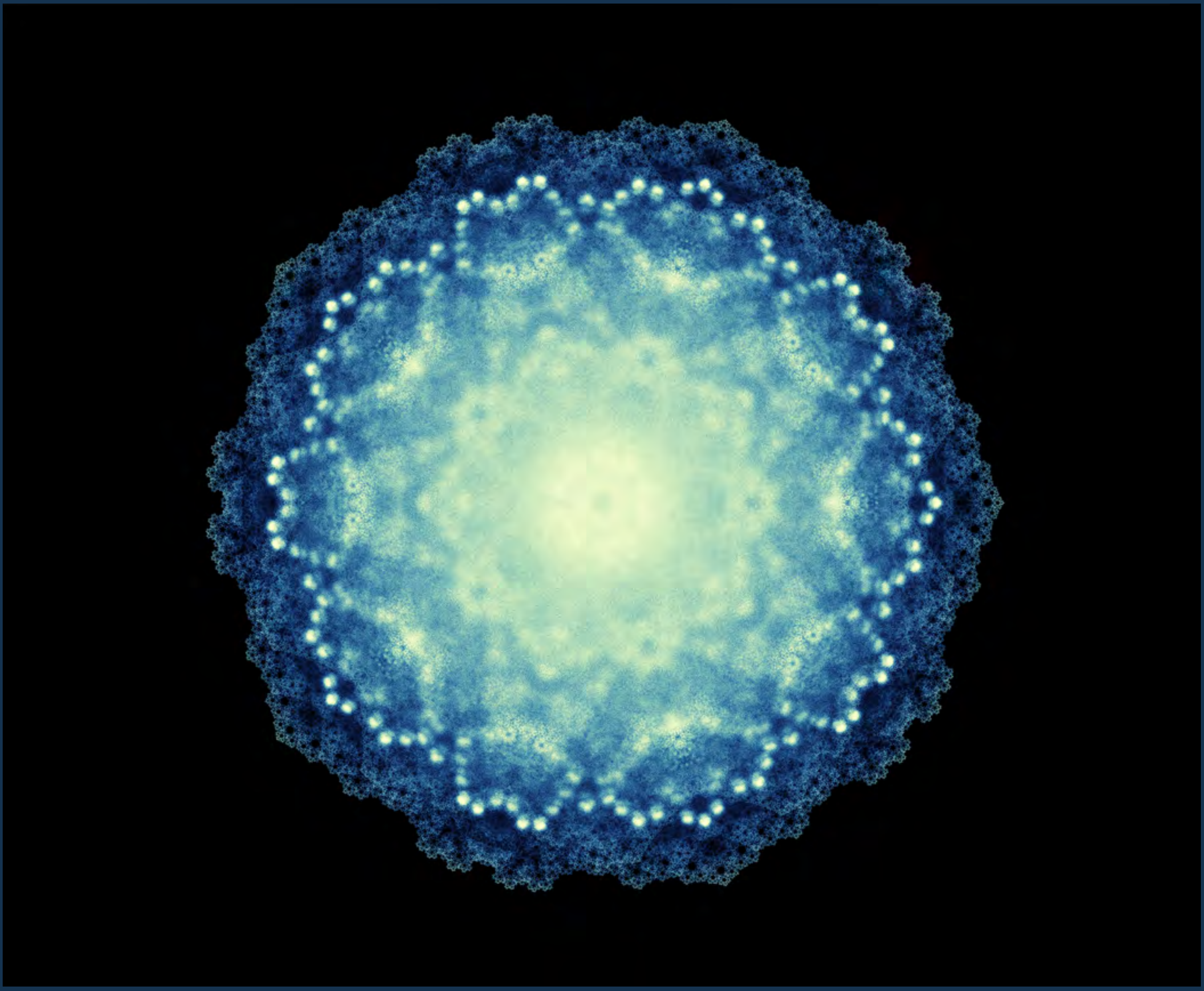
LONDON MATHEMATICAL SOCIETY
EST. 1865

NEWSLETTER

Issue: 491 - November 2020



COXETER FRIEZES AND GEOMETRY

THE PROBABILIST'S URN

ROYAL INSTITUTION CHRISTMAS LECTURER

EDITORIAL OFFICE

London Mathematical Society
De Morgan House
57–58 Russell Square
London WC1B 4HS
newsletter@lms.ac.uk

Charity registration number: 252660

COVER IMAGE

See the article ‘Bohemian Matrices’ on page 16.

Do you have an image of mathematical interest that may be included on the front cover of a future issue? Email images@lms.ac.uk for details.

ISSN: 2516-3841 (Print)
ISSN: 2516-385X (Online)
DOI: 10.1112/NLMS

NEWSLETTER WEBSITE

The Newsletter is freely available electronically at lms.ac.uk/publications/lms-newsletter.

MEMBERSHIP

Joining the LMS is a straightforward process. For membership details see lms.ac.uk/membership.

SUBMISSIONS

The Newsletter welcomes submissions of feature content, including mathematical articles, career related articles, and microtheses, from members and non-members. Submission guidelines and LaTeX templates can be found at lms.ac.uk/publications/submit-to-the-lms-newsletter.

Feature content should be submitted to the editor-in-chief at newsletter.editor@lms.ac.uk.

News items should be sent to newsletter@lms.ac.uk.

Book reviews should be sent to reviews@lms.ac.uk.

Notices of events should be prepared using the template at lms.ac.uk/publications/lms-newsletter and sent to calendar@lms.ac.uk.

For advertising rates and guidelines see lms.ac.uk/publications/advertise-in-the-lms-newsletter.

CONTENTS

WELCOME

As a recent recruit to the *LMS Newsletter* Board I am delighted to have been offered the opportunity to write the editorial welcome to this issue. The *Newsletter* is the main route for the Society to communicate with its members and the wider mathematics community, and the redesign initiated in 2017 has made it even more effective in this role in my view. I've particularly enjoyed the variety of feature articles over the last few years, and some more recent introductions such as 'Notes of a Numerical Analyst'.

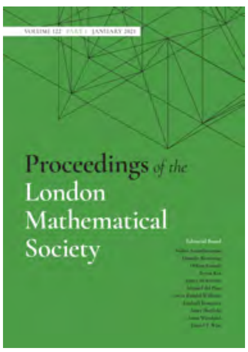
Amongst the objectives of the LMS is the promotion of equality of opportunity and a culture of fairness across all career stages. We aim to model this within the *Newsletter*, with articles written by early career researchers to balance content from more senior mathematicians, as well as articles aimed at addressing issues faced by those at an early career stage. For example, in this issue an interview with Dr Hannah Fry covers her experience as a mathematician, writer and broadcaster who is also active on social media, and we also have a microthesis from recent PhD graduate Luke Jeffreys.

