Bristol Algebra Seminar 16 November.

An introduction to plethysms of symmetric functions and representations of general linear groups

Thanks, dictionary / interplay

Polynomial reps of $GL_d(\mathbb{C})$ degree r; $E = \langle e_1 \cdots e_d \rangle_{\mathbb{C}}$	$\leftrightarrow$ Symmetric functions of degree r in $x_1 \cdots x_d$
$\mathrm{Sym}^r E$	$s_{(r)}$ complete
$\Lambda^r E$	$s_{(1^r)}$ elementary
$\nabla^\lambda E$	s_λ Schur func
$U \oplus V$	$\mathrm{tr}\,U + \mathrm{tr}\,V$
$U \otimes V$	$\mathrm{tr}\,U \times \mathrm{tr}\,V$
Composition: $\nabla^\nu \nabla^\mu E$	plethysm $s_\nu \circ s_\mu$
weight space decomposition $V = \bigoplus_{\substack{(\alpha_1 \cdots \alpha_d) \\ \alpha_1 + \cdots + \alpha_d = r}} V_\alpha$	$\mathrm{tr}_V \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_d \end{pmatrix} = \sum_\alpha \dim V_\alpha \, x_1^{\alpha_1} \cdots x_d^{\alpha_d}$

§1 Polynomial reps

Examples $E = \langle e_1, \ldots, e_d \rangle \circlearrowleft GL_d(\mathbb{C})$

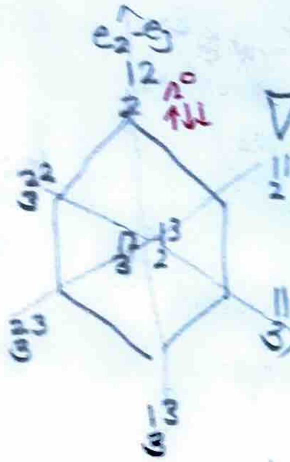
$$E \otimes E = \langle e_i \otimes e_j + e_j \otimes e_i \rangle \oplus \langle e_i \otimes e_j - e_j \otimes e_i \rangle$$

$$\cong \mathrm{Sym}^2 E \oplus \Lambda^2 E$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{array}{ccc} e_1^2 & e_1 e_2 & e_2^2 \\ \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{array}$$

'polynomial'

$d = 3$: $\underset{27}{E \otimes E \otimes E} \cong \underset{10}{\mathrm{Sym}^3 E} \oplus \underset{1}{\Lambda^3 E} \oplus ?$



$$\nabla^{(2,1)} E = \left\langle F\left(\begin{array}{cc} a \leq b \\ \wedge \\ c \end{array} \right) : a, b, c \in \{1 \cdots d\} \right\rangle \subseteq \mathrm{Sym}^2 E \otimes E$$

where $F\left(\begin{array}{|c|c|} \hline a & b \\ \hline c \\ \cline{1-1} \end{array}\right) = e_a e_b \otimes e_c - e_c e_b \otimes e_a$

$$\nabla^\lambda E = \langle F(t) : t \in SSYT(\lambda) \rangle$$

Irreducibles

- $\nabla^{(r)} E = \langle F(\boxed{a_1 a_2 \cdots a_r}) \rangle = \mathrm{Sym}^r E$
- $\nabla^{(1^r)} E = \langle F\begin{pmatrix} a_1 \\ \wedge \\ \vdots \\ \wedge \\ a_r \end{pmatrix} \rangle \cong \Lambda^r E$

bosons / fermions.

Composition of reps $d = 2$

$$\nabla^{(2,2)} E \longrightarrow \mathrm{Sym}^2(\mathrm{Sym}^2 E) \longrightarrow \mathrm{Sym}^4 E$$

$$\langle (e_1^2)(e_2^2) - (e_1 e_2)^2 \rangle \longrightarrow \langle (e_1^2)(e_1^2), (e_1^2)(e_1 e_2), (e_1^2)(e_2^2), (e_1 e_2)(e_1 e_2), (e_1 e_2)(e_2^2), (e_2^2)(e_2^2) \rangle \longrightarrow e_1^2 e_2^2$$

$$= F\left(\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}\right)$$

$$\mathrm{Sym}^2(\mathrm{Sym}^2 E) \cong \nabla^{(2,2)} E \oplus \mathrm{Sym}^4 E$$

Weight space decomp $V = \bigoplus_{\substack{(\alpha_1 \cdots \alpha_d) \\ \alpha_1 + \cdots + \alpha_d = r}} V_\alpha$ where $V_\alpha = \{ v \in V : \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_d \end{pmatrix} v = x_1^{\alpha_1} \cdots x_d^{\alpha_d} v \}$

$v \in V_\alpha$ is highest weight if $\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} v = v$.

