Some examples on duality

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1 An example on dual Specht modules

Let ${}_{\mathbb{Z}}S^\lambda = S^\lambda$ denote that Specht module corresponding to $\lambda \vdash n$ viewed as a module for $\mathbb{Z}S_n$. Let $\left(S^\lambda\right)^*$ denote its dual, $\mathrm{Hom}_{\mathbb{Z}}\left(S^\lambda, \mathbb{Z}\right)$ which is a $\mathbb{Z}S_n$ module *via* the rule $x\phi g = xg^-1\phi$ where $x \in S^\lambda$, $\phi \in \left(S^\lambda\right)^*$ and $g \in S_n$.

Theorem 1 *There is an inclusion of $\mathbb{Z}S_n$-modules, $\left(S^\lambda\right)^* \hookrightarrow S^\lambda$.*

An immediate corollary is that it is always possible to pick a $\mathbb{Z}$-lattice inside ${}_{\mathbb{Q}}S^\lambda$ so that the corresponding p-modular reduction is isomorphic to the *dual* of the conventional p-modular reduction (coming from the standard basis and written ${}_{\mathbb{F}_p}S^\lambda$). Thus our theorem gives a rich source of examples of non-isomorphic p-modular reductions. It would be interesting to know how far it is possible, by picking a suitable basis of ${}_{\mathbb{Q}}S^\lambda$, to make the p-modular reduction *semisimple*.

The theorem is not hard to prove, the main idea being the following trivial generalisation of James' Submodule Theorem.

Lemma 2 *Let U be a $\mathbb{Z}S_n$-submodule of ${}_{\mathbb{Z}}M^\lambda$. Let t be a fixed λ-tableaux and let b_t by the corresponding signed column sum. Then either $Ub_t \neq 0$ in which case U contains mS^λ for some $m \in \mathbb{Z}$ or $Ub_t = 0$ and then U is contained in $\left(S^\lambda\right)^\perp$.* □

Proof of theorem 1: Our first step is to follow the proof of theorem 8.15 in James [3]. Fix a λ-tableaux, u and define a map

$$\phi : M^\lambda \to S^{\lambda'} \otimes sgn \quad \text{by extending} \quad \mathbf{u} \to e_{u'} \otimes 1.$$

ϕ is well defined, for if the λ-tableaux s differ by a row permutation from u then $e_{t'}$ and $e_{u'}$ are equal up to a sign, which is taken care of by multiplying the right hand side by the sign representation. Also ϕ is obviously onto. If t is any λ-tableau then

$$e_t\phi = \sum_{\sigma \in C_t} \mathbf{t}\phi\sigma = e_{t'}\left|C_t\right| \neq 0$$

so by the Submodule Theorem, the kernel of ϕ must be contained in $S^{\lambda^\perp}$. By looking at $\mathbb{Z}$-ranks, we see $\ker\phi$ is of full rank in $S^{\lambda^\perp}$. Finally, as it is impossible for $x\phi \neq 0$ but $mx\phi = 0$ for $x \in M^\lambda, m \in \mathbb{Z}$, $\ker\phi$ must be all of $S^{\lambda^\perp}$. We have shown that

$$M^\lambda / S^{\lambda^\perp} \cong S^{\lambda'} \otimes \epsilon.$$

On the other hand, it is clear that $S^{\lambda^\perp}$ is the kernel of the map

$$\psi : M^\lambda \to S^{\lambda^*}$$

given by

$$x\psi = [y \to \langle x, y \rangle] \,.$$

Now the difficulty (such as it is) lies in proving that ψ is onto. This follows from a remark of James (see corollary 8.12 in [3]) that there is a $\mathbb{Z}$-basis of S^λ each of whose elements involve a unique standard tableaux precisely once. (Thus if the basis is $(f_1, \ldots, f_m)$ and f_i is unique in involving $\mathbf{t}_i$ then $\mathbf{t}_i\psi = \pm f_i^*$.) This shows that, over $\mathbb{Z}$, and hence over any field, $S^{\lambda^*} \cong S^{\lambda'} \otimes sgn$.

The second step in the proof is to identify $S^{\lambda'} \otimes sgn$ with Fulton's module $\widetilde{S^\lambda}$ constructed in §7.4 of Fulton [1]. Then the *proof* of Lemma 5 in his §7.4 gives us the necessary inclusion. The map is as follows: $S^{\lambda'} \otimes sgn$ is naturally a submodule of $M^{\lambda'} \otimes sgn$. S^λ is a quotient of this 'twisted' permutation module *via* $\mathbb{Z}S_n$ extension of the map sending

$$\mathbf{u}' \otimes 1 \to e_u$$

where u is any fixed λ-tableaux. Now one can check directly that the composition $S^{\lambda'} \otimes sgn \to S^\lambda$ is injective and so complete the proof. (The aproach in Fulton §7.4 [1] avoids the choice of u and gives a better reason why the composite map is injective.)