Another aspect of our commitment to equality of opportunity is to carefully consider the role models visible within the *Newsletter*. Whilst the LMS has been very active in thinking about women in mathematics and taking steps to improve their representation we have some way to go in terms of other forms of diversity. We might reasonably expect the ethnic diversity in mathematics to reflect the general population, but this has not always been the image presented. If our early career mathematicians do not see 'people like me' in their departments or in the mathematics community, it is easy to understand the discouragement caused. There are many successful Black, Asian and minority ethnic mathematicians and it is important that we raise awareness of this. I'm very much looking forward to the first "Black Heroes of Mathematics" conference, jointly supported by the LMS, which will be held online on 26–27 October. We aim to have a full report of this conference in a future issue.

Professor Cathy Hobbs
LMS Vice-President and Deputy Editor (Newsletter)

LMS NEWS

Proceedings relaunch: update



Founded in 1865, the *Proceedings* is today seen by many as the flagship journal of the LMS. With the support of the LMS Publications Committee, the Managing Editors, Timothy Browning and Oscar Randal-Williams, took on the task of elevating the position of the journal further and strengthening its position with relation to other top-flight mathematics journals.

A new, independent Editorial Board of international experts was appointed to identify papers of the highest quality and significance across a broad spectrum of mathematics and of any length. The first such papers handled by this Board have been published online at tiny.cc/PROC_LMS.

The complete Editorial Board is: Nalini Anantharaman, Timothy Browning, Hélène Esnault, James McKernan, Bryna Kra, Manuel del Pino, Oscar Randal Williams, Raphaël Rouquier, Scott Sheffield, Anna Wienhard and Daniel T. Wise.

To mark the first issue of the relaunched *Proceedings* the LMS is planning several events over the course of 2021, including at the virtual Joint Mathematics Meeting in January.

To find out more, including how to submit to the *Proceedings*, please visit lms.ac.uk/PROC.

IDM 2021

The theme for the next International Day of Mathematics (IDM), on 14 March 2021, is 'Mathematics for a Better World', and the LMS Education Committee would like to hear from members on how the LMS might celebrate this event. If you have an idea which you would like us to consider, please email Katherine Wright, Society and Research Officer, at education@lms.ac.uk. For more information on the IDM, including activities which have been featured in the past, see idm314.org.

In addition, school students and general mathematics lovers are invited to participate in the worldwide Poster Challenge, which will explore mathematics as a universal language. Participants should create a poster sharing one idea on how to make the world better using mathematics. Images should be used instead of words, combined with numbers, formulae, geometric shapes, or other mathematical elements to express your ideas. The deadline for submission is February 15, 2021, and the best submissions will be shared on the IDM website. Learn more at idm314.org/2021-poster-challenge.html.

OTHER NEWS

Breakthrough Prize in Mathematics

The 2021 Breakthrough Prize in Mathematics has been awarded to LMS member Professor Sir Martin Hairer FRS, for 'transformative contributions to the theory of stochastic analysis, particularly the theory of regularity structures in stochastic partial differential equations'.

Sir Martin is a world leader in the field of stochastic analysis, which provides conceptual and computational methods for rigorously modelling the interactions within high dimensional random systems. His work at the interface between probability theory and partial differential equations has created a body of work that is widely recognised as having revolutionised an entire field of research.

He obtained his first degree in Mathematics at the University of Geneva, where he also studied for his Master's degree and PhD in Physics under the supervision of Jean-Pierre Eckmann. After completing his PhD in 2002, Sir Martin moved to the University of Warwick as a Postdoctoral Fellow. He went on to became Regius Professor of Mathematics at Warwick in 2014. Since 2017 he has been Professor of Pure Mathematics at Imperial College London.

Sir Martin has received many distinctions and honours, including the Fields Medal (2014). In the same year he was elected a Fellow of the Royal Society (FRS) and was awarded the LMS Fröhlich Prize. He received the LMS Whitehead Prize in 2008. In 2019 he was made a substantive Knight Commander of the British Empire. He presented the LMS Popular Lectures in 2015, the Society's 150th Anniversary year.

Professor Jon Keating, LMS President, commented: "The LMS offers its very warmest congratulations to our colleague Sir Martin Hairer on his award of this year's Breakthrough Prize in Mathematics, which recognises his seminal research in stochastic analysis, his inspirational leadership of the field, and his many contributions to the wider mathematics community."

More information about Breakthrough Prizes and full citations are available at tinyurl.com/yxnhy73l.

The Royal Society's Mathematical Futures Programme

In 2017, the *Report of Professor Sir Adrian Smith's review of Post-16 mathematics* (tinyurl.com/yanh8988) considered ways of improving mathematics education for 16–18 year olds in England with the aim of ensuring that the future workforce is appropriately skilled and competitive. Adrian used the term mathematics in its broadest sense, referring to mathematical and quantitative skills, including numeracy, statistics and data analysis, stressing that "there is strong demand for mathematics, quantitative and data skills in the labour market at all levels and consistent under-supply" and "the importance of mathematics to a wide range of future careers".

Adrian made a compelling case for students continuing with the study of some form of mathematics to age 18, which was supported by ministers who responded: "The report presents clear evidence for the value of mathematical and quantitative skills to students, whichever route they take", and which included continuing and enhanced support for the recently reformed post-16/level 3/advanced mathematics curriculum and qualifications — AS/A-level Mathematics and Further Mathematics, and Core Maths (see tinyurl.com/y5wbjc6r).

Much activity has taken place since Adrian's review, including addressing his recommendation that "the Department for Education should work with UK learned societies to encourage universities to better signal and recognise the value of level 3 mathematics qualifications for entry to undergraduate courses with a significant quantitative element" through the Royal Society's work on the importance of studying mathematics post-16 (see tinyurl.com/y98e9thz).

Adrian concluded his report with an eye to the future under the heading 'Big Data and Data Science', arguing that "the increasing sophistication of technology is driving change to the economy and the nature of work. This is not only increasing the demand for mathematics and quantitative skills but is also changing the nature of required skillsets, in particular those relating to the analysis and use of 'big data'", with an accompanying final recommendation: "The Department for Education and the Department for Business, Energy & Industrial Strategy should commission a study into the long-term implications of the rise of data science as an academic and professional field, looking at skills required for the future and the specific implications for education and training in mathematics and quantitative skills."

It is therefore timely that the Royal Society has joined forces with Arm, Google and GSK on a two-year programme, *Mathematical Futures*, (tinyurl.com/y4rxlhob) which will explore the mathematical skills that will be needed by all citizens to thrive in the future and how education systems should develop those skills. The programme aims to shape the future of mathematics education by identifying the skills that will be in high demand by employers in the next 20 years.

