Character deflations, wreath products and Foulkes' Conjecture

Mark Wildon (joint work with Anton Evseev and Rowena Paget)

Royal Holloway
University of London

Outline

§1 Foulkes' Conjecture

§2 Deflations

§3 Combinatorial rule for deflated character values

§4 Applications

§1 Foulkes' Conjecture

Let S_r be the group of all permutations of $\Omega = \{1, 2, \ldots, r\}$. It is often useful to consider actions of S_r on other sets.

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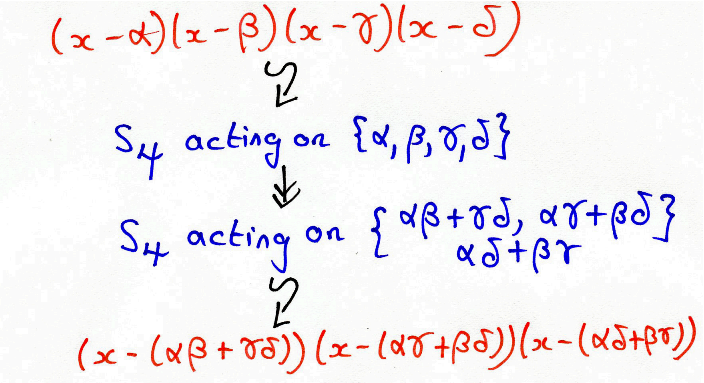

Here we can find S_4 acting on (amongst other things):

- the set $\{\alpha, \beta, \gamma, \delta\}$
- the set $\{\alpha\beta + \gamma\delta, \alpha\gamma + \beta\delta, \alpha\delta + \beta\gamma\}$ of size 3,
- the field extension $\mathbf{Q}(\alpha, \beta, \gamma, \delta)$,
- the 4-dimensional $\mathbf{Q}$-vector space $\langle \alpha, \beta, \gamma, \delta \rangle_{\mathbf{Q}}$.

Linear representations

Let $\mathbf{C}\Omega = \langle e_1, e_2, \ldots, e_r \rangle$. This is the natural permutation representation of S_r where the elements of S_r act by permutation matrices.

Vector space decomposition:

$$\mathbf{C}\Omega = \langle e_1 + e_2 + \cdots + e_r \rangle \bigoplus \langle e_i - e_j : 1 \leq i < j \leq r \rangle .$$

Each summand is preserved (i.e. mapped into itself) by the action of S_r. No proper subspace of either summand is preserved, so each summand is an irreducible representation of S_r.

Different permutation representations of S_r can be compared by looking at the multiplicities of their irreducible constituents.

Example: comparing different linear representations

- S_4 acting on $\{1, 2, 3, 4\}$, point stabiliser S_3
- S_4 acting on $\big\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\big\}$, point stabiliser $S_2 \times S_2$

$$(12) \mapsto \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

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$$(12) \mapsto \begin{pmatrix} 1 & & & \\ & -1 & 0 & 0 \\ & -1 & 1 & 0 \\ & -1 & 0 & 1 \end{pmatrix} \quad (12) \mapsto \begin{pmatrix} 1 & & & & & \\ & -1 & 0 & 0 & & \\ & -1 & 1 & 0 & & \\ & -1 & 0 & 1 & & \\ & & & & 1 & -1 \\ & & & & 0 & -1 \end{pmatrix}$$

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Remarkable fact: any representation $\rho : S_r \to \mathrm{GL}(V)$ is determined (up to a suitable notion of isomorphism) by its character

$$\phi(g) = \mathrm{tr}\big(\rho(g)\big) \quad \text{for } g \in S_r.$$

Moreover, the multiplicity of an irreducible representation with character χ in ρ is

$$\langle \phi, \chi \rangle = \frac{1}{r!} \sum_{g \in S_r} \phi(g)\chi(g).$$

Foulkes' Conjecture

- Let a, $b \in \mathbf{N}$.
- Let $\Omega^{(a^b)}$ be the collection of set partitions of $\{1, 2, \ldots, ab\}$ into b sets each of size a, acted on by S_{ab}.
- Let $\mathbf{C}\Omega^{(a^b)}$ the the corresponding permutation representation of S_{ab}.
- Let $\phi^{(a^b)}$ be the character of $\mathbf{C}\Omega^{(a^b)}$. So if $g \in S_{ab}$ then $\phi^{(a^b)}$ is the number of set partitions in $\Omega^{(a^b)}$ that are fixed by g.