In summary, we fix a λ-tableaux, u and send $\langle -, \mathbf{u} \rangle$ to $e_{u'} \otimes 1 \in S^{\lambda'} \otimes sgn$, and then to $\sum_{\sigma \in R_u} e_{u\sigma}$. $\mathbb{Z}S_n$ extension of this map then gives us an injection, $S^{\lambda^*} \hookrightarrow S^\lambda$. $\square$

Example 3 *We exhibit an explicit inclusion $S^{(n-1,1)^*} \hookrightarrow S^{(n-1,1)}$. As usual we omit the redundant top row from $(n-1,1)$-tabloids.*

Let $e_i = \mathbf{i} - \mathbf{1} \in S^{(n-1,1)}$, so $S^{(n-1,1)}$ has $\mathbb{Z}$-basis $(e_i : i \in [2..n])$. Let $(e_i^* : i \in [2..n])$ be the dual basis of $S^{(n-1,1)}$. If u is the standard $(n-1,1)$-tableau with an n in the bottom row then $\langle -, \mathbf{u} \rangle$ corresponds to e_n^*. We are told to map

$$e_n^* \to \sum_{\sigma \in S_{n-1}} e_{u\sigma}$$

so $e_n \to (n-2)!\,((n-1)\mathbf{n} - \mathbf{1} - \ldots - (\mathbf{n-1}))$.

2 Further examples of duality

This theorem is not expected to be orignal.

Theorem 4 *Let G be a finite group and let F be a field. Then any finite dimensional uniserial FG-module is cyclic.*

Proof: Let u be a non-cyclic finite dimensional FG-module. We will construct an ascending chain of submodules of U as follows. Take any $u_1 \in U$. Since $u_1FG \neq U$ we can find $u_2 \in U \backslash u_1FG$. Now if $u_1 \notin u_2FG$ we are done, for the module $u_1FG + u_2FG \subseteq U$ has two different composition series. On the other hand, if $u_1 \in u_2FG$ then we can continue the process, reaching a chain of modules

$$u_1FG \subset u_2FG \subset u_3FG \subset \ldots$$

each strictly included in the previous one. But this chain cannot reach U (as then U would be cyclic) so two of these supposedly different modules must coincide — contradiction. $\square$

Example 5 *The finite-dimensionality assumption in theorem 4 cannot be removed. Let $S_{\mathbb{N}}$ denote the* finitary *symmetric group on the natural numbers. (A permutation is finitary if it has finite support, i.e. if its fixed point set is cofinite.) Let F be a finite or countable field. Then $U = \mathbb{F}^{\mathbb{N}}$ is a cyclic permutation module for $FS_{\mathbb{N}}$ in an obvious way.*

By standard linear algebra we know that as an F-vector space, U is of uncountably-infinite dimension. Now if θ is any vector in U, the submodule of $U^\star$ generated by θ is countable, because both F and $S_{\mathbb{N}}$ are countable. Thus $U^\star$ cannot be cyclic.

Specht modules are cyclic (generated by any one polytabloid), and so are their duals (see §1 above). Theorem 4 above shows that the dual of a cyclic uniserial module is cyclic. Another partial result if that if U is a quotient of FG, by V say, and V has an FG-complement, then U is cyclic. Below we show that, in general, the dual of a cyclic module need not be cyclic.

Example 6 *Let $G = C_2 \times C_2$ and let $F = \mathbb{F}_2$, the finite field with 2 elements. Let N be the permutation module of C_2 acting on 2 points, and let $M = N \otimes N$, with F-basis $e_i \times f_j$ for $i = 1, 2$ and $j = 1, 2$. Let $U = M/\langle (e_1 + e_2) \otimes (f_1 + f_2)\rangle$. As a quotient of a cyclic module, U is certainly cyclic. Let*

$$V = \langle (e_1 + e_2) \otimes (f_1 + f_2), (e_1 + e_2) \otimes f_1, e_1 \otimes (f_1 + f_2)\rangle$$

Since M has radical layers F, $F \oplus F$, F, V is isomorphic to U^* (this can also be checked directly). On the other hand, V is not cyclic — one checks directly by careful calculation that although there are 8 vectors in V, counting 0, *none of them generate V as a FG-module).*

One thing that has puzzled me in the past is the distinction made between the two possible symmetric power constructions. Let V be a left-FG module with F-basis $(v_1, \ldots, v_n)$. $V^{\otimes r}$ has a basis given by $(v_i : i \in [n]^{[}r])$ where $v_i = v_{i_1} \otimes \ldots \otimes v_{i_r}$. S_r acts on the right on the set of all 'multi-indexes', i.e. functions $[r] \to [n]$ by place permutation, and hence on $E^{\otimes r}$: if $\sigma \in S_r$ then

$$v_i\sigma = v_{i_{1\sigma^{-1}}} \otimes \ldots \otimes v_{i_{r\sigma^{-1}}}.$$

Our first version of the symmetric power is the submodule of $V^{\otimes r}$ given by symmetrising this basis: let $f_i = \sum v_j$ where the sum is over all v_j that can be obtained from the action of S_r on e_i, each occuring once. Alternatively, $f_i = \sum v_i\sigma$ where the sum is taken over a set of coset representatives for $\mathrm{Stab}\, i$ in S_r. Let S_nV be the F-span of all such elements. The action of GL_n (on the left) commutes with the action of S_r (on the right), i.e. $(gv_i)\sigma = g(v_i\sigma)$ so S_nV is a representation of $GL_n(F)$.