In common with the 2017 Smith review, the term 'mathematical' is intended to be as inclusive as possible — arithmetical/numerical/functional, along with traditional and modern areas of mathematics, and applications of mathematics in the mathematical and quantitative sciences. The programme will therefore also consider competencies in working with data literacy, statistics, computing and data science, which will ensure young people are not only equipped to meet the needs of a technological economy, but are also empowered, well-rounded citizens.

The Royal Society's 2019 *Dynamics of Data Science Skills* report argued that: "Education should provide a grounding to ensure that all young people develop underpinning data science knowledge and skills" drawing on data experts who "highlighted a range of core skills and disciplines that need to be developed early on including coding, computer science, mathematics, machine learning, statistics, and more."

The report made clear that "Post-16 curriculum change within the next ten years is vital to ensure young people leave education with the broad and balanced range of skills they will need to flourish in a changing world of work", citing the Royal Society's 2017 review report on *The integration of data science in the primary and secondary curriculum*, which reviewed how data science skills are nurtured in England's curriculum, identifying some barriers to embedding data skills into the curriculum as well as some opportunities for further development.

Key areas of work of the *Mathematical Futures* programme include identifying the skills and training that teachers will need in classrooms of the future, and addressing the challenges of achieving an improved gender balance in maths, including an exploration of why the majority of girls do not continue with maths post-16.

The programme is chaired by the eminent mathematician and former LMS President Sir Martin Taylor FRS, and is overseen by the Royal Society Advisory Committee on Mathematics Education (ACME), also chaired by Sir Martin, and in association with the Royal Statistical Society (RSS), the LMS, and the Institute of Mathematics and Its Applications (IMA).

Sir Martin said: “The challenge we all face is that mathematics, statistics, technology and data are rapidly shaping the world we live in, and, particularly, our working lives. That is why it is important to have a workforce that is mathematically and data literate and capable of doing the new jobs that will emerge in the years to come.”

Input to the programme is currently being sought through a Call for Views (tinyurl.com/y5d88lqn) from sectors including higher education, government and employers in business and industry, and from a wide range of stakeholders more generally. This naturally includes the learned societies associated with the programme — IMA, LMS, RSS — who will also have the opportunity to feed into the work of the programme in other ways over the coming months.

Professor Paul Glaister
University of Reading

Sir Roger Penrose wins Nobel Prize

The LMS would like to congratulate Sir Roger Penrose on his award of the 2020 Nobel Prize for Physics. The prize was divided, one half awarded to Roger Penrose “for the discovery that black hole formation is a robust prediction of the general theory of relativity”, the other half jointly to Reinhard Genzel and Andrea Ghez “for the discovery of a supermassive compact object at the centre of our galaxy”.

Stokes at 200

Royal Society Publishing has recently published a special double issue of *Philosophical Transactions A* entitled ‘Stokes at 200 (Parts 1 & 2) compiled and edited by Silvana Cardoso, Julyan Cartwright, Herbert Huppert and Christopher Ness. The articles can be accessed at bit.ly/TransA2174 and bit.ly/TransA2179. The issues are freely available online until 30 November 2020 or you can purchase the print issues at the reduced price of £35 each by contacting debbie.vaughan@royalsociety.org.

Prestigious Honours for LMS Members

Julia Gog (photograph by Lionel D'Souza) and Bryan Birch

The Sylvester Medal has been awarded to LMS member Professor Bryan Birch (University of Oxford). As noted in the citation, Professor Birch's work has played a "major role in driving the theory of elliptic curves, through the Birch–Swinnerton-Dyer conjecture and the theory of Heegner points". He has been an LMS member since 1958 and a Fellow of the Royal Society since 1972. His support for the LMS has included editing the *Proceedings of the London Mathematical Society* from 2000 to 2002. He has also received several LMS honours including the Senior Whitehead Prize in 1993 and the Society's highest honour, the De Morgan Medal, in 2007.

The Rosalind Franklin Award and Lectureship has been awarded to LMS member Professor Julia Gog (University of Cambridge) for her outstanding work in mathematics and disease modelling. Her mathematical work in the study of infectious diseases has led her to being an important member of the government's Scientific Advisory Group for Emergencies (SAGE), providing advice and insight during the current covid-19 pandemic. Professor Gog has also contributed significantly to the work of the LMS. She presented the 2014 LMS Popular Lectures and was the LMS–NZMS Forder Lecturer in 2016. She was awarded the LMS Whitehead Prize in 2017.

Professor Jon Keating, LMS President, commented "The LMS offers its very warmest congratulations to both Professor Birch and Professor Gog for these major awards, which recognise the outstanding contributions they have made, and continue to make, in terms of their important mathematical research and intellectual leadership."

Other notable Royal Society awards to mathematicians include Professor Herbert Huppert (University of Cambridge) who has been awarded a Royal Medal. More information is available at tinyurl.com/y5xdffea.

In further news, Professor Julia Gog has been made OBE in the Queen's Birthday Honours for services to academia and the covid-19 response.

Association for Women in Mathematics: Misidentification

I am grateful to the two readers who pointed out two errors in one of the three photo captions in my article Association for Women in Mathematics in the September 2020 issue of the LMS *Newsletter*, pages 42–43. Alberto (not Roberto, my first mistake) Calderón was a well-known Argentinian mathematician who worked at top universities in the US, was the husband of Alexandra Bellow, and looked quite a bit like Walter Rudin (my second mistake). But I should have realised that there was at least a 50:50 chance that the man in the middle was the husband of Mary Ellen Rudin rather than of Alexandra Bellow!

Below (left) is a photo of Alberto Calderón and Alexandra Bellow from the 1991 Joint Mathematics Meetings in San Francisco, the same mathematics conference at which the photo (right) was taken of Alexandra Bellow, Walter Rudin and Mary Ellen Rudin.

Janet Beery
AWM Clerk (Secretary) and Membership Chair
University of Redlands, USA

MATHEMATICS POLICY DIGEST

Programme to Boost Maths Skills for the Careers of 2040

The Royal Society has joined forces with Arm, Google and GSK on a two-year programme — *Mathematical Futures* — aimed at shaping the future of mathematics education by identifying the arithmetical, data and digital skills that will be in high demand by employers in the next 20 years, as well as the skills and training needed by teachers in the classrooms of the future.

The programme will be chaired by Sir Martin Taylor and overseen by the Royal Society Advisory Committee on Mathematics Education (ACME), in association with the Royal Statistical Society, the London Mathematical Society, and the Institute of Mathematics and Its Applications. Input will also be sought from sectors including higher education, government and employers in business and industry. More information about the programme is available at tinyurl.com/y4rxlhob

A full article by Paul Glaister, a member of the LMS Education Committee, is on page 6.

