An introduction to modular plethysms

Mark Wildon

MFO August 2022

Outline

§1 Motivation: the Wronskian isomorphism

§2 Schur functors, Schur functions and a definition of plethysm

§3 Modular plethysms for $\mathrm{SL}_2(F)$

§4 Modular plethysms for the symmetric group

§1 Motivation: A modular Wronskian isomorphism

Let V be a vector space.

- $$\begin{aligned}\mathrm{Sym}^2 V &= V^{\otimes 2}/\langle v \otimes w - w \otimes v : v, w \in V\rangle \\ &= \langle vw : v \in V, w \in V\rangle\end{aligned}$$
- $$\begin{aligned}\bigwedge^2 V &= V^{\otimes 2}/\langle v \otimes v : v \in V\rangle \\ &= \langle v \wedge w : v \in V, w \in V\rangle\end{aligned}$$

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Observation. $\mathrm{Sym}^2\mathbb{C}^d$ and $\bigwedge^2 \mathbb{C}^{d+1}$ both have dimension $\binom{d+1}{2}$.

- For instance, if $v_1, \ldots, v_d$ is a basis for $\mathbb{C}^d$ then $\mathrm{Sym}^2\mathbb{C}^d$ has basis $v_1^2, \ldots, v_d^2, v_1 v_2, \ldots, v_{d-1} v_d$ of size $d + \binom{d}{2}$.

Question. Asked by მამუკა ჯიბლაძე on MathOverflow: Is there a natural isomorphism between these vector spaces?

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Question. Asked by მამუკა ჯიბლაძე on MathOverflow: Is there a natural isomorphism between these vector spaces?

Answer. Yes!

§1 Motivation: the Wronskian isomorphism

Are there nice isomorphisms $S^2(k^n) \cong \Lambda^2(k^{n+1})$?

Asked 1 year, 1 month ago Active 1 year, 1 month ago Viewed 349 times

12

This might be forced to migrate to math.SE but let me still risk it.

The spaces $S^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

⋮

share cite edit close flag edited Jan 15 '19 at 10:52

asked Jan 15 '19 at 9:45
მამუკა ჯიბლაძე
13.9k ●3 ●50 ●125

19

Let E be a 2-dimensional k-vector space. The Wronksian isomorphism is an isomorphism of $\mathrm{SL}(E)$-modules $\bigwedge^m S^{m+r-1}(E) \cong S^m S^r(E)$. It is easiest to deduce it from the corresponding identity in symmetric functions (specialized to 1 and q), but it can also be defined explicitly: see for example Section 2.5 of this paper of Abdesselam and Chipalkatti.

In particular, identifying $S^n(E)$ with the homogeneous polynomial functions on E of degree n, their definition becomes the map $\wedge^2 S^n(E) \to S^2 S^{n-1}(E)$ defined by

$$f \wedge g \mapsto \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Now $S^n(E) \cong k^{n+1}$ and $S^{n-1}(E) \cong k^n$, so we have the required isomorphism $S^2 k^n \cong \wedge^2 k^{n+1}$.

share cite edit delete flag edited Jan 15 '19 at 11:49

answered Jan 15 '19 at 11:09
Mark Wildon
8,018 ●1 ●32 ●51

Action of $\mathrm{SL}_2(F)$ on $\bigwedge^2 \mathrm{Sym}^2 E$ where $E = \langle X, Y\rangle$

$$
\begin{array}{cc} X & Y \\ \end{array}
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto
\begin{array}{c}
\begin{array}{ccc} X^2 \wedge XY & Y^2 \wedge XY & X^2 \wedge Y^2 \end{array} \\
\begin{pmatrix}
\alpha^3\delta - \alpha^2\beta\gamma & \alpha\beta^2\delta - \alpha\beta^2\gamma & 2\alpha^2\beta\delta - 2\alpha\beta^2\gamma \\
-\alpha\gamma^2\delta + \beta\gamma^3 & \alpha\delta^3 - \beta\gamma\delta^2 & 2\beta\gamma^2\delta - 2\alpha\gamma\delta^2 \\
\alpha^2\gamma\delta - \alpha\gamma^2\beta & \beta^2\gamma\delta - \alpha\beta\delta^2 & \alpha^2\delta^2 - \beta^2\gamma^2
\end{pmatrix}
\end{array}
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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$. The matrices are not even conjugate if $\mathrm{char}\, F = 2$! Instead
$$\mathrm{Sym}_2 E = \langle X \otimes X, Y \otimes Y, X \otimes Y + Y \otimes X \rangle \cong (\mathrm{Sym}^2 E)^\star.$$
- So $(\mathrm{Sym}^2 E)^\star \cong_{\mathrm{SL}_2(F)} \bigwedge^2 \mathrm{Sym}^2 E$ and the duality is essential.

