrm{Sym}^{p^e + b} E$$

by
- $\nabla^{(a+1,1^b)} \to \nabla^{(b+1,1^a)}$ and $p^e + b \to p^e + a$
- $\mathrm{Sym}^{p^e+b} E \to \mathrm{Sym}_{p^e+b} E$
- $\nabla^{(a+1,1^b)} \to \Delta^{(a+1,1^b)}$

are non-isomorphic.

<u>Problem</u> Which plethystic isos for $SL_2(\mathbb{C})$ have modular generalisations?