Digest prepared by Dr John Johnston
Society Communications Officer

Note: items included in the Mathematics Policy Digest are not necessarily endorsed by the Editorial Board or the LMS.

EMS 30th Anniversary

The celebratory event to mark the European Mathematical Society's 30th anniversary, originally scheduled to take place in the last weekend of October at the ICMS Edinburgh, has sadly been postponed until next Spring because of the current coronavirus situation in Europe.

Oberwolfach Research Fellows

In response to international travel restrictions and reduced scientific contact within universities, the *Mathematisches Forschungsinstitut Oberwolfach gGmbH* (MFO) offers in the period 1 July 2020 – 31 March 2021 the opportunity to apply for an Oberwolfach Research Fellowship. Single researchers or groups up to four people can apply for a research stay (2 weeks – 3 months). Junior researchers and post-docs can apply for full scholarships and travel support. Decisions on the proposals are planned within 3–4 weeks. The MFO offers free lodging and board. As usual, the full MFO infrastructure (in particular the library, but also modern video conference tools) is available. The MFO has developed new hygiene measures for hand disinfection, accommodation, lecture halls and individual meal services to make a research stay as safe as possible. For further details visit tinyurl.com/yy8gyorr.

EMS Newsletter

Fernando Manuel Pestana da Costa, from Lisbon, is the new editor-in-chief of the Newsletter of the EMS. He writes: "Following decisions taken by the Executive Committee of the EMS, the newsletter will soon undergo a deep restructuring, reflected in the change of its name from *Newsletter* to *Magazine*, the introduction of an online first publishing policy, the removal of news from the printed version, and the introduction of new subjects such as 'Mathematics and the Arts', and 'Mathematics and Industry', for which new editors have been added to the previously existing editorial team. All of us, the editorial team and the EMS Press staff, hope the result will be an enhanced newsletter/magazine that will continue to keep the high standards set by the previous editors-in-chief, and will continue to serve the mathematically interested European community in progressively better ways." The new format will appear as the first issue of 2021.

Ramanujan Mathematical Society

The conference *Number Theory and Discrete Mathematics* will be held online from 11 to 14 December 2020. It is dedicated to mark the hundredth year of the passing away of Srinivasa Ramanujan. Several distinguished European mathematicians will deliver lectures. For further details visit tinyurl.com/y3b8hras.

David Chillingworth
LMS/EMS Correspondent

Note: items included in the European Mathematical Society News represent news from the EMS are not necessarily endorsed by the Editorial Board or the LMS.

OPPORTUNITIES

LMS Prizes 2021: Call for Nominations

The LMS would like to invite nominations for the following prizes in 2021, which are intended to recognise and celebrate achievements in and contributions to mathematics:

- the Pólya Prize, which is awarded in recognition of outstanding creativity in imaginative exposition of, or distinguished contribution to, mathematics within the UK;
- the Senior Whitehead Prize, for work in, influence on or service to mathematics, or recognition of lecturing gifts in the field of mathematics;
- the Naylor Prize and Lectureship, for work in, influence on, and contributions to applied mathematics and/or the applications of mathematics, and lecturing gifts;
- the Berwick Prize, awarded to the author(s) of a piece of research published by the Society between January 2013 and December 2020;
- the Anne Bennett Prize, for work in and influence on mathematics, particularly acting as an inspiration for women mathematicians;
- the Whitehead Prizes, which are awarded for work in and influence on mathematics to mathematicians with fewer than 15 years' experience at post-doctoral level (up to six may be awarded).

We have in various years found ourselves with rather few nominations for the Pólya Prize and the Anne Bennett Prize, and would particularly welcome nominations for these. We would also strongly welcome more nominations for all prizes for women and other underrepresented groups in the mathematical community. In some cases, our prizes are intended to celebrate contributions to the community, and we intend to place a good deal of weight on this aspect in forthcoming rounds. In all cases the Prizes Committee interprets the criteria broadly, so if in doubt please submit a nomination.

Regulations and nomination forms can be found at tinyurl.com/lmsprizes2021. Please return nomination forms to Katherine Wright, Society and Research Officer: prizes@lms.ac.uk.

The timetable for awarding LMS prizes has moved forward this year and the deadline for nominations is 31 December 2020. Any nominations received after that date will be considered in the next award round.

Hirst Prize and Lectureship 2021: Call for Nominations

Originally launched in 2015 as part of the 150th anniversary celebrations of the LMS, the Hirst Prize and Lectureship is now jointly awarded by the LMS and the British Society for the History of Mathematics (BSHM). The prize is named after Thomas Archer Hirst, who was the fifth President of the LMS (1872–74).

The Hirst Prize and Lectureship is intended to recognise contributions to the study of the history of mathematics. The prize shall be awarded in recognition of original and innovative work in the history of mathematics, which may be in any medium.

The prize is open to any mathematician or historian of mathematics. In a given year, the members of the Hirst Prize Committee, and the members of the LMS and BSHM Councils shall be ineligible for the award of the prize. There is no requirement for the winner to be based in the UK.

The award will be considered by the Councils of the LMS and BSHM in March 2021, and the winner will be announced in summer 2021. The winner will be invited to give a lecture on the history of mathematics at a meeting of the LMS in the year following the award.

Nominations are now invited; download a nomination form at tinyurl.com/hirst2021. Nominations should be sent to the Secretary to the Hirst Prize Committee, Katherine Wright (prizes@lms.ac.uk), by 31 December 2020.

David Crighton Medal 2021: Call for Nominations

The David Crighton Medal was established by the Councils of the LMS and IMA in 2002 in order to pay tribute to the memory of Professor David George Crighton FRS. The silver gilt medal will be awarded to an eminent mathematician for services both to mathematics and to the mathematical community, who is normally resident in the mathematical community represented by the two organisations on the 1st January of the year of the award.

The award is considered biennially by the Councils of the LMS and IMA. The medal-winner will normally be presented with the award at a joint meeting of the IMA and the LMS, and will be invited to give a lecture.

The most recent winner of the David Crighton Medal was Professor Ken Brown CBE FRSE, in 2019. Previous winners of the Medal are Professor I. David Abrahams (2017), Professor Frank Kelly CBE FRS (2015), Professor Arieh Iserles and Dr Peter Neumann, OBE (2012), Professor Keith Moffatt, FRS (2009), Professor Sir Christopher Zeeman, FRS (2006) and Professor Sir John Ball, FRS (2003).

Nominations are now invited and must be received by 28 February 2021. These should be made on a nomination form available on both Societies' websites (tinyurl.com/crighton21) or from the Secretary to the David Crighton Committee (prizes@lms.ac.uk).