Duality and the modular Wronskian isomorphism

Theorem (McDowell–W 2020)

Let F be any field. Let $E \cong F^2$ be the natural representation of $\mathrm{SL}_2(F)$. There is an explicit isomorphism

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(F)} \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

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As a corollary we obtain a modular version of Hermite reciprocity.

Corollary (Hermite 1854 over $\mathbb{C}$, McDowell–W 2020)

Let F be any field. Let $m, \ell \in \mathbb{N}$ and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(F)$. Then

$$\mathrm{Sym}_m \mathrm{Sym}^\ell E \cong \mathrm{Sym}^\ell \mathrm{Sym}_m E$$

by an explicit map.

§2 Schur functors and Schur functions

- Polynomial representations of $\mathrm{GL}(E)$ with $E = \langle e_1, e_2, e_3 \rangle \cong \mathbb{C}^3$.

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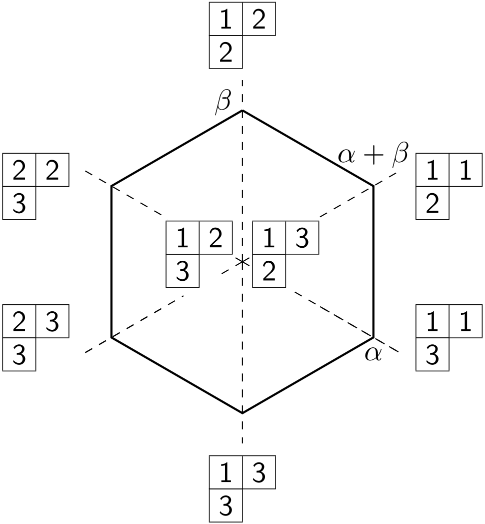

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- Tensor product: $\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E$
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- Symmetric functions
  - $s_{(2)}(y_1, y_2, y_3) \ = y_1^2 + y_2^2 + y_3^3 + y_1 y_2 + y_1 y_3 + y_2 y_3$
  - $s_{(2,1)}(x_1, x_2, x_3) = x^{\begin{smallmatrix}1&1\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&1\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} + x^{\begin{smallmatrix}1&3\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&2\\3&\end{smallmatrix}} + x^{\begin{smallmatrix}2&3\\3&\end{smallmatrix}}$

    $= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + \cdots + x_2 x_3^2$

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    $= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + \cdots + x_2 x_3^2$
  - Multiplication: $s_{(2)}(x_1, x_2)^2 = (x_1^2 + x_2^2 + x_1 x_2)^2$
  - Evaluate $s_{(2)}(y_1, y_2, y_3)$ at monomials in $s_{(2)}(x_1, x_2)$ to get

    $$s_{(2)}(x_1^2, x_2^2, x_1 x_2) = (x_1^2)^2 + (x_1^2)(x_2^2) + (x_1^2)(x_1 x_2) + \cdots + (x_1 x_2)^2.$$
  - **Plethysm**: $(s_\nu \circ s_\mu)(x_1, x_2, \ldots, x_d) = s_\nu$ evaluated at monomials in $s_\mu(x_1, \ldots, x_d)$. Equivalently: formal character of $\nabla^\nu\big(\nabla^\mu(E)\big)$

Combinatorial definitions

Given a tableau t let $x^t = x_1^{a_1} x_2^{a_2} \ldots$ where a_i is the number of entries of t equal to i.

Definition (Schur function)

Let μ be a partition. The *Schur function* s_μ is the generating function enumerating semistandard μ-tableaux by weight: $\sum_{t \in \mathrm{SSYT}(\mu)} x^t$.

For example $x^{\begin{smallmatrix} 1 & 1 & 2 \\ 3 & 5 & \end{smallmatrix}} = x_1^2 x_2 x_3 x_5$ and

$$\begin{aligned} s_{(2)}(x_1, x_2, \ldots) &= x^{\boxed{1\,1}} + x^{\boxed{1\,2}} + x^{\boxed{2\,2}} + x^{\boxed{1\,3}} + \cdots \\ &= x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots \end{aligned}$$

Definition

Let μ and ν be partitions. Let $\mathrm{SSYT}(\mu) = \{t(1), t(2), \ldots\}$. The *plethystic product* of s_ν and s_μ is $s_\nu \circ s_\mu = s_\nu(x^{t(1)}, x^{t(2)}, \ldots)$.