Conjecture (Foulkes' Conjecture)

If $a < b$ and χ is an irreducible character of S_{ab} then

$$\langle \phi^{(a^b)}, \chi \rangle \geq \langle \phi^{(b^a)}, \chi \rangle.$$

Murnaghan–Nakayama Rule

Let λ be a partition of r and let $\gamma = (\gamma_1, \dots, \gamma_k)$ be such that $\gamma_1 + \cdots + \gamma_k = r$. A border-strip tableau of shape λ and type γ is an assignment of the numbers from the set $\{1, 2, \dots, k\}$ to the boxes of the diagram of λ such that

(i) The boxes labelled i form a border-strip of length γ_i;

(ii) The boxes labelled by numbers $\leq i$ form the diagram of a partition.

Let $\lambda = (5, 4, 2, 1)$ and let $\gamma = (6, 3, 3)$. To find one border-strip tableau of shape λ and type γ:

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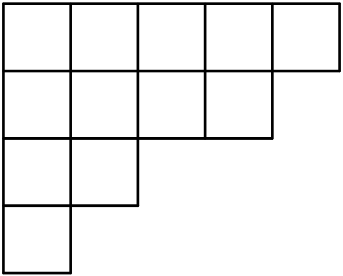

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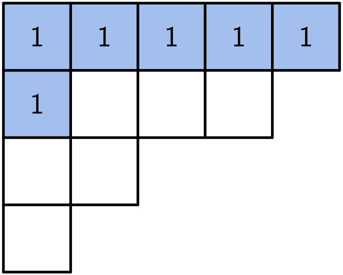

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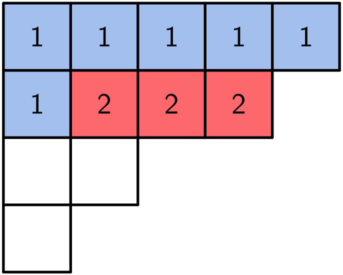

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1	1	1	1	1
1	2	2	2	
3	3			
3				

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Theorem (Murnaghan–Nakayama)

Let $g \in S_r$ have cycle type γ. Then

$$\chi^\lambda(g) = \sum_T \operatorname{sgn}(T)$$

where the sum is over all border-strip tableaux of shape λ and type γ, and $\operatorname{sgn}(T) = (-1)^{\text{sum of all leg lengths in } T}$.

Example: $a = 2$, $b = 6$, $\lambda = (6, 3, 3)$, $\gamma = (1, 2, 3)$

1_1	1_2	3_1	3_2	3_2	3_2
2_1	2_2	3_1			
2_1	2_2	3_1			

sign $+1$

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1_1	1_2	2_1	2_1	2_2	2_2
3_1	3_1	3_2			
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1_1	1_2	2_1	2_1	2_2	2_2
3_1	3_1	3_2			
3_1	3_2	3_2			

sign $+1$

1_1	1_2	2_2	3_2	3_2	3_2
2_1	2_1	2_2			
3_1	3_1	3_1			

sign -1

§4: Applications

- $c_{\lambda,\gamma}$ is independent of the order of the parts of γ. For example, if $\lambda = (6,3,3)$ then $c_{\lambda,(1,2,3)} = +1 + 1 - 1 = 1$ and correspondingly $c_{\lambda,(2,3,1)} = 1$.

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- (A special case of) Young's Rule: let $\pi^{(a^b)}$ be the permutation character of S_{ab} acting on all ordered set partitions of $\{1, 2, \ldots, ab\}$ into b sets each of size a. Then $\langle \pi^{(a^b)}, \chi^\lambda \rangle$ is equal to the number of semistandard λ-tableaux of type (a^b).

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- A new recursive formula for Foulkes multiplicities. Fix $a \in \mathbf{N}$. Then

$$\left\langle \phi^{(a^b)}, \chi^\lambda \right\rangle = \frac{1}{b} \sum_{\ell=1}^{b} \sum_{\mu} \mathrm{sgn}(\lambda/\mu) \left\langle \phi^{(a^{b-\ell})}, \chi^\mu \right\rangle$$

where the second sum is over all partitions μ obtainable by removing a border-strips of length ℓ from λ, subject to the constraint in the main theorem.

