Minimal and maximal constituents of plethysms of Schur functions

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Algebraic Combinatorics and Group Actions

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Outline

- §1 Motivation: Examples of plethysms
- §2 Main result: Minimal and maximal constituents of $s_\mu \circ s_\nu$

§1 Polynomial representations of $\mathrm{GL}(V)$

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Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$: $(uv)(u'v') \mapsto uvu'v'$.

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  - let $\mathcal{C}$ be the image of the squaring map $V \hookrightarrow \mathrm{Sym}^2 V$,

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- Vanish doubly on $\mathcal{C}$: $(Y_{11}Y_{22} - Y_{12}^2)^2$
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- Such functions are in kernel of $\mathrm{Sym}^4(\mathrm{Sym}^2 V) \to \mathrm{Sym}^8 V$, so

$$\mathrm{Sym}^4(\mathrm{Sym}^2 V) \cong \Delta^{(4,4)}V \oplus \Delta^{(6,2)}V \oplus \Delta^{(8)}V.$$

Plethysm: Symmetric groups and wreath products

Take $\dim V \geq 4$. So $S_4 \leq \mathrm{GL}(V)$: $(13) \mapsto \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$.

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- Weight space $(1,1,1,1)$ inside $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is

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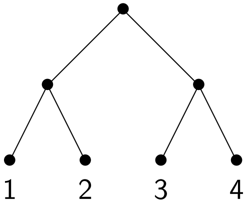
- Identify $(v_1 v_2)(v_3 v_4)$ with the set partition $\big\{\{1,2\},\{3,4\}\big\}$.
- Stabiliser $S_2 \wr S_2 = (S_2 \times S_2) \rtimes S_2 = \langle (12), (34) \rangle \rtimes \langle (13)(24) \rangle$.



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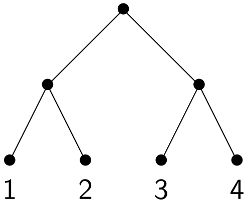

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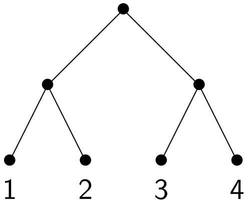

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- Character $\chi^{(2,2)} + \chi^{(4)}$, corresponding to $\Delta^{(2,2)} \oplus \Delta^{(4)}$.

Foulkes' Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into n sets each of size m.

Conjecture (Foulkes)

If $m \leq n$ then there is an injective map of S_{mn}-representations $\langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$.

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Equivalently, there is an injective map of $\mathrm{GL}(V)$-representations

$$\mathrm{Sym}^m(\mathrm{Sym}^n V) \to \mathrm{Sym}^n(\mathrm{Sym}^m V).$$

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Equivalently, if $\phi^{(m^n)}$ is the character of $\langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$, then $\langle \phi^{(n^m)}, \chi^\lambda \rangle \leq \langle \phi^{(m^n)}, \chi^\lambda \rangle$ for all $\lambda \in \mathrm{Par}(mn)$.

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- These are the only multiplicity-free Foulkes characters for $mn \geq 18$ (Saxl, 1980).

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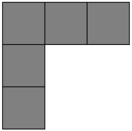
$$\phi^{(2^n)} = \sum_{\lambda \in \mathrm{Par}(n)} \chi^{2\lambda}$$

- Hence FC holds when $m = 2$.
- These are the only multiplicity-free Foulkes characters for $mn \geq 18$ (Saxl, 1980).
- Giannelli, MW 2014: results on decomposition numbers of symmetric groups obtained from local structure of $\langle \Omega^{(2^n)} \rangle$ over fields of prime characteristic.

Decomposition Numbers: 3-block of S_{12} with core $(3, 1, 1)$

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2) = \langle 2\rangle$	1									
$(9,4,1) = \langle 2,2\rangle$	1	1								
$(9,3,2) = \langle 2,1\rangle$	2	1	1							
$(8,4,2) = \langle 1\rangle$	1	1	1	1						
$(6^2,2) = \langle 1,2\rangle$				1	1					
$(6,4^4) = \langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2) = \langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1) = \langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1) = \langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2) = \langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5) = \langle 2,3\rangle$			1							
$(6,4,1^4) = \langle 2,2,3\rangle$							1			
$(6,3,2,1^3) = \langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2) = \langle 3,2\rangle$								1		
$(6,1^8) = \langle 2,3,3\rangle$						1				
$(5,4,2,1^3) = \langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2) = \langle 3,1\rangle$	1		1			1				1
$(3^2,2^4) = \langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4) = \langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6) = \langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5) = \langle 3,3\rangle$					1				1	
$(3,1^{11}) = \langle 3,3,3\rangle$					1					