Warning. I haven't defined a general plethysm. Note $\circ$ is not linear in its second component: $f \circ (g + h) \neq f \circ g + f \circ h$.

§3 Plethysms and Stanley's Hook Content Formula

Theorem

Let $E = \langle X, Y\rangle$ be natural representation of $\mathrm{SL}_2(\mathbb{C})$. Let λ and μ be partitions and let $\ell, m \in \mathbb{N}$. The following are equivalent:

(i) $\nabla^\lambda \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(\mathbb{C})} \nabla^\mu \mathrm{Sym}^m E$;

(ii) $(s_\lambda \circ s_{(\ell)})(q, q^{-1}) = (s_\mu \circ s_{(m)})(q, q^{-1})$;

(iii) $s_\lambda(q^\ell, q^{\ell-2}, \ldots, q^{-\ell}) = s_\mu(q^m, q^{m-2}, \ldots, q^{-m})$;

(iv) $s_\lambda(1, q, \ldots, q^\ell) = s_\mu(1, q, \ldots, q^m)$ *up to a power of* q;

(i) *representations of* $\mathrm{SL}_2(\mathbb{C})$	$\longleftrightarrow$	(ii) *plethysms of symmetric functions*	$\longleftrightarrow$	(iv) *combinatorial enumeration*

Example of (iv) $\iff$ (i): Hermite reciprocity over $\mathbb{C}$.

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(v) $C(\lambda) + \ell + 1 \big/ H(\lambda) = C(\mu) + m + 1 \big/ H(\mu)$

where $\big/$ is difference of multisets (negative multiplicities okay) and

- $C(\lambda) = \{j - i : (i, j) \in [\lambda]\}$ *is the multiset of contents of* λ;
- $H(\lambda) = \{h_{(i,j)} : (i, j) \in [\lambda]\}$ *is the multiset of hook lengths of* λ.

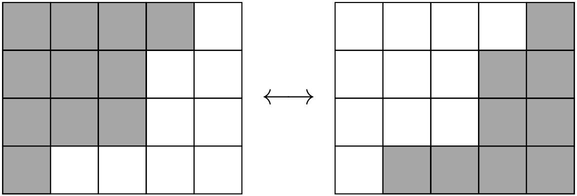
Part (v) is a corollary of Stanley's Hook Content Formula.

Example of (v) $\Longleftrightarrow$ (i): Wronksian isomorphism over $\mathbb{C}$.

Plethystic complement isomorphism for $\mathrm{SL}_2(\mathbb{C})$

Let λ be a partition contained in a box with d rows and s columns. Let $\lambda^{\bullet d}$ be its complement. For example if $s = 5$, $d = 4$ then

$$(4, 3, 3, 1)^{\bullet 4} = (4, 2, 2, 1).$$



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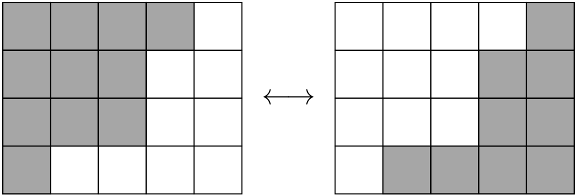


Theorem (King 1985 [if], Paget–W 2019 [only if])

Let E be the natural representation of $\mathrm{SL}_2(\mathbb{C})$. Let λ have at most d parts. Then

$$\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^{\lambda^{\bullet d}} \mathrm{Sym}^\ell E$$

if and only if $\lambda = \lambda^{\bullet d}$ or $\ell = d - 1$.

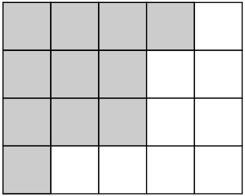
From the complement isomorphism to combinatorics

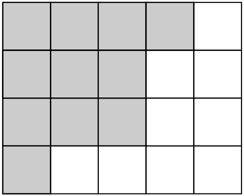
For example, using a rectangle with 4 rows and 5 columns,

$$\nabla^{(4,3,3,1)}\mathrm{Sym}^3 E \cong \nabla^{(4,2,2,1)}\mathrm{Sym}^3 E.$$

By (i) $\Longrightarrow$ (v) taking $\lambda = (4,3,3,1)$, $\lambda^{\bullet 4} = (4,2,2,1)$

$$C(\lambda) + 4/H(\lambda) = C(\lambda^{\bullet 4}) + 4/H(\lambda^{\bullet 4}).$$





