Commuting conjugacy classes in groups: an overview

Mark Wildon (joint work with John Britnell)

Outline

(1) Introduction
(2) Finite symmetric groups
(3) General linear groups

§1 Introduction

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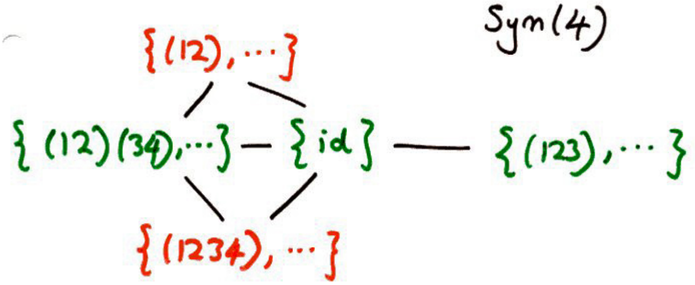

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But the conjugates of a proper subgroup of G cannot cover G. Hence $\mathrm{Cent}_G(x) = G$. $\square$

Traité des substitutions

Note that $\mathrm{Cent}_G(x)^g$ is the stabiliser of x^g in the conjugacy action of G on x^G. So

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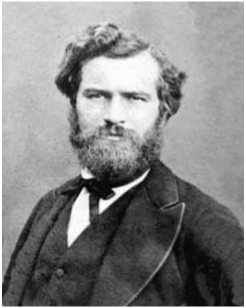
In 1870, Jordan showed that any non-trivial finite transitive permutation group contains an element without fixed points. So unless $\mathrm{Cent}_G(x) = G$, when the action is trivial, the conjugates of $\mathrm{Cent}_G(x)$ do not cover G.



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(3) If G is infinite then $Z(G)$ cannot be determined by $\sim$. Let X be an infinite set and let

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Then *any* two classes $x^G, y^G \in G$ commute. But G is not abelian.

§2 Commuting in finite symmetric groups

Conjugacy classes in $\mathrm{Sym}(n)$ are labelled by partitions of n.

For example, if $g = (2345)(67) \in \mathrm{Sym}(7)$ then $g^{\mathrm{Sym}(7)}$ consists of all permutations whose cycle decomposition has a 4-cycle, a 2-cycle and a fixed point. The labelling partition is $(4, 2, 1)$.

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Definition

If λ and ν are partitions of n, say that ν is a coarsening of λ, if ν can be obtained from λ by combining parts of the same size.

Theorem

The classes in $\mathrm{Sym}(n)$ corresponding to partitions λ and μ commute if and only if there is a partition ν which is a coarsening of both λ and μ.

Probabilistic questions

This part is joint work with Simon Blackburn (RHUL).

Theorem

Let $P(n)$ be the probability that if two elements $g, h \in \mathrm{Sym}(n)$ are chosen uniformly at random then $g^{\mathrm{Sym}(n)} \sim h^{\mathrm{Sym}(n)}$. Then there is a constant $C \approx 6.2$ such that $P(n) \sim \frac{C}{n^2}$ as $n \to \infty$.

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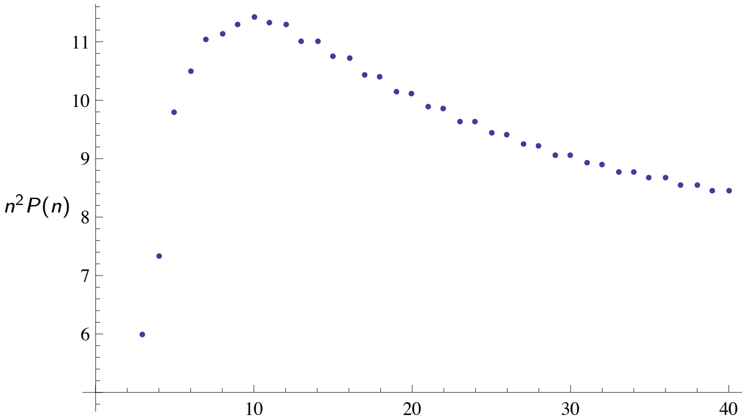

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Sketch proof: Most permutations in $\mathrm{Sym}(n)$ have a long cycle, of length $> n/\log n$. If g has a long cycle of length ℓ and $g^{\mathrm{Sym}(n)} \sim h^{\mathrm{Sym}(n)}$ then, almost always, h also has a long cycle of length ℓ. We use this to get a recurrence for $P(n)$. Some analysis then shows that $P(n) \sim C/n^2$ where

$$C = \sum_{n=0}^{\infty} P(n)$$

Marrying in symmetric groups

Say that an even permutation is marriable if it commutes with an odd permutation.