The alternative definition is to identify v_i with $v_i\sigma$, so define $S^rV = E^{\otimes r}/K$ where K is the submodule generated by all $v_i - v_i\sigma$. Let e_i be the image of v_i in the quotient. The latter definition is probably the one most people mean by 'symmetric power'.

Using the convenient notation of Green's book, say that if k and i are multi-indices then $k \sim i$ if k and i have the same content, i.e. if k and i are in the same S_r orbit. It is easy to see that one can obtain a basis of either S_rV by choosing exactly one f_i from each orbit, and similarly for S^rV.

Lemma 7 *S_rV is isomorphic to the contravariant dual of S^rV.*

Proof: For the definition of contravariant dual see e.g. §2.7 of Green [2]. It is easy to find the action of $GL_n(F)$ on S^rV — if $g \in GL_n(F)$ then $ge_j = \sum_k g_{k_1j_1} \dots g_{k_rj_r} e_k$. Thus if $A(g)$ is the matrix of g acting on S^rV with respect to the basis of e_i's given above, we have

$$A(g)_{ij} = \sum_{k \sim i} g_{k_1j_1} \dots g_{k_rj_r}.$$

The action of $GL_n(F)$ on S_rV is only slightly more fiddly. We have that

$$gf_j = \sum_l \sum_\sigma g_{l_1i_1} \dots g_{l_rj_r} e_k \sigma = \sum_l \sum_\sigma g_{l_1 i_{1\sigma^{-1}}} \dots g_{l_r j_{r\sigma^{-1}}} e_k$$

where σ runs over a set of coset represenatives for $\operatorname{Stab} j$ in S_r. We read the coefficient of v_k off from the second form above, showing that if $B(g)$ is the matrix of g acting on S_rV with respect to the basis of f_i's given above then:

$$B(g)_{ij} = \sum_{k \sim j} g_{i_1k_1} \dots g_{i_rk_r}.$$

Comparing coefficients we can now see that $A(g)^{tr} = B(g^{tr})$, proving the lemma. □

Proof(2): Say $f : V \to W$ is a map of $GL_n(F)$-modules. Then we have an induced map $f^\circ : W^\circ \to V^\circ$ on the contravariant duals, given by

$$(f.\theta)v = \theta(fv).$$

By the same argument that works for the normal dual, we show that f° is a module-homomorphism.

We now state universal properties for S_rV and S^rV. Firstly we look at S^rV. Given any map $f : V^{\otimes r} \to W$ of $GL_n(F)$-modules with the property that f vanishes on the ideal K defined above, there is a unique map $g : S^rV \to W$ such that the following diagram commutes:

$$\begin{tikzcd} V^{\otimes r} \arrow[rr, "f"] \arrow[dr, "q"'] & & W \\ & S^rV \arrow[ur, "g"'] & \end{tikzcd}$$

where q is the quotient map $V^{\otimes r} \twoheadrightarrow S^rV$.

Similarly, given any map $f : W \to V^{\otimes r}$ of $GL_n(F)$-modules with the property that for all $w \in W$ and $\sigma \in S_r$, $(fw)\sigma = fw$ there is a unique map $g : W \to S_rV$ such that this diagram commutes:

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "g"'] & & V^{\otimes r} \\ & S_rV \arrow[ur, "\iota"'] & \end{tikzcd}$$

where ι is the inclusion of S_rV into $V^{\otimes r}$.

Since these diagrams are dual to each other, we see that $(S^rV)^\circ$ solves the same problem as S_rV (here we have used that V, and hence $V^{\otimes r}$ are their own contravariant duals), so by general nonsense we are allowed to deduce that $(S^rV)^\circ \cong S_rV$. □

In characteristic 0 the two versions of the symmetric power are isomorphic. This can be seen directly by scaling one of the bases appropriately. Alternatively one could be more sophisticated and observe that in characteristic 0, GL_n-modules are semisimple and that any simple-module is self-dual, so *any* $GL_n(K)$ module is its own contravariant-dual. In characteristic p though, the is a difference. A simple yet instructive example is to look at S^2E where E is the usual 2-dimensional module for $GL_2(F)$ in the case where F has characteristic 2 (see e.g. the section $S_K(2,2)$, revisited in Martin [5]). In general it is not hard to show from the definitions (see e.g. Green [2] §4.4 and §5.1) that if E is the usual n-dimensional module for $GL_n(F)$ then $S^rE \cong \nabla(r)$ and $S_r \cong \Delta(r)$.

References

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