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$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
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$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
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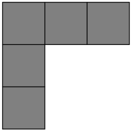

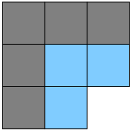

Decomposition Numbers: 3-block of S_{12} with core $(3,1,1)$

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
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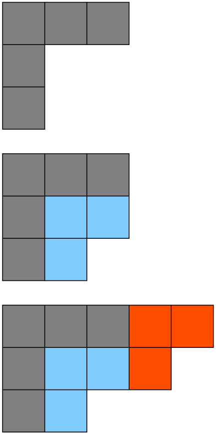

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$(6,4,2^2) = \langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1) = \langle 1,1,2\rangle$	2	1	1				1	1		
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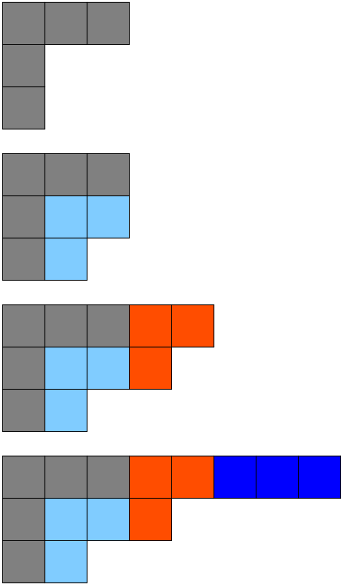

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$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
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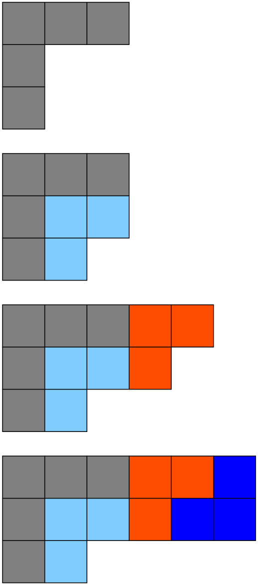

Foulkes' Conjecture and Howe's Conjecture

Let $\Omega^{(m^n)}$ be the set of all set partitions of $\{1, 2, \ldots, mn\}$ into n sets each of size m.

Conjecture (Howe 1987)

The $\mathbb{C}S_{mn}$-homomorphism $\theta^{(m^n)} : \langle \Omega^{(n^m)} \rangle_{\mathbb{C}} \to \langle \Omega^{(m^n)} \rangle_{\mathbb{C}}$ defined by

$$\{A_1, \ldots, A_m\} \mapsto \sum \{B_1, \ldots, B_n\},$$

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- Cheung, Ikenmeyer, Mkrtchyan 2015: $\theta^{(5^6)}$ is injective, hence FC is true for $m = 5$.

Open problem

Problem

Decompose $\phi^{(3^n)}$ into irreducible characters of S_{3n}.

Equivalently, decompose $\mathrm{Sym}^n(\mathrm{Sym}^3 V)$ into irreducible representations of $\mathrm{GL}(V)$.

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It is not hard to show that

$$\phi^{(3^n)}\big\downarrow_{S_{3n-1}} = \left(\phi^{(3^{n-1})} \times 1_{S_2}\right)\Big\uparrow^{S_{3n-1}}.$$

Computational evidence suggests that this property, together with $\langle \phi^{(3^n)}, 1_{S_{3n}} \rangle = 1$, determines $\phi^{(3^n)}$ uniquely.