LMS Undergraduate Research Bursaries in Mathematics 2021

The Undergraduate Research Bursary scheme provides an opportunity for students in their intermediate years to explore the potential of becoming a researcher. The award provides £215 per week to support a student undertaking a 6–8 week research project over Summer 2021, under the direction of a project supervisor.

Students must be registered at a UK institution for the majority of their undergraduate degree, and may only take up the award during the summer vacation between the intermediate years of their course. Students in the final year of their degree intending to undertake a taught Masters degree immediately following their undergraduate degree may also apply. Applications must be made by the project supervisor on behalf of the student.

For further information and to download the application form, visit tinyurl.com/y3leo4lh. Queries may also be addressed to Lucy Covington (urb@lms.ac.uk). The closing date for receipt of applications is 5pm on Monday 1 February 2021.

LMS Research Schools and Knowledge Exchange Research School 2022

Grants of up to £15,000 are available for LMS Research Schools, one of which will be focused on Knowledge Exchange. The LMS Research Schools provide training for research students in contemporary areas of mathematics. The Knowledge Exchange Research Schools will primarily focus on Knowledge Exchange and can be in any area of mathematics.

The LMS Research Schools take place in the UK and support participation of research students from both the UK and abroad. The lecturers are expected to be international leaders in their field. The LMS Research Schools are often partially funded by the Heilbronn Institute for Mathematical Research (heilbronn.ac.uk). Information about the submission of proposals can be found at tinyurl.com/ychr4lwm along with a list of previously supported Research Schools. Applicants are strongly encouraged to discuss their ideas for Research Schools with the Chair of the Early Career Research Committee, Professor Chris Parker (research.schools@lms.ac.uk) before submitting proposals. Proposals should be submitted to Lucy Covington (research.schools@lms.ac.uk) by 22 February 2021.

Clay Mathematics Institute Enhancement and Partnership Program

To extend the international reach of the Research School, prospective organisers may also wish to consider applying to the Clay Mathematics Institute (CMI) for additional funding under the CMI's Enhancement and Partnership Program. Further information about this program can be found at tinyurl.com/y72byonb. Prospective organisers are advised to discuss applications to this program as early as possible by contacting the CMI President, Martin Bridson (president@claymath.org). There is no need to wait for a decision from the LMS on your Research School application before contacting the CMI about funding through this program.

LMS Early Career Fellowships: 2020–21 Awards

To support early career mathematicians in the transition between PhD and a postdoctoral position, the London Mathematical Society offers up to eight Fellowships of between 3 and 6 months to mathematicians who have recently or will shortly receive their PhD. The award will be calculated at £1,200 per month plus a travel allowance. The fellowships may be held at one or more institutions but not normally at the institution where the fellow received their PhD.

The 2020–21 round is now open to applications. The deadline is 14 January 2021. For further details, including how to apply, visit the Society's website: tinyurl.com/y7npy2q7. Contact Tammy Tran — fellowships@lms.ac.uk or 020 7291 9979 — with any queries.

LMS Prospects in Mathematics Meeting 2021: Call for Expressions of Interest

Up to £7,000 is available to support the annual two-day event (usually taking place face to face in September) for Finalist Mathematics Undergraduates who are considering apply for a PhD after they have completed their current studies. This includes funding to cover fares and accommodation for up to 50 students, travel and accommodation for speakers and subsistence for participants including a social event.

LMS Prospects in Mathematics Meetings should feature speakers from a wide range of mathematical fields across the UK who discuss their current research and what opportunities are available to prospective PhD students. Prospective organisers should send an Expression of Interest (max. one A4 side in length) to the LMS Early Career Research Committee (ECR.grants@lms.ac.uk) by 15 November 2020 with the following details:

- Confirmation of support from the department.
- Reasons for wanting to host the LMS Prospects in Mathematics Meeting.
- A provisional list of speakers should be included. Speakers should be representative of the UK research landscape both in geographical terms and in scientific terms.
- Speakers from under-represented groups should be included and women speakers should account for at least 40% of the invited speakers.
- Confirmation that prospective organisers have read and understood the terms and conditions in the Guidelines for Organisers.
- The Early Career Research Committee is interested in innovative approaches to the LMS Prospects in Mathematics Meeting.

For further details about the LMS Prospects in Mathematics Meetings, see tinyurl.com/y9yn2ryo. The most recent Prospects in Mathematics Meeting 2020 was hosted online from 10 to 11 September by the University of Bath. Further details about this event are available at tinyurl.com/y2ez9wbz.

IMA Hedy Lamarr Prize 2021: Call for Nominations

Nominations for the inaugural award of the IMA Hedy Lamarr Prize for Knowledge Exchange in Mathematics and its Applications are invited from 1 October 2020 until 31 January 2021. This new IMA prize, named for the inventor and film actress Hedy Lamarr, is intended to reward knowledge exchange in the mathematical sciences in all its possible forms. Further information and the nomination form can be found at: tinyurl.com/y63k482t.

IMA John Blake University Teaching Medal 2021

Nominations for the inaugural award of the IMA John Blake University Teaching Medal are invited by 31 January 2021. This recently introduced IMA prize is named in honour of the late Professor John Blake, who was widely known for his passionate support of teaching and learning within the mathematical sciences in higher education. It will be awarded to individuals who might be working within any role and who have made a significant and sustained contribution to the teaching of the mathematical sciences at university level. Successful nominees will need to demonstrate a proven ability to inspire, enhance and support learning in a diverse range of students as well as to engage with the scholarship of teaching and learning and to influence the practice of colleagues, both within their own institution and beyond. Further information and nomination instructions can be found at tinyurl.com/y96t7cre.

Heilbronn Research Fellowships

Salary: £38,017-£42,792
3 years fixed term, full-time

The Heilbronn Institute for Mathematical Research invites applications for Heilbronn Research Fellowships. Fellows divide their time equally between their own research and the internal research programme of the Heilbronn Institute, which offers opportunities to engage in collaborative work as well as individual projects. Fellowships are available at any of our partner institutions: **Imperial College London**, **King's College London**, **University of Bristol**, **University College London**, **University of Manchester**.

Research areas of interest include, but are not restricted to, Algebra, Algebraic Geometry, Combinatorics, Computational Statistics, Data Science, Number Theory, Probability, and Quantum Information. These areas are interpreted broadly: Fellows have previously been appointed with backgrounds in most areas of Pure Mathematics and Statistics, and in several areas of Mathematical/Theoretical Physics.

Due to the nature of the Heilbronn Institute's work, Fellows will be required to obtain and maintain a Developed Vetting (DV) security clearance. UK resident UK nationals will normally be able to meet this condition: other potential applicants should consult the Heilbronn Manager, himr-recruitment@bristol.ac.uk about eligibility before applying.