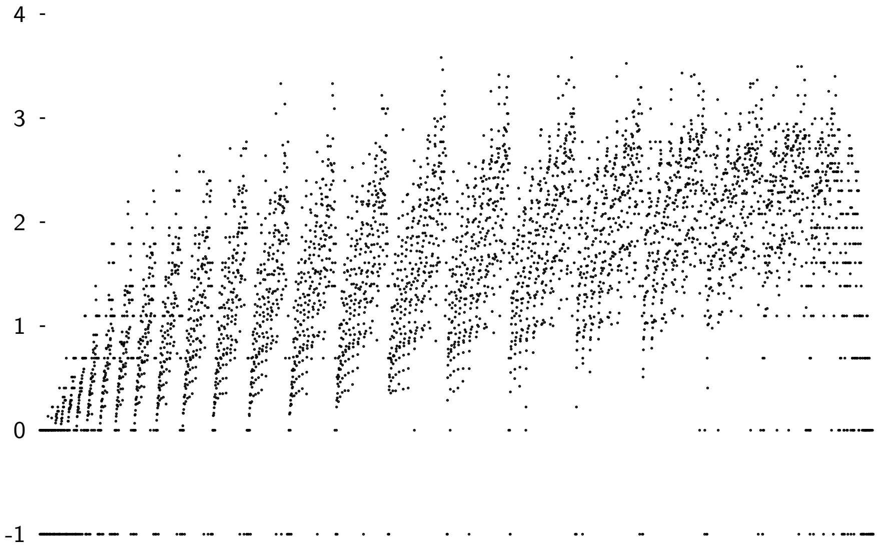
Explanation of graphs

These graphs show Foulkes multiplicities for all partitions with at most b parts, arranged in lexicographic order. The y axis shows

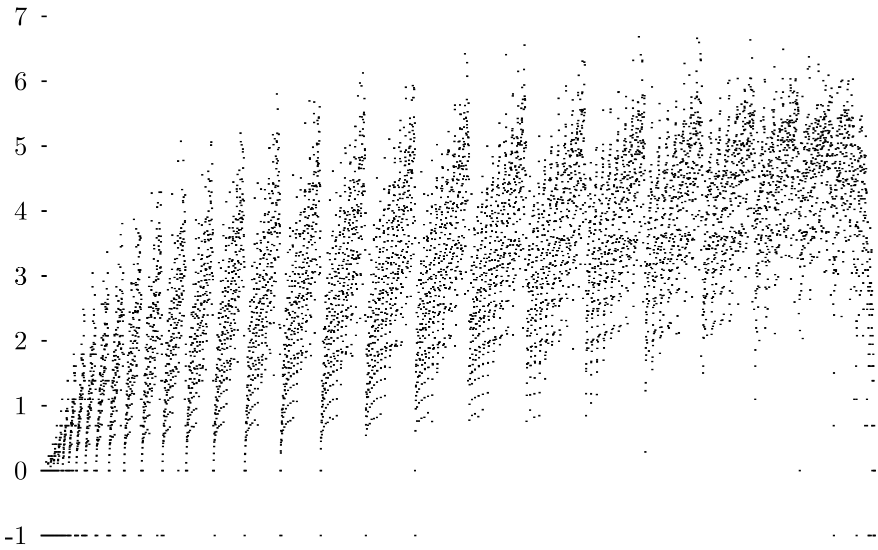
$$\log \frac{\left\langle \phi^{(a^b)}, \chi^\lambda \right\rangle}{\left\langle \phi^{(b^a)}, \chi^\lambda \right\rangle}.$$

If both numerator and denominator are 0 then the point is artificially placed at -1. If the denominator is 0 but not the numerator then $\log \left\langle \phi^{(a^b)}, \chi^\lambda \right\rangle$ is shown.

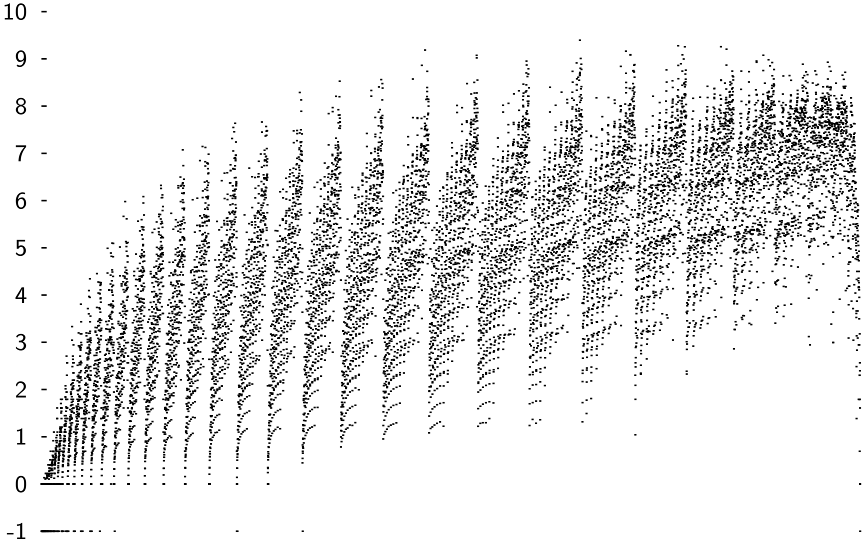
$a = 6$, $b = 7$, log comparison of multiplicities