Foulkes' Conjecture: computational results

- Müller, Neunhöffer 2005: FC is true if $m + n \leq 17$.
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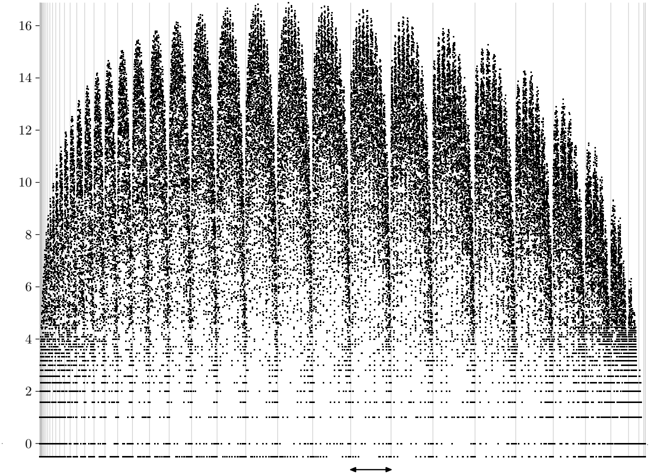

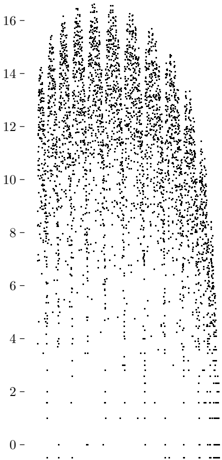

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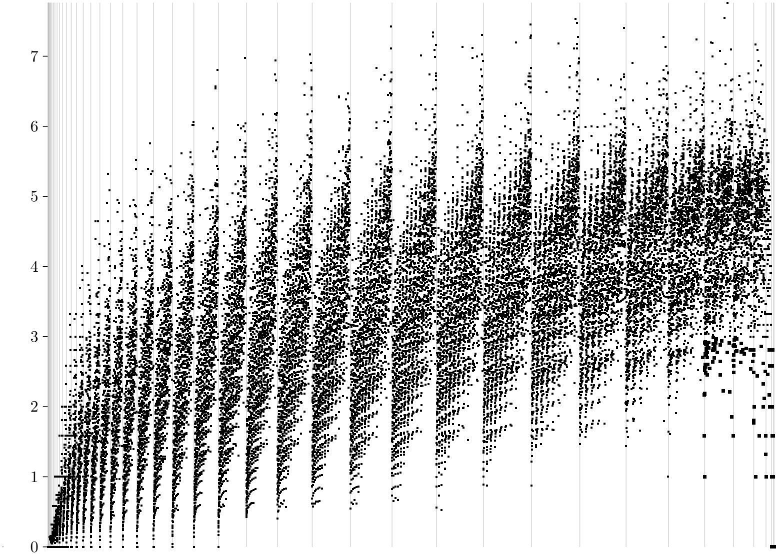

Plethysm: Symmetric polynomials.

Suppose $\dim V = d$.

- A basis of weight vectors for $\mathrm{Sym}^2 V$ is

$$\begin{array}{cccccc} v_1 v_1, & v_1 v_2, & v_2 v_2, & v_1 v_3, & \dots & v_d v_d \\ x_1^2 & x_1 x_2, & x_2^2, & x_1 x_3, & \dots & x_d^2 \end{array}$$

- The *formal character* of $\mathrm{Sym}^2 V$ is

$$s_{(2)}(x_1, \dots, x_d) = x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots + x_d^2.$$

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  Formal characters are symmetric polynomials.
- The formal character h of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is obtained by evaluating $s_{(2)}$ at the monomials $x_1^2, x_1 x_2, \dots$

$$\begin{array}{ccccc} (v_1 v_1)(v_1 v_1), & (v_1 v_1)(v_1 v_2), & (v_1 v_2)(v_1 v_2), & (v_1 v_1)(v_2 v_2), & \dots \\ x_1^2 x_1^2 & x_1^2 x_1 x_2, & x_1 x_2 x_1 x_2 & x_1^2 x_2^2, & \dots \end{array}$$

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$$\begin{aligned} h(x_1, \dots, x_d) &= x_1^4 + x_1^3 x_2 + 2x_1^2 x_2^2 + 2x_1^2 x_2 x_3 + 3x_1 x_2 x_3 x_4 \\ &= \left(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2 x_3 x_4 + \cdots\right) \\ &\qquad + \left(x_1^2 x_2^2 + x_1^2 x_2 x_3 + 2x_1 x_2 x_3 x_4 + \cdots\right) \\ &= s_{(4)}(x_1, \dots, x_d) + s_{(2,2)}(x_1, \dots, x_d) \end{aligned}$$

$\pi\lambda\eta\theta\upsilon\sigma\mu o\sigma$: Stanley's Problem 9

Let f and g be symmetric polynomials. Assume g has coefficients in $\mathbb{N}_0$ when expressed in the monomial basis. The *plethysm* $f \circ g$ is defined by evaluating f at the monomials of g.