For more information about the Heilbronn Institute and these Fellowships, see **http://heilbronn.ac.uk/fellowships-2021/**

The salary range for these Fellowships is £38,017-£42,792 depending on previous experience, plus the London weighting where appropriate. There is a salary supplement of £3.5K pa, in recognition of the distinctive nature of these Fellowships. Payment of this supplement is conditional on a finished PhD thesis having been accepted in final form. In addition, a fund of at least £2.5K pa to pay for research expenses will be available to each Fellow.

The Fellowship will be for three years, with a preferred start date in October 2021, though another date may be possible by prior agreement. Extensions of up to three years may be available, which can be held at a wider range of universities in the UK.

The Heilbronn Institute is a supporter of the LMS Good Practice Scheme aimed at advancing women's careers in mathematics and we particularly welcome applications from women and other under-represented groups for this post. For more information, please see **https://heilbronn.ac.uk/diversity/**

Applications will be made through mathjobs.org. Applicants will need to submit a cover letter, indicating the institutions where they wish to hold their Fellowship from the above list of partner universities. In addition, we require a CV and a one-page statement of proposed research. Applicants must also arrange for three letters of reference to be uploaded to the mathjobs.org website, prior to the closing date.

Candidates only need to make one application, even if they are interested in applying for a Fellowship in two or more of our host institutions.

We anticipate holding interviews during the week of January 18th 2021.

The application deadline is 11.59pm GMT, Sunday November 15th 2020.

Vacancies on LMS Committees

The detailed business of the LMS is run by about 23 committees and working groups, each usually having 10 people. Altogether this comes to a large number of people, to whom the Society is extremely grateful for this vital work. It is Council's responsibility to make the appointments to all these committees and to turn their membership over regularly, so that (a) the broadest possible spectrum of our membership is represented, and (b) the committees remain fresh and energetic. Of course when forming a committee, account must be taken of many things, such as maintaining subject and demographic balance, which means that on a given occasion otherwise very strong candidates may not always be able to be appointed. So we are always looking for new people!

The list of committees can be found at lms.ac.uk/about/committees. If you are interested, or would like to recommend a colleague, please contact James Taylor at james.taylor@lms.ac.uk in order that Council can maintain a good list of potential members of its various committees. It is not necessary to specify a particular committee. If you would like to know what is involved, you could in the first instance ask your LMS Departmental Representative.

On this occasion we are in particular looking for new members of the Research Policy Committee.

Stephen Huggett
LMS General Secretary

Annual LMS Subscription 2020–21

Members are reminded that their annual subscription, including payment for publications, for the period Nov 2020 – Oct 2021 becomes due on 1 November 2020 and should be paid no later than 1 December 2020. In October, the Society sent a reminder to all members to renew their subscription for 2020–21. If you have not received a reminder, please email membership@lms.ac.uk.

Members can now view and pay their membership subscriptions online via the Society's website: lms.ac.uk/user. Further information about subscription rates for 2020-21 and a subscription form may also be found at lms.ac.uk/content/paying-your-subscription. The Society encouragees payment by direct debit, and also accepts credit or debit card and cheque. You can set up a direct debit and make payements via your online membership record: lms.ac.uk/user.

Benefits of LMS membership include free online access to selected Society journals, a complimentary bi-monthly *Newsletter*, discounts on selected Society publications and much more: lms.ac.uk/membership/member-benefits.

Elizabeth Fisher
Membership & Grants Manager

Report: Categorifications in Representation Theory

When lockdown first hit back in March, September seemed a long way off and it was unclear whether our conference could happen in person. But, as time wore on and it became clearer that normality would not return for some time, we decided to take the plunge and hold the conference online instead.

We tried to replicate as many aspects of a physical conference as we could, holding online coffee breaks using the gather.town platform and a quiz in lieu of a conference dinner. One notable positive of the online format was that people could participate from all over the world. We were pleased to have speakers addressing notions of categorifications from a range of perspectives. We had talks on 2-representation theory, on categorification of cluster algebras, as well as on combinatorial and geometric aspects of categorification.

The event received LMS funding for Postgraduate Research Conferences (Scheme 8).

Nicholas Williams & Aran Tattar
PhD students, University of Leicester

Correction

Eagle-eyed members will have noticed an error on Page 39 of the September issue, which we have now corrected on the online edition. The function $f(x) = x + 8\sin x$ is not periodic. The function $8\sin x$ would do as an example of a function periodic, surjective and not injective. In addition, the Editor-in-Chief would like to note that Page 40 of the September issue was a draft example by the author of the article which was mistakenly included during production.

Cover Image: Bohemian Matrices

R.M. CORLESS, N.J. HIGHAM AND S.E. THORNTON

A family of Bohemian matrices is a set of matrices in which the matrix entries belong to a discrete set of numbers bounded independently of the matrix dimensions. The term is a contraction of BOunded HEight Matrix of Integers. Such matrices arise in many applications, and include $\{0,1\}$ graph incidence matrices and $\{-1,1\}$ Bernoulli matrices.

Many interesting and often easy-to-pose questions arise in connection with a given $n \times n$ Bohemian matrix family $\mathcal{B}$, either for general n or for a particular n, including: what fraction of matrices in $\mathcal{B}$ are singular?; what are the most ill conditioned matrices in $\mathcal{B}$?; and how many distinct characteristic polynomials are there for matrices in $\mathcal{B}$?

The study of Bohemian matrices is not new; Taussky and others were considering them sixty years ago. Today, though, the computational power available at our fingertips can provide illumination in various ways, including through brute force calculations for small dimensions, through solving carefully posed optimization problems, and through visualising various quantities of interest.

Our example visualisation on the front cover of this *Newsletter* is a density plot over the complex plane of the eigenvalues of 10 million 13×13 upper Hessenberg Toeplitz matrices with diagonal entries fixed at 0, subdiagonal entries fixed at 1, and all other entries sampled randomly from the set of fifth roots of unity. The image raises several questions, in particular about how the restriction to upper Hessenberg Toeplitz structure affects the distribution of the eigenvalues as the dimension n goes to infinity.

For matrices M_n from an *unstructured* matrix family with entries drawn independently and identically distributed from a population with zero mean and unit variance the so-called circular law [3, Thm. 1.10] says that the eigenvalue distribution (called the empirical spectral distribution) of $M_n/\sqrt{n}$ approaches the uniform distribution on the unit disk. This universality is quite remarkable: only the mean and variance matter. However, when the matrices are required to be unit upper Hessenberg with zero diagonal the circular law no longer appears to hold, and the eigenvalues may sometimes better be described as being distributed over a lozenge or other polygonal figure instead of a disk [1].