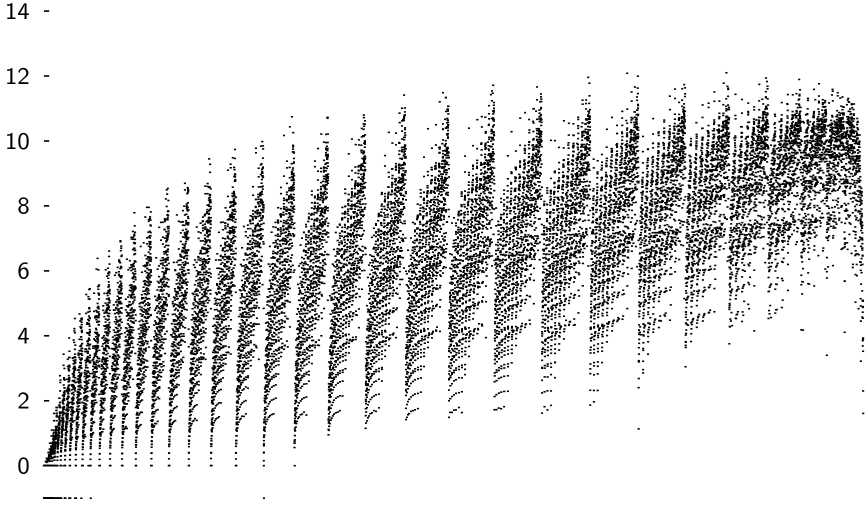
$a = 6$, $b = 8$, log comparison of multiplicities



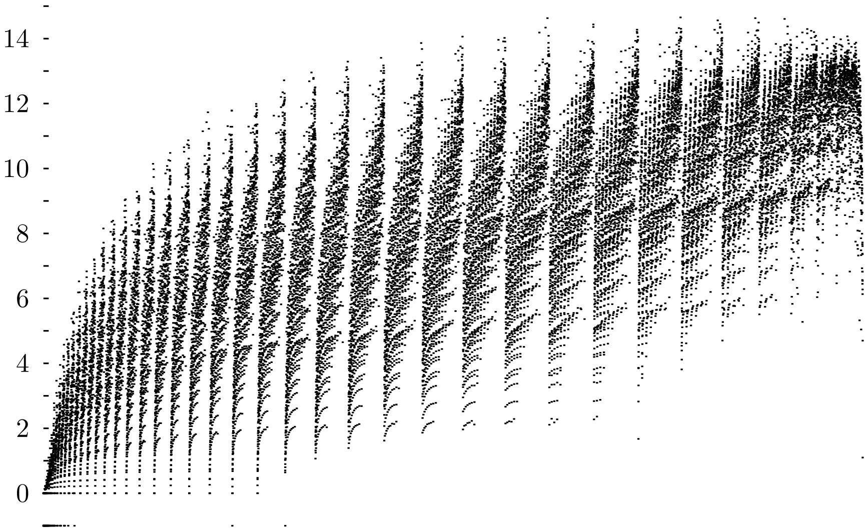
$a = 6$, $b = 9$, log comparison of multiplicities



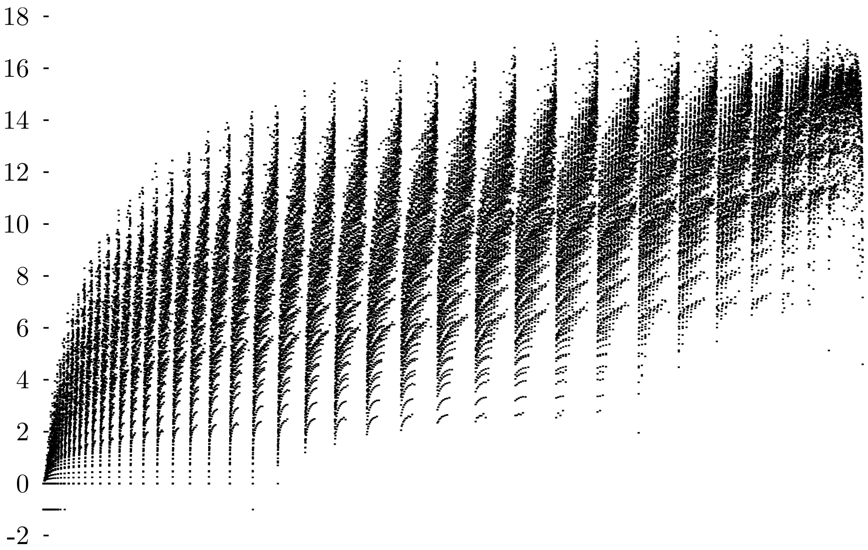
$a = 6$, $b = 10$, log comparison of multiplicities