- The formal character of $\Delta^\nu(\Delta^\mu V)$ is $s_\nu \circ s_\mu$.
- The corresponding character of S_{mn} is

$$\left(\widetilde{(\chi^\mu)^{\times n}} \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu\right)\big\uparrow_{S_m \wr S_n}^{S_{mn}}$$

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Problem (Weak Foulkes' Conjecture)

Show that if $m \leq n$ then $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ has non-negative coefficients.

Equivalently, $S_m \wr S_n$ has at least as many orbits as $S_n \wr S_m$ on the coset space $S_{mn}/S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, for each $\lambda \in \mathrm{Par}(mn)$.

$\pi\lambda\eta\theta\upsilon\sigma\mu o\sigma$: Stanley's Problem 9

Let f and g be symmetric polynomials. Assume g has coefficients in $\mathbb{N}_0$ when expressed in the monomial basis. The *plethysm* $f \circ g$ is defined by evaluating f at the monomials of g.

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Problem (Stanley, 2000)

Let $\mu \in \mathrm{Par}(m)$, $\nu \in \mathrm{Par}(n)$, $\lambda \in \mathrm{Par}(mn)$. Find a combinatorial interpretation of the coefficient of s_λ in $s_\nu \circ s_\mu$.

Plethysms and enumeration

Theorem (Read 1959)

$\langle s_{(2m)} \circ s_{(3)}, s_{(3m)} \circ s_{(2)} \rangle$ is the number of 3-regular graphs (with loops and multiple edges permitted) on $2m$ vertices.

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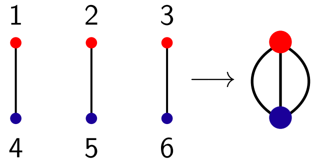

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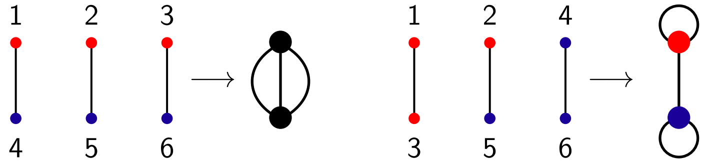

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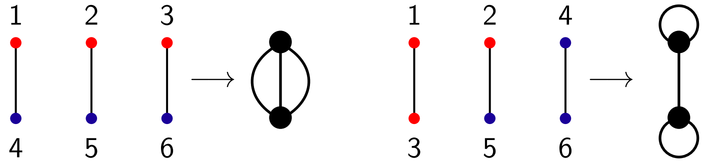

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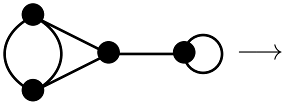

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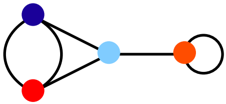 $\longrightarrow$

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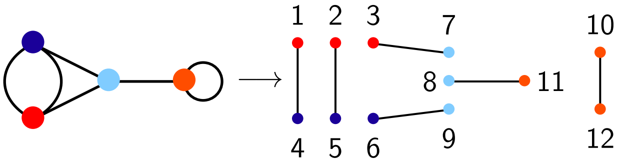

§2: Minimal and maximal constituents of plethysms

Let $\lambda, \lambda^\star \in \mathrm{Par}(r)$. We say λ *dominates* $\lambda^\star$, and write $\lambda \trianglerighteq \lambda^\star$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all j. For example

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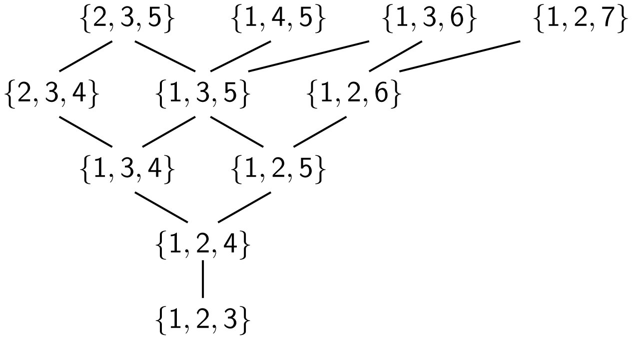
This solves a special case of Stanley's Problem 9.