Little seems to be known yet about this effect of matrix structure. Looking closely at the edges of our plot we see a suggestion of an emerging fractal, which we have seen similar signs of in other upper Hessenberg Toeplitz Bohemian families. Finally, if we compute eigenvalues exhaustively (which we can only do for small dimensions) then we see evidence of what are being called “algebraic starscapes” [2], which are images of algebraic numbers and their Diophantine approximations.

For more details, see www.bohemianmatrices.com.

FURTHER READING

[1] E.Y.S. Chan, R.M. Corless, L. Gonzalez-Vega, J.R. Sendra, J. Sendra and S.E. Thornton, Upper Hessenberg and Toeplitz Bohemians, Linear Algebra Appl. 601 (2020) 72–100.

[2] E. Harriss, K.E. Stange and S. Trettel, Algebraic Number Starscapes, arXiv:2008.07655.

[3] T. Tao, V. Vu and M. Krishnapur, Random Matrices: Universality of ESDs and the Circular Law, Ann. Probab. 38(5) (2010) 2023–2065.

Robert Corless

Rob is an Emeritus Distinguished University Professor at Western University, Canada. His research interests include computational dynamical systems, computational algebra, and computational special functions.

Nicholas Higham

Nick is a Royal Society Research Professor and Richardson Professor of Applied Mathematics at the University of Manchester. His research focuses on numerical linear algebra.

Steven Thornton

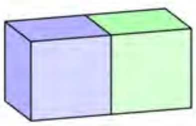
Steven is a recent PhD graduate of Western University, Canada. His research interests extend beyond Bohemian matrices into mathematical finance including algorithms for portfolio construction, and statistical arbitrage.

Notes of a Numerical Analyst

Two Cubes

NICK TREFETHEN FRS

Here's a mathematical problem. You have two unit cubes face to face against each other, each a uniform solid of mass 1. According to Newton's inverse-square gravitational law with gravitational constant 1, what's the force of attraction F between them?



I doubt you've encountered this problem, and it's easy to see why. It looks pointless. There are no serious applications, and no deeper mathematical ideas are lurking here. The force is given by a six-dimensional integral with respect to the coordinates x, y, z of one cube and ξ, η, ζ of the other:

$$F = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_1^2 f \; dx\, dy\, dz\, d\xi\, d\eta\, d\zeta$$

with

$$f = \frac{x - \xi}{\left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{3/2}}.$$

The integral is intractable, and it seems there's not much to say about it.

I cooked up this problem some years ago as a challenge for the graduate students in Oxford's Numerical Analysis "Problem Solving Squad". Could they calculate F to 10 digits of accuracy? Six-dimensional integrals are easy enough if the integrand is smooth — you can use six-fold Gauss quadrature, for example. But this integrand is singular since the cubes are touching, and calculating F accurately is hard. The elegant trick we found, devised by Alex Prideaux, was to break each cube into eight sub-cubes, whereupon a step of recursion eliminates the singularity.

The answer is $F \approx 0.9259812605\ldots$, and I know just one interesting thing about this number: it is less than 1. If the cubes were balls or spheres, F would be exactly 1, as Newton proved. Here we get $F < 1$ because cubes have some of their mass further out in the corners.

But then came a surprise from Bengt Fornberg of the University of Colorado, who is a wizard of formulae as well as algorithms. Fornberg spent several days gnawing on the integral. It's not hard to reduce the dimensionality from 6 to 3, but he managed to reduce it further to dimension 2, and then 1. Finally, he astonished us with an exact result:

$$\begin{aligned} F = \frac{1}{3}\Bigg(&\frac{26\pi}{3} - 14 + 2\sqrt{2} - 4\sqrt{3} + 10\sqrt{5} - 2\sqrt{6} \\ &+ 26\log(2) - \log(25) + 10\log(1+\sqrt{2}) \\ &+ 20\log(1+\sqrt{3}) - 35\log(1+\sqrt{5}) \\ &+ 6\log(1+\sqrt{6}) - 2\log(4+\sqrt{6}) \\ &- 22\tan^{-1}(2\sqrt{6})\Bigg). \end{aligned}$$

Look at this mess! There are 14 terms, each one as arbitrary as $10\log(1+\sqrt{2})$. How could we even be confident it was correct? Of course, ironically, by checking it against the numerical approximation. Yes, this freight train of a formula really is the right answer.

Does it matter if a problem has an exact solution? If there were no explicit formula, would that make a difference to the status of the two cubes problem? Or if the answer were simply $1/\sqrt{2}$?

FURTHER READING

[1] F. Bornemann et al., The SIAM 100-Digit Challenge: A Study in High-Accuracy Numerical Computing, SIAM, 2004.

[2] W. Hackbusch, Direct integration of the Newton potential over cubes, Computing, 68 (2002), 193–216.

[3] L. N. Trefethen, Ten digit problems, in D. Schleicher and M. Lackmann, eds., Invitation to Mathematics: From Competitions to Research, Springer, 2011.

Nick Trefethen

Trefethen is Professor of Numerical Analysis and head of the Numerical Analysis Group at the University of Oxford.

Coxeter Friezes and Hyperbolic Geometry

PETER JØRGENSEN

Coxeter friezes are beautiful objects at the cusp of art and mathematics. Conway and Coxeter showed that triangulated polygons give Coxeter friezes with integral entries. But is there a way to get Coxeter friezes from p-angulated polygons for a general integer $p \geqslant 3$?

Introduction

Coxeter friezes were defined by Coxeter 50 years ago and subsequently named after him. A few years later, Conway and Coxeter proved a classic theorem showing how triangulated polygons give rise to Coxeter friezes.

For the past decade, I have worked on these beautiful objects with my coauthors at the University of Hannover. My visit to Hannover in January 2017 was special: Thorsten Holm and I discovered a generalisation of Conway and Coxeter's Theorem, showing how p-angulated polygons give rise to Coxeter friezes for any integer $p \geqslant 3$.

Conway and Coxeter's proof was non-trivial but elementary in the sense of using only basic arithmetic. Thorsten and I were surprised to discover that we needed the (not very elementary) theory of symmetries of the hyperbolic plane for our generalisation.