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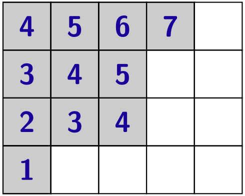
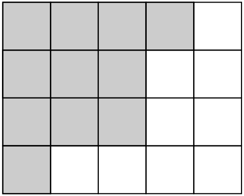
By (i) $\implies$ (v) taking $\lambda = (4,3,3,1)$, $\lambda^{\bullet 4} = (4,2,2,1)$

$$C(\lambda) + 4 \cup H(\lambda^{\bullet 4}) = C(\lambda^{\bullet 4}) + 4 \cup H(\lambda).$$

$C(\lambda) + 4$





From the complement isomorphism to combinatorics

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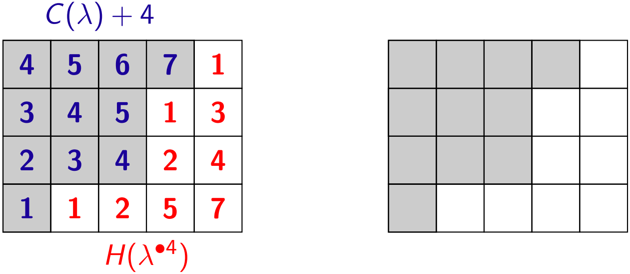

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$C(\lambda) + 4$

4	5	6	7	1
3	4	5	1	3
2	3	4	2	4
1	1	2	5	7

$H(\lambda^{\bullet 4})$

$H(\lambda)$

7	5	4	1	
5	3	2		
4	2	1		
1				

From the complement isomorphism to combinatorics

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3	4	5	1	3
2	3	4	2	4
1	1	2	5	7

$H(\lambda^{\bullet 4})$

$H(\lambda)$

7	5	4	1	1
5	3	2	3	2
4	2	1	4	3
1	7	6	5	4

$C(\lambda^{\bullet 4}) + 4$

Either way have same multiset: $\{1^4, 2^3, 3^3, 4^4, 5^3, 6, 7^2\}$

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By (i) $\implies$ (v) taking $\lambda = (4,3,3,1)$, $\lambda^{\bullet 4} = (4,2,2,1)$

$$C(\lambda) + 4 \cup H(\lambda^{\bullet 4}) = C(\lambda^{\bullet 4}) + 4 \cup H(\lambda).$$

$C(\lambda) + 4$

4_0	5_1	6_2	7_3	1_0
3_0	4_1	5_2	1_0	3_1
2_0	3_1	4_2	2_0	4_1
1_0	1_0	2_1	5_2	7_3

$H(\lambda^{\bullet 4})$

$H(\lambda)$

7_3	5_2	4_1	1_0	1_0
5_2	3_1	2_0	3_1	2_0
4_2	2_1	1_0	4_1	3_0
1_0	7_3	6_2	5_1	4_0

$C(\lambda^{\bullet 4}) + 4$

Either way have same multiset: $\{1^4, 2^3, 3^3, 4^4, 5^3, 6, 7^2\}$

Problem. A 2001 theorem of Christine Bessenrodt implies a stronger version with arm-lengths. Interpret this as a plethysm of Jack symmetric functions.

Modular complements

Theorem (McDowell–W 2020)

- *Let G be a group;*
- *Let V be a d-dimensional representation of G over an arbitrary field;*
- *Let $s \in \mathbb{N}$, and let λ be a partition with $\ell(\lambda) \leq d$ and first part at most s.*
- *Recall that $\lambda^{\bullet d}$ denotes the complement of λ in the $d \times s$ rectangle.*

There is an explicit isomorphism

$$\nabla^{\lambda} V \cong \nabla^{\lambda^{\bullet d}} V^{\star} \otimes (\det V)^{\otimes s}.$$

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This generalizes the complementary partition isomorphism from $\mathrm{SL}_2(\mathbb{C})$ to arbitrary fields and groups. In fact, over any ring.

A ‘no-go’ result in positive characteristic

Theorem (King 1985)

Let E be the natural representation of $\mathrm{SL}_2(\mathbb{C})$ and let $m \in \mathbb{N}$. For a large class of partitions λ, there is an explicit isomorphism

$$\nabla^{\lambda}\mathrm{Sym}^{m+\ell(\lambda)}E \cong_{\mathrm{SL}(E)} \nabla^{\lambda'}\mathrm{Sym}^{m+\ell(\lambda')}E.$$

- In particular, King's result holds when λ is a hook; that is $\lambda = (a+1, 1^b)$ for some $a, b \in \mathbb{N}_0$.
- Just for this talk: say that a partition in the class for King's theorem is ‘*royal*’.