Special case $\mu = (m)$ for minimals

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be m-subsets of $\mathbb{N}$, written so that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. We say that A *majorizes* B, and write $A \preceq B$, if

$$a_1 \leq b_1, \ldots, a_m \leq b_m.$$



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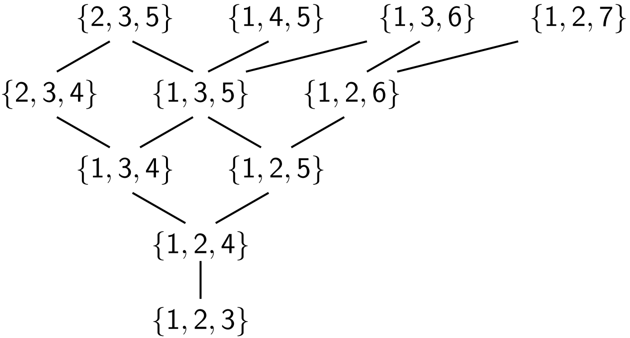

- A *closed set family of size r* is a family $\mathcal{P}$ of m-subsets of $\mathbb{N}$ such that $|\mathcal{P}| = r$ and if $B \in \mathcal{P}$ and $A \preceq B$ then $A \in \mathcal{P}$.

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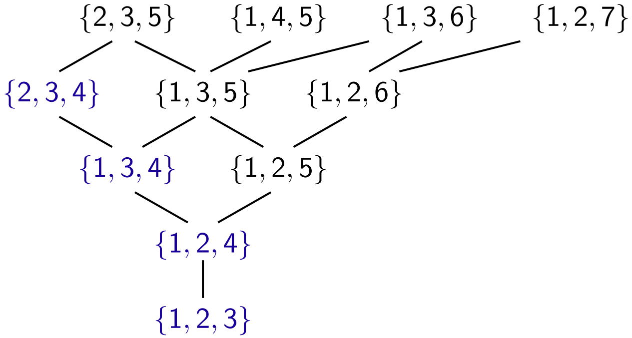

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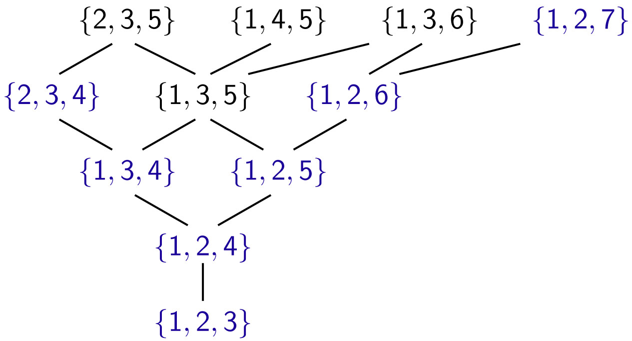

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- For example,

$$\big(\big\{\{1,2,3\},\{1,2,4\},\{1,3,4\}\big\},\big\{\{1,2,3\}\big\}\big)$$

  is a closed set family tuple of size $(3,1)$, weight $(4,3,3,2)$ and type $(4,4,3,1)$.

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Theorem (Paget, MW, 2014)

Let m be odd. The minimal partitions λ such that s_λ has non-zero coefficient in $s_\nu \circ s_{(m)}$ are precisely the minimal types of the closed set family tuples of size ν.

Special case $\nu = (n)$ for minimals

- A μ-tableau is *conjugate-semistandard* if its rows are strictly increasing and its columns are non-decreasing. When $\mu = (m)$ such tableaux correspond to m-subsets: $\{1, 3, 4\} \leftrightarrow \boxed{1\,|\,3\,|\,4}$.
- The majorization order generalizes to a partial order on conjugate-semistandard μ-tableaux.
- We define closed μ-tableau families and their weights and types analogously.

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is a closed $(2,1)$-tableau family of size 3, weight $(5,3,1)$ and type $(3,2,2,1,1)$.

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Theorem (Paget, MW, 2016)

Let m be odd and let $\mu \in \mathrm{Par}(n)$. The minimal partitions λ such that s_λ has non-zero coefficient in $s_{(n)} \circ s_\mu$ are precisely the minimal types of the closed μ-tableau families of size n.

This determines all minimal λ such that $\Delta^\lambda V$ appears in the coordinate ring of $\Delta^\mu V$.