Wonderful fact! If $v \in V_\lambda$ is highest weight then v generates a subrep $\nabla^\lambda E$.

(e.g. v above is hw wt $(2,2)$)

Exercise Show $\overset{\text{delay}}{\downarrow}$ that $v^a (e_1^2)^{n-2a}$ is highest weight $(2a, 2a) + (2(n-2a), 0)$ and hence

$$\mathrm{Sym}^2 \mathrm{Sym}^n E \cong \nabla^{2n} E \oplus \nabla^{(2n-2)} E \oplus \cdots$$

§2 ~~Some~~ $SL_2(\mathbb{C})$ ~~reps~~ isomorphism

$$\dim \mathrm{Sym}^3 \mathbb{C}^3 = \dim \Lambda^3 \mathbb{C}^5$$

$$\left(\!\!\binom{3}{3}\!\!\right) \qquad \binom{5}{3}$$

Generally $\left(\!\!\binom{a}{b}\!\!\right) = \binom{a+b-1}{b}$.

Categorification: $\mathrm{Sym}^b(\mathrm{Sym}^{a-1}\mathbb{C}^2) \underset{SL_2(\mathbb{C})}{\cong} \Lambda^b \mathrm{Sym}^{a+b-2}\mathbb{C}^2$.

Thm [McDowell – Wildon] Holds for an arbitrary field $\mathbb{F}$ provided 'Sym^{a-1}' is replaced with a dual functor Sym_{a-1}.

§3 Plethysms of symmetric functions

$d=2$ $\mathrm{tr}_{\mathrm{Sym}^2 E}\begin{pmatrix}\alpha & \\ & \delta\end{pmatrix} = \mathrm{tr}\begin{pmatrix}\alpha^2 & & \\ & \alpha\delta & \\ & & \delta^2\end{pmatrix} = \alpha^2+\alpha\delta+\delta^2 = s_{(2)}(\alpha,\delta)$

$\mathrm{tr}\, \mathrm{Sym}^2 E\begin{pmatrix}x_1 & & \\ & \ddots & \\ & & x_d\end{pmatrix} = \sum_i x_i^2 + \sum_{i<j} x_i x_j = s_{(2)}(x_1 \dots x_d)$

$d=3$ $\mathrm{tr}\, \nabla^{(2,1)}E\begin{pmatrix}x_1 & & \\ & x_2 & \\ & & x_3\end{pmatrix} = x^{\boxed{1\,1}\atop\boxed{3}} + x^{1\,2 \atop 2} + \cdots + x^{2\,3 \atop 3}$

$= x_1^2 x_2 + x_1 x_2^2 + \cdots + x_2 x_3^2$

Wonderful fact 2 The traces of diagonal matrices determine the rep (closure...?)

Defn $s_\lambda(x_1 \dots x_d) = \sum_{\substack{t \in SSYT(\lambda) \\ \text{entries} \le d}} x^t \quad \left(= \mathrm{tr}\, \nabla^\lambda E\begin{pmatrix}x_1 & & \\ & \ddots & \\ & & x_d\end{pmatrix}\right)$

πληθυσμος $d=2$

$\mathrm{tr}_{\mathrm{Sym}^2(\mathrm{Sym}^2 E)} = \mathrm{tr}_{\mathrm{Sym}^2 E}\begin{pmatrix}\alpha^2 & & \\ & \alpha\delta & \\ & & \delta^2\end{pmatrix} = (\alpha^2)(\alpha^2) + (\alpha^2)(\alpha\delta) + \cdots + (\delta^2)(\delta^2)$

$= s_{(2)}(y_1, y_2, y_3)\big|_{y_1=\alpha^2,\, y_2=\alpha\delta,\, y_3=\delta^2} = (s_{(2)} \circ s_{(2)})(\alpha,\delta)$

