NOTES ON MURPHY OPERATORS AND NAKAYAMA'S CONJECTURE

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What follows are notes on G. E. Murphy's paper *The idempotents of the symmetric group and Nakayama's Conjecture* [3], read in the Representation Theory Advanced Class, Trinity 2008, Oxford. If you use these notes and have any corrections or suggestions for improvement, please let me know.

1. Residues and cores

Given a partition λ, we define the *residue* of the node in row i and column j to be $j - i$. We define its p-*residue* (also known as its p-*class*)to be its residue taken mod p. Early in §1 of [3], Murphy states the following non-obvious relationship between residues and p-cores, which later turns out to be critical to the success of his proof of the Nakayama conjecture.

Proposition 1. *Two partitions have the same multiset of p-residues if and only if they have the same p-core.*

Recall that the p-core of a partition λ is the partition obtained by repeatedly removing rim-p-hooks from λ, until no more can be removed. In the diagram below showing $(6, 3, 3, 1)$, the 2-core is hatched, and nodes of 2-residue 0 are shaded. The thick lines show two of the three 2-hooks that can be immediately removed; the thin lines show the remaining three 2-hooks we remove *en route* to the core.

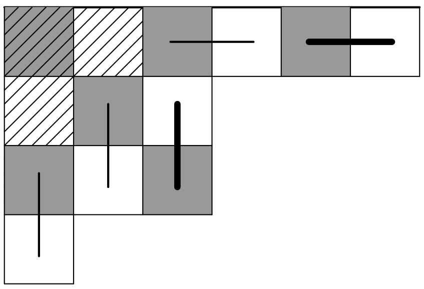

Figure 1. The 2-core of $(6, 3, 3, 1)$.

Assuming for the moment that the p-core is well-defined, it is quite easy to prove the 'if' direction of the proposition. For, if λ and μ are partitions of n with the same p-core, say γ, then we can obtain λ and μ from γ by repeatedly adding p-hooks. The result now follows from the observation that the set of residues of the nodes in any p-hook is always $\{0, 1, \ldots, p-1\}$.

The 'only if' direction is harder. To prove it, we shall need G. D. James' abacus notation for partition.

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Review of abacus notation. Let λ be a partition of n. Starting in the southwest corner of the Young diagram of λ walk along its rim, heading towards the northeast corner. For each step right, put a space, indicated $\circ$, and for each step up, put a bead, indicated $\bullet$. For example, the partition $(6, 3, 3, 1)$ has sequence $\circ\bullet\circ\circ\bullet\bullet\circ\circ\circ\bullet$. It is useful to allow such a sequence to begin with any number of beads, and to finish with any number of spaces; these must be stripped off before the partition is reconstructed from its bead sequence.

One then arranges the bead sequence in p columns, known as the *runners* of the abacus. For instance $(6, 3, 3, 1)$ is represented on a 2-abacus as follows.

$$\begin{array}{cc} \circ & \bullet \\ \circ & \circ \\ \bullet & \bullet \\ \circ & \circ \\ \circ & \bullet \end{array}$$

The reader should have little difficulty in seeing that the 2-hooks in $(6, 3, 3, 1)$ correspond to the beads on the above abacus having a space immediately above them. More generally, the north-eastern-most node in a rim-p-hook of λ corresponds to a bead on a p-abacus display for λ with a space immediately above it. An abacus display for the partition obtained by removing this hook may be obtained by sliding the bead one step up.

It follows that the p-core of λ is obtained by pushing all the beads in an associated abacus up as far as they will go. The abacus makes it obvious that the p-core we reach is independent of the manner in which we remove hooks, and hence that the the p-core of a partition is well-defined. For example, $(6, 3, 3, 1)$ has the 2-core $(2, 1)$, shown in the abacus below.

$$\begin{array}{cc} \bullet & \bullet \\ \circ & \bullet \\ \circ & \bullet \\ \circ & \circ \\ \circ & \circ \end{array}$$

Remainder of proof. To prove the 'only if' part of Proposition 1 we need to show that a p-core is determined by its multiset of p-residues. (Thanks to the triangular shape of 2-cores, this is obvious when $p = 2$, but it is already non-obvious when $p = 3$.) The following proof is adapted from that of Theorem 2.7.41 in [1].

