# A tour of Foulkes' Conjecture

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# Outline

(1) Foulkes' Conjecture
(2) Irreducible representations of symmetric groups
(3) Set families and maps

# §1: Foulkes' Conjecture

Let $S_r$ be the symmetric group on $\Omega = \{1, 2, \ldots, r\}$.

Let $\mathbf{C}\Omega = \langle e_1, e_2, \ldots e_r \rangle$. This is the natural permutation representation of $S_r$ where element of $S_r$ act by permutation matrices.

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$$\mathbf{C}\Omega = \langle e_1 + e_2 + \cdots + e_r \rangle \bigoplus \langle e_i - e_j : 1 \leq i < j \leq r \rangle .$$

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Each summand is preserved by the action of $S_r$. No proper subspace of either summand is preserved, so each is an irreducible representation of $S_r$.

# Foulkes' Conjecture

- Let $a, b \in \mathbf{N}$.
- Let $\Omega_a^b$ be the collection of set partitions of $\{1, 2, \ldots, ab\}$ into $b$ sets each of size $a$, acted on by $S_{ab}$.
- Let $\mathbf{C}\Omega_a^b$ be the corresponding permutation representation of $S_{ab}$.

If $U$ an irreducible representation of $S_{ab}$, let $[\mathbf{C}\Omega_a^b : U]$ denote the number of summands of $\mathbf{C}\Omega_a^b$ isomorphic to $U$.

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### Conjecture (Foulkes' Conjecture)

*If $a < b$ and $U$ is an irreducible representation of $S_{ab}$ then*

$$[\mathbf{C}\Omega_a^b : U] \geq [\mathbf{C}\Omega_b^a : U].$$

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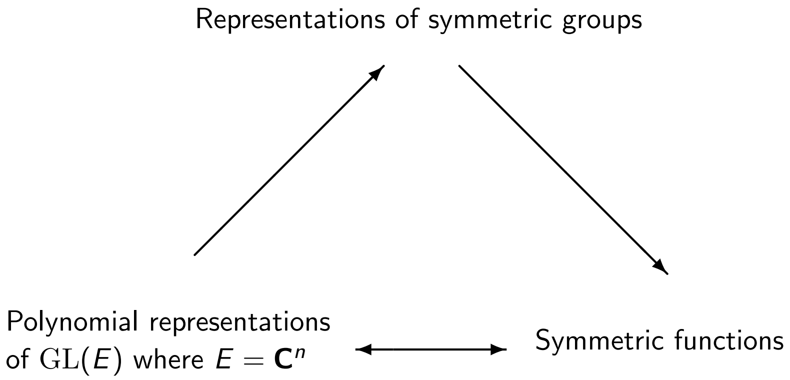
- $a = 4$, McKay 2008.

# Other settings for Foulkes' Conjecture



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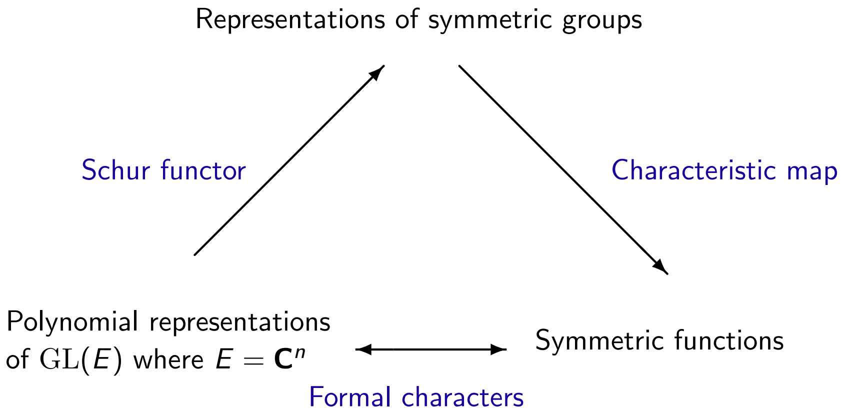

# Other settings for Foulkes' Conjecture

Representations of symmetric groups

$1_{S_r}$

Schur functor

Characteristic map

$S^r E$

$h_r$

$h_r(x, y) = x^r + x^{r-1}y + \cdots + y^r$

Polynomial representations of $\mathrm{GL}(E)$ where $E = \mathbf{C}^n$

Symmetric functions

Formal characters

# §2 Irreducible representations of $S_r$

Indexed by partitions of $r$, e.g. $(5,2,2) \in \mathrm{Par}(9)$.