Theorem (Paget-W 2019)

Let E be the natural representation of $\mathrm{SL}_2(\mathbb{C})$. There is a plethystic isomorphism

$$\nabla^{\lambda}\mathrm{Sym}^{m}E \cong_{\mathrm{SL}(E)} \nabla^{\lambda'}\mathrm{Sym}^{m'}E$$

if and only if λ is royal and $m - m' = \ell(\lambda) - \ell(\lambda')$.

A 'no-go' result in positive characteristic

Let F be an infinite field of prime characteristic p and let E be the natural representation of $\mathrm{SL}_2(F)$.

Theorem (McDowell–W 2020)

There exist infinitely many pairs (a, b) such that, provided e is sufficiently large, the eight representations of $\mathrm{SL}_2(F)$ obtained from $\nabla^{(a+1,1^b)}\mathrm{Sym}^{p^e+b}E$ by

- *Replacing ∇ with Δ (duality)*
- *Replacing $(a+1, 1^b)$ with $(b+1, 1^a)$ and $p^e + b$ with $p^e + a$ (King conjugation);*
- *Replacing $\mathrm{Sym}^\ell E$ with $\mathrm{Sym}_\ell E$ (another duality);*

are all non-isomorphic.

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are all non-isomorphic.

Problem

What plethystic isomorphisms of representations of $\mathrm{SL}_2(\mathbb{C})$ have modular analogues?

§4 Modular plethysms for the symmetric group

Problem (Decomposition numbers)

Determine the composition factors of Specht modules over fields of prime characteristic.

	(6)	(5,1)	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
$(5,1)$	1	1					
$(4,2)$	·	·	1				
$(3,3)$	·	1	·	1			
$(4,1,1)$	·	1	·	·	1		
$(3,2,1)$	1	1	·	1	1	1	
$(2,2,1,1)$	·	·	·	·	·	·	1
$(2,2,2)$	1	·	·	·	·	1	·
$(3,1,1,1)$	·	·	·	·	1	1	·
$(2,1,1,1,1)$	·	·	·	1	·	1	·
$(1,1,1,1,1,1)$	·	·	·	1	·	·	·

For instance the Specht module $S^{(3,3)}$ has composition factors labelled by $(5,1)$ and $(3,3)$.

Even partitions and plethysms

For $n \in \mathbb{N}$,

$$s_n \circ s_2 = \sum_{\lambda \in \mathrm{Par}(n)} s_{2\lambda}$$

where 2λ is the *even* partition obtained by doubling each part of λ. Equivalently, for the symmetric group,

$$\mathbb{C}\big\uparrow_{S_2 \wr S_n}^{S_{2n}} = \bigoplus_{\lambda \in \mathrm{Par}(n)} S^{2\lambda}.$$

Given a p-core γ, let $\mathcal{E}(\gamma)$ be the set of even partitions obtained from γ by adding the least possible number of disjoint p-hooks.

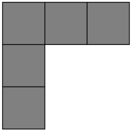
- For example if $p = 3$ then $\mathcal{E}\left(\ydiagram{1,1}\right) = \left\{(6,2), (4,4), (4,2,2)\right\}$

Theorem (Giannelli–W 2014)

Let p be an odd prime and let γ be a p-core. Let $\lambda \in \mathcal{E}(\gamma)$ be maximal. The column of the decomposition matrix labelled by λ has entries 0 *and* 1*. Moreover its non-zero entries are in rows labelled by $E(\gamma)$*

Decomposition Numbers: 3-block of S_{12} with core $(3,1,1)$

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



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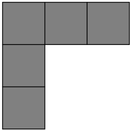
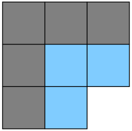
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					