Let γ be a p-core. We may represent γ on a p-abacus using a multiple of p beads, say rp in all. Thanks to this convention, if we label the runners 0, 1, ..., $p - 1$, then a bead on runner i corresponds to a step up past a removable node of residue $\equiv i \pmod p$. For example, the 3-core $(6, 4, 2, 1, 1)$ may be represented on the 3-abacus shown below.

$$\begin{array}{ccc} \bullet & \circ & \bullet \\ \bullet & \circ & \bullet^{*} \\ \circ & \circ & \bullet \\ \circ & \circ & \bullet \end{array}$$

The starred bead on runner 2 corresponds to the removable node of residue 2 in row 3 and column 2. By deleting rows of beads at the top of the abacus, we may assume, as is the case above, that at least one runner is empty.

Now we repeatedly remove removable nodes from γ, until we reach the empty partition. Removing a node of residue $\equiv i \pmod p$ corresponds to moving a bead one space left, *from* runner i *to* runner $i-1$. (With the obvious modification that if $i = 0$ then $i-1$ is taken to be $p-1$.) If there are x_i nodes of residue i then this manoeuver must occur exactly x_i times. By the time we reach the empty partition, which is represented by an abacus with exactly r beads on each runner, we have moved x_i beads *from* runner i, and x_{i+1} beads *to* runner i. Hence, if we started with c_i beads on runner i then

$$x_i - x_{i+1} = c_i - r \quad \text{(indices taken mod } p\text{)}.$$

Obviously, all the c_i are non-negative, and, by our choice of abacus display, at least one is zero. Thus, given the x_i, we may use the last equation to uniquely determine the c_i, and hence recover the p-core γ. This completes the proof.

2. Murphy operators

The Murphy operators L_u, for $u \in \{1, \ldots, n\}$, are defined by

$$L_u = (1, u) + (2, u) + \cdots + (u-1, u) \in \mathbf{Z}S_n.$$

It is easy to see that the Murphy operators commute; in fact L_u commutes with any element of S_{u-1}. (We shall see very shortly, that the vanishing of L_1 is due to the fact that 1 can only appear in a standard tableau in a node of residue 0.)

It is increasingly clear that the subalgebra of $\mathbf{Z}S_n$ generated by the Murphy operators plays a critical role in the representation theory of the symmetric groups—this subalgebra appears to be analogous in many ways to a Cartan subalgebra of a complex semisimple Lie algebra. For example, by Lemma 4 below, any $\mathbf{Q}S_n$-module decomposes as a direct sum of weight-spaces for the L_u.

Given a tableau t, and a number u between 1 and n, let $\mathrm{res}(t, u)$ denote the residue of the node of t_i containing u. The first unmistakable sign that the Murphy operators are of interest is the following proposition, which shows that they can pick out these residues. In it we use the total order $<$ defined on the set of row-standard tableaux of a fixed shape by setting $s < t$ if the greatest number that appears in a different place in s to t appears higher up in s than in t.

Proposition 2. *Let t be a standard λ-tableau. Then*

$$e_t L_u = \mathrm{res}(t, u) e_t + e_{<t}$$

where $e_{<t}$ denotes a $\mathbf{Z}$-linear combination of polytabloids e_s for tableaux s such that $s < t$.

This proposition is proved in Murphy's earlier paper [2]. (In fact Murphy proves the stronger result that has $e_{\lhd t}$ with $e_{<t}$ in the above.) Here is an

example, intended to give to illustrate the way in which Garnir relations are used in Murphy's proof. Take

$$t = \begin{array}{cccc} 1 & 2 & 3 & 7 \\ 4 & 6 & 9 & \\ 5 & 8 & & \end{array}.$$

We consider the effect of L_8 on e_t. Easiest are the actions of (28) and (68), for it is immediate from the definition of the polytabloid e_t that $e_t(28) = e_t(68) = -e_t$.