Specht module $S^\lambda$ is irreducible representation labelled by $\lambda$. Linearly spanned by all $\lambda$-tableaux: e.g. if $\lambda = (4,2)$ then $S^{(4,2)}$ consists of all linear combinations of

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 4&3&5&6\\\hline 1&2\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 6&5&4&3\\\hline 2&1\\\cline{1-2}\end{array}\,,\ldots$$

Satisfies Garnir relations:

- Column swaps:

$$\begin{array}{|c|c|c|c|}\hline 4&3&5&6\\\hline 1&2\\\cline{1-2}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 4&2\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 4&3\\\cline{1-2}\end{array}$$

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- Shuffles. Related to determinantal identities:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} a & \beta \\ c & \delta \end{vmatrix} \begin{vmatrix} b & \alpha \\ d & \gamma \end{vmatrix} - \begin{vmatrix} a & \alpha \\ c & \gamma \end{vmatrix} \begin{vmatrix} \beta & b \\ \delta & d \end{vmatrix}$$

# Hook formula

The set of tableaux whose rows increase from left to right, and whose columns increase from top to bottom form a basis for $S^\lambda$.

The hook formula states that if $\lambda$ is a partition of $r$

$$\dim S^\lambda = \frac{r!}{\prod_{\alpha \in \lambda} h_\alpha}$$

where $h_\alpha$ is the hook-length of the box $\alpha$. For example, the red box has hook-length 6 and $\dim S^{(5,4,2,1)} = \frac{12!}{8.6.4.3.1.6.4.2.1.3.1.1}$.

| | | | | |
|---|---|---|---|---|
| 8 | 6 | 4 | 3 | 1 |
| 6 | 4 | 2 | 1 | |
| 3 | 1 | | | |
| 1 | | | | |

# §3: Set families and maps

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- Say that $\mathcal{P}$ has type $\lambda$ if there are $\lambda'_i$ sets containing $i$.

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- Say that $\mathcal{P}$ has type $\lambda$ if there are $\lambda'_i$ sets containing $i$.
- If $X$, $Y$ are sets of size $a$, say that $X$ is majorized by $Y$ if one can write $X = \{x_1, \ldots, x_a\}$ and $Y = \{y_1, \ldots, y_a\}$ where $x_1 \leq y_1, \ldots, x_a \leq y_a$.
- Say that $\mathcal{P}$ is closed if $Y \in \mathcal{P},\ X \preceq Y \implies X \in \mathcal{P}$

**Theorem**

*Let $a$ be odd. If there is a closed set family of shape $(a^b)$ and type $\lambda$ then $[\mathbf{C}\Omega_a^b : S^\lambda] \geq 1$.*

# Minimal constituents

- Say that a set family $\mathcal{P}$ is minimal if $\mathcal{P}$ has minimal type (in the dominance order) for its shape.
- Say that $S^\lambda$ is a minimal constituent of $\mathbf{C}\Omega_a^b$ if $[\mathbf{C}\Omega_a^b : S^\lambda] \geq 1$ and $\lambda$ is minimal with this property.

### Theorem

*If $a$ is odd then $S^\lambda$ is a minimal constituent of $\mathbf{C}\Omega_a^b$ if and only if there is a minimal set family of shape $(a^b)$ and type $\lambda$.*

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**Theorem**

*Let $\mathcal{P}$ be a set family. Then*

$$\mathcal{P} \text{ unique of its type} \implies \mathcal{P} \text{ minimal} \implies \mathcal{P} \text{ closed.}$$

*None of these implications is reversible.*