Decomposition Numbers: 3-block of S_{12} with core $(3,1,1)$

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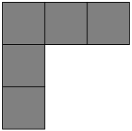
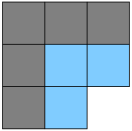
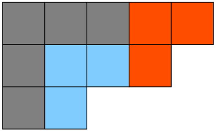
$(12,1^2) = \langle 2\rangle$	1									
$(9,4,1) = \langle 2,2\rangle$	1	1								
$(9,3,2) = \langle 2,1\rangle$	2	1	1							
$(8,4,2) = \langle 1\rangle$	1	1	1	1						
$(6^2,2) = \langle 1,2\rangle$				1	1					
$(6,4^4) = \langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2) = \langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1) = \langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1) = \langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2) = \langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5) = \langle 2,3\rangle$			1							
$(6,4,1^4) = \langle 2,2,3\rangle$							1			
$(6,3,2,1^3) = \langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2) = \langle 3,2\rangle$								1		
$(6,1^8) = \langle 2,3,3\rangle$						1				
$(5,4,2,1^3) = \langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2) = \langle 3,1\rangle$	1		1			1				1
$(3^2,2^4) = \langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4) = \langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6) = \langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5) = \langle 3,3\rangle$					1				1	
$(3,1^{11}) = \langle 3,3,3\rangle$					1					

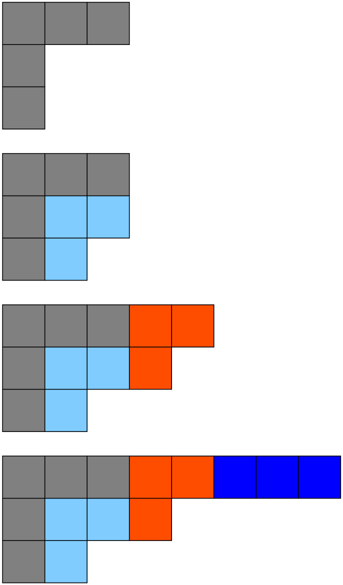






Decomposition Numbers: 3-block of S_{12} with core $(3, 1, 1)$

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2) = \langle 2\rangle$	1									
$(9,4,1) = \langle 2,2\rangle$	1	1								
$(9,3,2) = \langle 2,1\rangle$	2	1	1							
$(8,4,2) = \langle 1\rangle$	1	1	1	1						
$(6^2,2) = \langle 1,2\rangle$				1	1					
$(6,4^4) = \langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2) = \langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1) = \langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1) = \langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2) = \langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5) = \langle 2,3\rangle$			1							
$(6,4,1^4) = \langle 2,2,3\rangle$							1			
$(6,3,2,1^3) = \langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2) = \langle 3,2\rangle$								1		
$(6,1^8) = \langle 2,3,3\rangle$						1				
$(5,4,2,1^3) = \langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2) = \langle 3,1\rangle$	1		1			1				1
$(3^2,2^4) = \langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4) = \langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6) = \langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5) = \langle 3,3\rangle$					1				1	
$(3,1^{11}) = \langle 3,3,3\rangle$					1					



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$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
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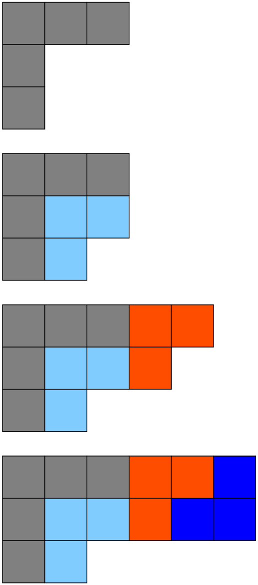
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					

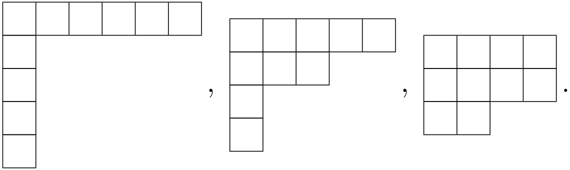


The plethysm $s_{(1^n)} \circ s_{(2)}$

Proposition (Paget-W 2022)

Let $\nu \in \mathrm{Par}(n)$ and let $\mu \in \mathrm{Par}(m)$ with $m, n \geq 2$. The only plethysms $s_\nu \circ s_\mu$ in which every constituent is both maximal and minimal in the dominance order are $s_{(1^n)} \circ s_{(2)}$ and $s_{(1^n)} \circ s_{(1^2)}$.

For example, $s_{(1^5)} \circ s_{(2)}$ has constituents



This is a special case of a much more general theorem extending joint work from 2019 and 2021 that gives an explicit combinatorial description of all maximal and minimal constituents in plethysms.

Problem

Get results on decomposition numbers from the monomial modules for the symmetric group corresponding to $s_{(k)}(s_{(1^n)} \circ s_{(1^2)})$.

Thank you!

Thank you!

Some more suggestions for problems on modular plethysms are in my MFO 'Research summary'.