Next we consider the action of (38). To help with this calculation, we shall use the (potentially highly misleading) shorthand that a tableau t stands for its associated polytabloid e_t; accordingly, we must write

$$\begin{array}{cccc} 1 & 2 & 8 & 7 \\ 4 & 6 & 9 & \\ 5 & 3 & & \end{array} = - \begin{array}{cccc} 1 & 2 & 8 & 7 \\ 4 & 3 & 9 & \\ 5 & 6 & & \end{array}$$

as a sum of standard polytabloids. An application of the Garnir relation permuting the entries $\{2,3\} \cup \{4,5\}$ gives

$$\begin{array}{cccc} 1 & 2 & 8 & 7 \\ 4 & 3 & 9 & \\ 5 & 6 & & \end{array} = - \begin{array}{cccc} 1 & 3 & 8 & 7 \\ 2 & 4 & 9 & \\ 5 & 6 & & \end{array} + \begin{array}{cccc} 1 & 3 & 7 & 8 \\ 2 & 5 & 9 & \\ 4 & 6 & & \end{array} + \begin{array}{cccc} 1 & 2 & 8 & 7 \\ 3 & 4 & 9 & \\ 5 & 6 & & \end{array}$$
$$- \begin{array}{cccc} 1 & 2 & 8 & 7 \\ 3 & 5 & 9 & \\ 4 & 6 & & \end{array} + \begin{array}{cccc} 1 & 4 & 8 & 7 \\ 2 & 5 & 9 & \\ 3 & 6 & & \end{array}.$$

A further application of the relation permuting the entries $\{8,9\} \cup \{7\}$ completes the rewriting. Note that if t' is a standard tableau in the resulting expression then $t' < t$. (For the general result behind this, see Lemma 2.1 in [2].)

To find the action of the remaining summands of L_8, we consider the Garnir relation permuting $\{1,4,5\} \cup \{8\}$ in the original tableau t. It gives

$$e_t = e_t(18) + e_t(48) + e_t(58).$$

Hence, adding up are results obtained so far, we find that

$$e_t L_8 = e_t - 2e_t + e_{<t} = \mathrm{res}(t, 8)e_t + e_{<t},$$

exactly as Proposition 2 predicts.

3. Idempotents

Key to Murphy's proof is the following construction of a complete set of primitive idempotents in $\mathbf{Q}S_n$. This first appeared in his earlier paper [2].

Theorem 3. *Let λ be a partition of n. Let $d = \dim S^\lambda$ and let $t_1, \ldots, t_d$ be the standard λ tableaux as ordered by the $<$ order on tableaux. Let*

$$E_i = \prod_{c=-n+1}^{n-1} \prod_{\substack{u \\ \mathrm{res}(t_i,u) \neq c}} \frac{L_u - c}{\mathrm{res}(t_i, u) - c} \in \mathbf{Q}S_n.$$

The E_i lie in the block of $\mathbf{Q}S_n$ corresponding to the representation S^λ. Moreover, they form a complete set of primitive orthogonal idempotents lying in this block.

This result, despite its apparent complexity, follows quite easily from Proposition 2. We start its proof by showing that S^λ decomposes as a direct sum of weight-spaces for the algebra generated by the Murphy operators.

Lemma 4. *There is a basis $\{f_1, \ldots, f_d\}$ of S^λ on which the Murphy operators act by $f_j L_u = \operatorname{res}(t_j, u) f_j$. Up to a scalar, f_i is uniquely determined by the equation $f_i = e_i + e_{<i}$.*

Proof. Since two standard tableaux with the same residues are equal, it is possible to find $x_1, \ldots, x_n \in \mathbf{Q}$ such that

$$x_1 \operatorname{res}(t_i, 1) + \ldots + x_n \operatorname{res}(t_n, i)$$

takes d different values as i varies from 1 to d. Now consider the linear map

$$T = x_1 L_1 + \ldots + x_n L_n \in \operatorname{End}(S^\lambda).$$

With respect to the basis $e_1, \ldots, e_d$, we have

$$T = \begin{pmatrix} y_1 & & & & \\ \star & y_2 & & & \\ \star & \star & y_3 & & \\ \vdots & \vdots & & \ddots & \\ \star & \star & \cdots & \star & y_d \end{pmatrix}$$

where $y_i = x_1 \operatorname{res}(t_i, 1) + \ldots + x_d \operatorname{res}(t_i, d)$. Now, by basic linear algebra, we may find a unique y_i-eigenvector f_i for T of the form $e_i + e_{<i}$. As T has distinct eigenvalues, and the L_u preserve the T-eigenspaces, the vector f_i is a common eigenvector for the L_u. □