Application to invariants of Riemann curvature tensor

mathoverflow

Questions Tags Users Badges Ur

A question on invariant theory of $GL_n(\mathbb{C})$.

12

1

Let ρ denote the irreducible algebraic representation of $GL_n(\mathbb{C})$ with the highest weight $(2,2,\underbrace{0,\dots,0}_{n-2})$.

Let $k \leq n/2$ be a non-negative integer. How to decompose into irreducible representations the representation $Sym^k(\rho)$?

More specifically, I am interested whether $Sym^k(\rho)$ contains the representation with the highest weight $(\underbrace{2,\dots,2}_{2k},\underbrace{0,\dots,0}_{n-2k})$, and if yes, whether the mutiplicity is equal to one.

A a side remark, the representation ρ has a geometric interpretation important for me: it is the space of curvature tensors, namely the curvature tensor of any Riemannian metric on $\mathbb{R}^n$ lies in ρ.

invariant-theory classical-invariant-theor dg.differential-geometry rt.representation-theory plethysm

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edited Oct 3 '12 at 19:28

asked Oct 3 '12 at 17:31

sva

4,239 • 18 • 43

Application to invariants of Riemann curvature tensor

14

The plethysm $\mathrm{Sym}^k\rho$ contains the irreducible representation with highest weight $(2,\ldots,2,0,\ldots,0)$ exactly once. It looks like a tricky problem to say much about its other irreducible constituents.

Let Δ^λ denote the Schur functor corresponding to the partition λ, and let E be an n-dimensional complex vector space. Using symmetric polynomials (or other methods) one finds

$$\mathrm{Sym}^2(\mathrm{Sym}^2 E) = \Delta^{(2,2)}E \oplus \mathrm{Sym}^4 E.$$

Therefore

$$\mathrm{Sym}^k\mathrm{Sym}^2\mathrm{Sym}^2 E \cong \sum_{r=0}^{k} \mathrm{Sym}^r(\Delta^{(2,2)}E) \otimes \mathrm{Sym}^{k-r}(\mathrm{Sym}^4 E).$$

The irreducible representations contained in the rth summand are labelled by partitions with at most $2r + (k-r) = k + r$ parts. So to show that $\mathrm{Sym}^k(\Delta^{(2,2)}(E))$ contains $\Delta^{(2^k)}E$, it suffices to show that $\Delta^{(2^k)}E$ appears in $\mathrm{Sym}^k\mathrm{Sym}^2\mathrm{Sym}^2 E$.

Let $U = \mathrm{Sym}^2 E$. There is a canonical surjection

$$\mathrm{Sym}^k(\mathrm{Sym}^2 U) \to \mathrm{Sym}^{2k} U.$$

given by mapping $(u_1u_1')\ldots(u_ku_k') \in \mathrm{Sym}^k(\mathrm{Sym}^2 U)$ to $u_1u_1'\ldots u_ku_k' \in \mathrm{Sym}^{2k}U$. Therefore $\mathrm{Sym}^k(\mathrm{Sym}^2 U)$ contains $\mathrm{Sym}^{2k}U = \mathrm{Sym}^{2k}(\mathrm{Sym}^2 E)$. It is well known that

$$\mathrm{Sym}^{2k}(\mathrm{Sym}^2 E) = \sum_{\lambda} \Delta^{2\lambda}(E)$$

where the sum is over all partitions λ of $2k$ and $2(\lambda_1,\ldots,\lambda_m) = (2\lambda_1,\ldots,2\lambda_m)$. Taking $\lambda = (1^{2k})$ we see that $\Delta^{(2^{2k})}E$ appears.

It remains to show that the multiplicity of $\Delta^{(2^k)}E$ in $\mathrm{Sym}^k(\Delta^{(2,2)}E)$ is 1. We work over $\mathbb{C}$, so there is a chain of inclusions

$$\mathrm{Sym}^k(\Delta^{(2,2)}(E)) \subseteq \mathrm{Sym}^k(\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E) \subseteq (\mathrm{Sym}^2 E)^{\otimes 2k}.$$

By the Littlewood–Richardson rule (or the easier Young's rule), the multiplicity of $\Delta^{(2^k)}E$ in the right-hand side is 1.

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answered Oct 4 '12 at 0:42
Mark Wildon
3,958 • 1 • 19 • 30

This is nice. – Dan Petersen Oct 4 '12 at 6:55