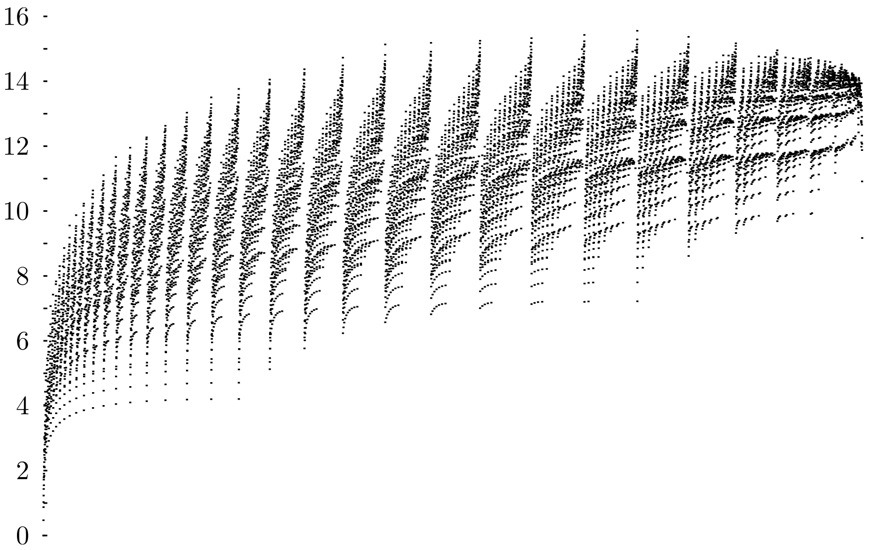
$a = 6$, $b = 11$, log comparison of multiplicities



$a = 6$, $b = 12$, log comparison of multiplicities



$a = 6$, $b = 9$, log comparison of Kostka multiplicities



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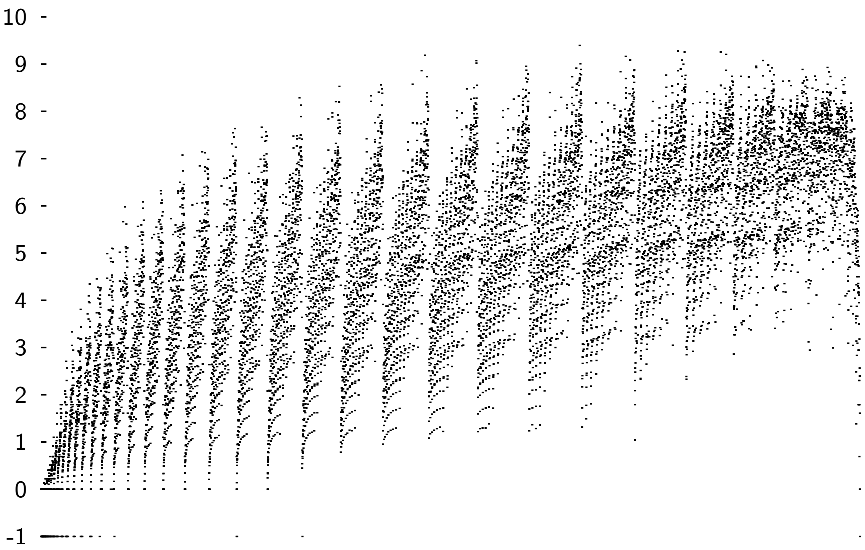

Timings

For $a = 6$, varying $b \geq 6$, here are the times to compute all Foulkes multiplicities.

SYMMETRICA is a specialised package for computing with symmetric functions developed by Adelbert Kerber, Axel Kohnert *et al*: it is usually much faster than MAGMA and other more general purpose computer algebra systems. See `www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/`

b	6	7	8	9	10	11	12
SYMMETRICA	0.4s	3.5s	22.6s	272.0s	2710.0s	426m8s	$>$ 2 days
Recurrence	0.6s	3.9s	25.6s	127.7s	454.3s	31m50s	117m3s
Memory used (Gb)	?	?	0.2	0.2	0.35	0.7	1
Speed up	0.6	1.1	0.88	2.1	6.0	13.4	$>$ 24.6

To test Foulkes' Conjecture only the multiplicities for partitions with $\leq a$ parts are needed. This leads to big savings: for example for $a = 6$ and $b = 13$ only 20 minutes are needed (rather than over a day to compute all multiplicities).