We can now prove Theorem 3 by calculating the action of the E_i on the basis f_i. Note first of all that, since the E_i are polynomials in the L_u, they preserve the weight-space decomposition given by Lemma 4. However, if $i \neq j$, then we may find $v \in \{1, \ldots, n\}$ such that $\operatorname{res}(t_j, v) \neq \operatorname{res}(t_i, v)$. Hence $L_v - \operatorname{res}(t_j, v)$ is a term in the product defining E_i, and it follows from Proposition 2 that $e_j E_i = e_{<j}$. Therefore $f_j E_i = 0$ if $i \neq j$. Similarly, one can show by direct calculation that $e_j E_j = e_j + e_{<j}$, and therefore $f_j E_j = f_j$.

Thus, assuming that the E_i do, as claimed, lie in the block of $\mathbf{Q}S_n$ corresponding to S^λ, we have found $\dim S^\lambda$ orthogonal idempotents in this block. By dimension counting, they form a complete set of primitive idempotents.

There is no particular difficulty in filling this gap in the proof, but as we have to consider e_j, f_j and E_i defined with respect to different partitions, the proof is inevitably slightly more involved; we refer the reader to Murphy's paper [2] for the details.

Notation change: From now on we will have to consider several different partitions at once, so we write E_i^λ rather than E_i for the idempotent just defined, and similarly we decorate the basis vectors e_i^λ and f_i^λ, and the tableaux t_i^λ.

Remark. Let R_λ be the multiset of residues of the partition λ. The denominator of E_i^λ is then

$$\prod_{r,c} (r - c)$$

where the product is over all $r \in R_\lambda$ and $c \in \{-n+1, \ldots, n-1\}$ such that $r \neq c$. Note that this depends only on the partition λ, and not on which tableaux t_i^λ we choose. A generalisation of this seems to be important to the modular case.

4. Some calculations for $(n-1,1)$

It is interesting to see the form of these idempotents in the case where $\lambda = (n-1,1)$. In this case there are $n-1$ standard tableaux; when $n = 4$ they are

$$t_1 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \; t_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \; t_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

Note that t_i has $i+1$ in its bottom row, not i. (This is annoying, but essential if we are to be consistent with the notation used so far.)

Using MAGMA to help with the calculations (code available at the webpage `http://www-maths.swan.ac.uk/staff/mjw/other.html`), one finds that:

$$\begin{aligned}
E_1^{(1,1)} &= \tfrac{1}{2} - \tfrac{1}{2}(12) \\
E_1^{(2,1)} &= \tfrac{1}{3} - \tfrac{1}{3}(12) + \tfrac{1}{6}(13) + \tfrac{1}{6}(23) - \tfrac{1}{6}(123) - \tfrac{1}{6}(132) \\
E_2^{(2,1)} &= \tfrac{1}{3} + \tfrac{1}{3}(12) - \tfrac{1}{6}(13) - \tfrac{1}{6}(23) - \tfrac{1}{6}(123) - \tfrac{1}{6}(132) \\
E_1^{(3,1)} &= \tfrac{1}{8} - \tfrac{1}{8}(12) + \tfrac{1}{16}(13) + \tfrac{1}{16}(14) + \tfrac{1}{16}(23) + \tfrac{1}{16}(24) + \tfrac{1}{8}(34) \\
&\quad - \tfrac{1}{16}(1234) - \tfrac{1}{16}(1243) - \tfrac{1}{16}(1342) - \tfrac{1}{16}(1432) - \tfrac{1}{8}(12)(34) \\
&\quad - \tfrac{1}{16}(123) - \tfrac{1}{16}(132) - \tfrac{1}{16}(124) - \tfrac{1}{16}(142) \\
&\quad + \tfrac{1}{16}(134) + \tfrac{1}{16}(143) + \tfrac{1}{16}(234) + \tfrac{1}{16}(243) \\
E_2^{(3,1)} &= \tfrac{1}{8} + \tfrac{1}{8}(12) - \tfrac{1}{16}(13) + \tfrac{5}{48}(14) - \tfrac{1}{16}(23) + \tfrac{5}{48}(24) + \tfrac{1}{24}(34) \\
&\quad - \tfrac{1}{48}(1234) - \tfrac{1}{48}(1243) - \tfrac{1}{12}(1324) - \tfrac{1}{48}(1342) - \tfrac{1}{12}(1423) - \tfrac{1}{48}(1432) \\
&\quad + \tfrac{1}{24}(12)(34) - \tfrac{1}{12}(13)(24) - \tfrac{1}{12}(14)(23) \\
&\quad - \tfrac{1}{16}(123) - \tfrac{1}{16}(132) + \tfrac{5}{48}(124) + \tfrac{5}{48}(142) \\
&\quad - \tfrac{1}{48}(134) - \tfrac{1}{48}(143) - \tfrac{1}{48}(243) - \tfrac{1}{48}(234) \\
E_3^{(3,1)} &= \tfrac{1}{8} + \tfrac{1}{8}(12) + \tfrac{1}{8}(13) - \tfrac{1}{24}(14) + \tfrac{1}{8}(23) - \tfrac{1}{24}(24) - \tfrac{1}{24}(34) \\
&\quad - \tfrac{1}{24}(1234) - \tfrac{1}{24}(1243) - \tfrac{1}{24}(1324) - \tfrac{1}{24}(1342) - \tfrac{1}{24}(1423) - \tfrac{1}{24}(1432) \\
&\quad - \tfrac{1}{24}(12)(34) - \tfrac{1}{24}(13)(24) - \tfrac{1}{24}(14)(23) \\
&\quad + \tfrac{1}{8}(123) + \tfrac{1}{8}(132) - \tfrac{1}{24}(124) - \tfrac{1}{24}(142) \\
&\quad - \tfrac{1}{24}(134) - \tfrac{1}{24}(143) - \tfrac{1}{24}(234) - \tfrac{1}{24}(243).
\end{aligned}$$

It is interesting that the numbers in the denominator of $E_1^{(3,1)}$ are all powers of 2, while factors of 3 appear in $E_2^{(3,1)}$ and $E_3^{(3,1)}$. (This does not contradict the remark at the end of §3, because the numerators of the E_i^λ certainly do depend on i.) The curious fraction 5/48 in $E_2^{(3,1)}$ is also worth noting. Taking the sum $E_1^{(n-1,1)} + \ldots + E_{n-1}^{(n-1,1)}$ we obtain the primitive central idempotent in $\mathbf{Q}S_n$ for $S^{(n-1,1)}$,

$$z^{(n-1,1)} = \frac{n-1}{n!} \sum_{\sigma} (|\mathrm{Fix}\,\sigma| - 1)\hat{\sigma}.$$

Here, σ runs over a set of representatives for the conjugacy classes of S_n, and $\hat{\sigma}$ denotes the sum of all elements conjugate to σ. For example,

$$\begin{aligned} z^{(1,1)} &= \tfrac{1}{2} - \tfrac{1}{2}(12) \\ z^{(2,1)} &= \tfrac{2}{3} - \tfrac{1}{3}\widehat{(12)} \\ z^{(3,1)} &= \tfrac{3}{8} + \tfrac{1}{8}\widehat{(12)} - \tfrac{1}{8}\widehat{(12)(34)} - \tfrac{1}{8}\widehat{(1234)} \end{aligned}$$

Using these calculations, we can give examples of some of Murphy's other results. For example, his Lemma 1.8 in [3] predicts that if $u-1$ and u are in the same row or column of t_i, then $(u-1, u)$ commutes with E_i. In particular, every E_i commutes with (12), as can easily be verified for the idempotents above. (This can also be seen in another way: every L_u commutes with (12), and the E_i are polynomials in the L_u.)

References

[1] James, G., and Kerber, A. *The representation theory of the symmetric group*, vol. 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[2] Murphy, G. E. A new construction of Young's seminormal representation of symmetric groups. *J. Alg. 69* (1981), 287–297.

[3] Murphy, G. E. The idempotents of the symmetric group and Nakayama's conjecture. *J. Alg. 81* (1983), 258–265.

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