Theorem

There is a bijection

$$\left\{ \begin{array}{c} \textit{marriable classes} \\ h^{\mathrm{Sym}(n)} \subseteq \mathrm{Alt}(n) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{all classes} \\ g^{\mathrm{Sym}(n)} \subseteq \mathrm{Sym}(n) \backslash \mathrm{Alt}(n) \end{array} \right\}$$

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Proof: show that given any r marriable classes, $C_1, \ldots, C_r$ there are r classes of odd elements $D_1, \ldots, D_r$ such that $C_i \sim D_i$ for each i.

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Proof: show that given any r marriable classes, $C_1, \ldots, C_r$ there are r classes of odd elements $D_1, \ldots, D_r$ such that $C_i \sim D_i$ for each i. Then apply Hall's Marriage Theorem.

Let $C = C_1 \cup \cdots \cup C_r$.

Let $X = \{(h, g) : h \in C,\ g \text{ odd},\ hg = gh\}$. So

$$|X| = \sum_{h \in C} \tfrac{1}{2}|\mathrm{Cent}(h)| = \frac{n!}{2} \sum_{h \in C} \frac{1}{|h^{\mathrm{Sym}(n)}|} = \frac{n!}{2} r.$$

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Another application of Hall's Marriage Theorem

Let G be a group with a finite index subgroup H. There exist $g_1, \ldots, g_n \in G$ such that

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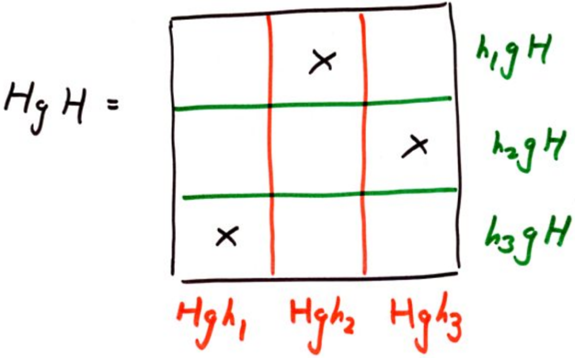

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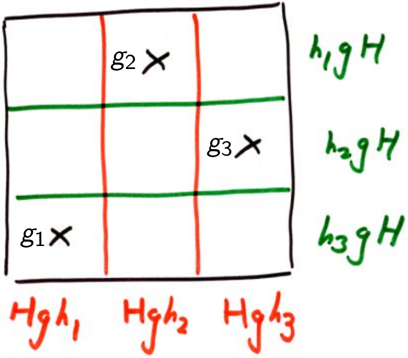

Background to results on $\mathrm{GL}_n(F)$

Fix a field F. Given a partition λ of n, Let $J(\lambda) \in \mathrm{GL}_n(F)$ be the unipotent Jordan matrix corresponding to λ.
A major open problem is to describe the conjugacy classses of $\mathrm{GL}_n(F)$ that meet $\mathrm{Cent}_{\mathrm{GL}_n(F)} J(\lambda)$. In our langauge: which classes commute with $J(\lambda)^{\mathrm{GL}_n(F)}$?

- Let $D(\lambda)$ be the largest partition such that $J(\lambda) \sim J\big(D(\lambda)\big)$. In 2009 Iarrobino proved that the map $\lambda \mapsto D(\lambda)$ is idempotent.
- In 2010, Kosir and Oblak found $D(\lambda)$ in the cases where it has at most two parts
- In 2008, Oblak defined a partition $Q(\lambda)$ and conjectured that $Q(\lambda) = D(\lambda)$. In 2012, Iarrobino and Khattami proved that $D(\lambda) \leq Q(\lambda)$.

Our results reduce the general problem of deciding which classes in $\mathrm{GL}_n(F)$ commute to the problem for nilpotent classes over field extensions of F.

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Theorem

Let $X, Y \in \mathrm{GL}_n(q)$. Then X and Y have the same type if and only if there exist polynomials $F, G \in \mathbf{F}_q[x]$ such that $F(X) \in Y^{\mathrm{GL}_n(q)}$ and $g(Y) \in X^{\mathrm{GL}_n(q)}$.

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Corollary

Suppose that $X^{\mathrm{GL}_n(q)} \sim Y^{\mathrm{GL}_n(q)}$. Then any class of the type of X commutes with any class of the type of Y.

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Theorem

Let p be a prime and let $r \geq 1$. There exists $n \in \mathbf{N}$ such that

$$U_{p^a}\big((n, n)\big) \sim U_{p^a}\big((n+1, n-1)\big)$$

if and only if $a > r$.

Future directions

- What is the correct generalization of type for matrices over infinite fields? Probably it involves isomorphism classes of Galois extensions.
- Find all possible determinants of a matrix of a given type. This leads to some interesting problems in arithmetic combinatorics.
- What is the probability that two classes chosen uniformally at random in $\mathrm{Sym}(n)$ commute?