Generally $s_\nu \circ s_\mu = s_\nu$ (monomials in s_μ)

Examples $s_{(2)} \circ s_{(2)} = (x_1^2)(x_1^2) + (x_1^2)(x_1x_2) + \cdots + (x_2^2)(x_2^2)$

$= x_1^4 + x_1^3 x_2 + 2x_1^2 x_2^2 + \cdots = s_{(4)} + s_{(2,2)}$

$s_{(1,1)} \circ s_{(2)} = (x_1^2)(x_1 x_2) + \cdots$

$= x_1^3 + x_1^2 x_2 + x_1 x_2^2 = s_{(3,1)}$.

Problem Decompose $s_\nu \circ s_\mu$ as a sum of Schur functions.

Eqv: find $\langle s_\nu \circ s_\mu, s_\lambda \rangle = [\nabla^\nu \nabla^\mu E : \nabla^\lambda E]$ for all λ. '$\Lambda^2 \mathrm{Sym}^2 \cong \nabla^{(3,1)}$'

(Stanley Problem 9: $\nu = (n)$, $\mu = (m)$)?

§4 Foulkes' Conjecture

Conjecture [Foulkes '50] If $n \geq m$ then $s_n \circ s_m - s_m \circ s_n$ is a positive linear combination of Schur functions.

'categorified' Eqv: $\mathrm{Sym}^n \mathrm{Sym}^m E$ has $\mathrm{Sym}^m \mathrm{Sym}^n E$ as a subrep.

Proved when:

$m=2$: $s_2 \circ s_n = s_{(2n)} + s_{(2n-2,2)} + \cdots$

$s_n \circ s_2 = \sum_{\lambda \in \mathrm{Par}(n)} s_{2\lambda}$

$m \leq 5$: Thrall 42, McKay 08, Cheung–Ikenmeyer–Mkrtchyan 17]

$m+n \leq 19$: Evseev–Paget–Wildon 14

Theorem [Carré–Thibon 92, Brion 93, Manivel 97, Bowman–Paget 18]
Let $\gamma \in \mathrm{Par}(r)$.

$$\lim_{\substack{m \to \infty \\ n \to \infty}} \langle s_{(n)} \circ s_{(m)}, s_{(mn-|\gamma|; \gamma)} \rangle = \lim_{\substack{m \to \infty \\ n \to \infty}} \langle s_{(m)} \circ s_{(n)}, s_{(mn-|\gamma|; \gamma)} \rangle$$

/end.

New proof [Paget–W 22]. Using

$$s_{(n)} \circ s_{(m)} = s_{(n)}(\text{monomials in } s_{(m)})$$
$$= s_{(n)}(x_{i_1} \cdots x_{i_m} : i_1 \leq \cdots \leq i_m)$$
$$= \sum_{\substack{\{A_1 \ldots A_n\} \\ A_i \text{ multi } m\text{-subset} \\ \text{of } \{1 \ldots d\}}} x^{A_1} \cdots x^{A_n} \quad (*) \qquad ; \text{ eg. } x^{\{1 1 3\}} = x_1^2 x_3$$

it is STP (using a few pages of nutritious algebraic combinatorics) that if $T_\gamma(m,n) = \{A_1 \ldots A_n\} : \text{content } (\underset{\#1s}{mn-|\gamma|}, \underset{\#2s}{\gamma_1}, \ldots, \underset{\#\ell+1s}{\gamma_\ell}) \}$ in all A_i

then $T_\gamma(m,n)$ has constant size for $m \geq r$, $n \geq r$. Let $\{A_1 \ldots A_n\} \in T_\gamma(m,n)$

- If $m > r$ then since only r entries ≥ 2 in all A_i, $A_i = \{1, 1, \ldots\}$ each i and $\{A_1 \ldots A_n\} \mapsto \{A_1 \setminus \{1\} \ldots A_n \setminus \{1\}\} : T_\gamma(m,n) \to T_\gamma(m-1,n)$ is bijection.
- If $n > r$ then, similarly, one of the multisets is $\{1, 1 \ldots 1\}$. Removing it gives a bijection $T_\gamma(m,n) \to T_\gamma(m, n-1)$. □