Coxeter friezes

Figure 1 shows part of a Coxeter frieze, which is an infinite strip consisting of three or more rows of positive real numbers.

$$\begin{array}{cccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & \\
 & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 \\
2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

Figure 1. A Coxeter frieze with integral entries

The first and last rows consist of ones, and there is an offset between consecutive rows. The offset means that a Coxeter frieze contains many small "diamonds", that is, neighbouring entries forming the pattern in Figure 2. Each such diamond must satisfy the *local condition* $ad - bc = 1$.

$$\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}$$

Figure 2. Each "diamond" in a Coxeter frieze must satisfy $ad - bc = 1$

It seems almost miraculous that Coxeter friezes, henceforth often just called friezes, exist at all. Not only do overlapping diamonds result in a delicate web of interlinked equations; there is also the condition of starting and ending with a row of ones. However, many friezes do exist, and they do not necessarily have integral entries, see Figure 3.

$$\begin{array}{cccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\sqrt{2} & & 3\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & \\
 & 5 & & 5 & & 1 & & 3 & & 3 & & 3 & & 3 & & 1 & & 5 \\
2\sqrt{2} & & 4\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & 4\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & \\
 & 3 & & 3 & & 3 & & 1 & & 5 & & 5 & & 1 & & 3 & & 3 \\
2\sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 3\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

Figure 3. A Coxeter frieze with non-integral entries

Coxeter friezes as a nexus

Coxeter gave the definition of frieze in a short paper [3] with little motivation. Subsequent developments have shown that friezes form a nexus between algebra, combinatorics, and mathematical physics.

Friezes can be viewed as solutions of the discrete Schrödinger equation and as shadows of so-called cluster algebras and cluster categories. They have an interface with multiple areas of contemporary mathematics, and there is an active community of research mathematicians working on them [6].

Glide reflections and quiddity rows

Friezes enjoy a surprising property established by Coxeter himself: each frieze has a fundamental domain whose glide reflections cover the entire frieze; see Figure 4.

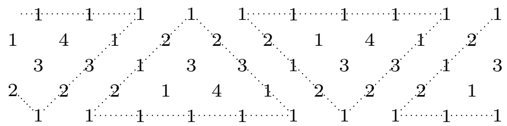


Figure 4. Glide reflections of a fundamental domain covering a Coxeter frieze

A less surprising but equally fundamental property is that a frieze is determined by its second row. The first row is known since it consists of ones. If we also know the second row, then we can determine the third row because the local condition $ad - bc = 1$ can be rewritten $c = \frac{ad-1}{b}$. Similarly, knowing the second and third rows enables the determination of the fourth row and so on.

The second row of a frieze is known as the *quiddity row*. "Quiddity" means "essence", and the quiddity row of the frieze in Figure 1 is $\ldots, 1, 4, 1, 2, 2, 2, 1, 4, 1, 2, \ldots$.

Conway and Coxeter's Theorem

Let $n \geqslant 4$ be an integer. Conway and Coxeter's Theorem says that there is a bijection from triangulated n-gons to friezes with integral entries and $n-1$ rows.

Given a triangulated n-gon, at each vertex count the number of incident triangles. Cycling clockwise around the polygon gives a doubly infinite sequence of integers, which is the quiddity row of the frieze corresponding to the polygon.

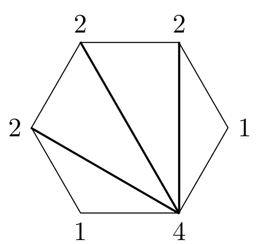


Figure 5. This triangulated hexagon corresponds to the Coxeter frieze in Figure 1

For instance, the triangulated hexagon in Figure 5 gives the sequence $\ldots, 1, 4, 1, 2, 2, 2, 1, 4, 1, 2, 2, 2, \ldots$, which is the quiddity row of the frieze in Figure 1. It is far from obvious that this recipe provides a bijection, and Conway and Coxeter's proof is indeed non-trivial. It is, however, elementary in the sense of using only basic arithmetic.

On Conway and Coxeter's proof

Using an original format, Conway and Coxeter did not write down a proof of their theorem but provided a collection of 35 problems [1] and subsequent hints [2]. Their theorem is the solution to problems 28 and 29. The easier part of the proof is to show that the recipe we have seen turns a triangulated polygon into a frieze with integral entries.

To go back from a frieze to a triangulated polygon, a key step is to show that the quiddity row of a frieze with integral entries must contain the integer 1. This implies that a frieze with m rows can be "reduced", resulting in a frieze with $m-1$ rows. This corresponds to removing an "ear" from a triangulated polygon. An *ear* is a triangle which can be removed without creating a gap in the triangulation. For instance, Figure 5 has two ears: they are the triangles with a vertex labelled 1.

Iterating this step eventually removes every triangle from a triangulated polygon, which can then be recovered by running the steps in reverse.

Generalising Conway and Coxeter's Theorem to p-angulated polygons

Are there friezes corresponding to p-angulated polygons for a general integer $p \geqslant 3$? This natural question had been considered by several mathematicians, but no answer was known when I travelled to Hannover in 2017. However, when I left, Thorsten Holm and I had proved that there is a bijection from p-angulated n-gons to friezes of type Λ_p with $n-1$ rows.

A frieze is *of type* Λ_p if its quiddity row has entries in the set

$$\{\ell\lambda_p \mid \ell \in \{1, 2, 3, \ldots\}\},$$

where

$$\lambda_p = 2\cos(\pi/p).$$

At first glance, this definition looks strange. The numbers λ_p are unmotivated, although some small instances are nice: $\lambda_3 = 1$, $\lambda_4 = \sqrt{2}$, $\lambda_5 = \frac{1+\sqrt{5}}{2}$ (the Golden Ratio), and $\lambda_6 = \sqrt{3}$. However, we obtain a bijection from p-angulated n-gons to friezes of type Λ_p with $n-1$ rows as follows.

Given a p-angulated n-gon, at each vertex count the number of incident p-angles. Cycling clockwise around the polygon gives a doubly infinite sequence of integers.

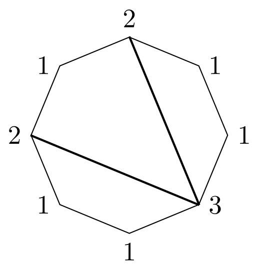


Figure 6. This 4-angulated octagon corresponds to the Coxeter frieze in Figure 3

Multiplying by λ_p gives the quiddity row of the frieze of type Λ_p corresponding to the polygon.

For instance, the 4-angulated octagon in Figure 6 gives the sequence $\ldots, 1, 3, 1, 1, 2, 1, 2, 1, \ldots$. Multiplying by $\lambda_4 = 2\cos(\pi/4) = \sqrt{2}$ gives the quiddity row of the frieze of type Λ_4 in Figure 3.

Note that since $\lambda_3 = 1$, a frieze of type Λ_3 is a frieze whose quiddity row has entries in the set $\{1, 2, 3, \ldots\}$. Although it is not obvious, these are precisely the friezes with integral entries, so if $p = 3$ then we recover Conway and Coxeter's Theorem.

A subtlety is that the n-gon has a p-angulation if and only if

$$n \geqslant p \text{ and } n \equiv 2 \bmod p-2 \qquad (*)$$

(it is an exercise for the reader to show this). Hence there is a frieze of type Λ_p with $n-1$ rows if and only if $(*)$ holds. For instance, there is a frieze of type Λ_4 with $n-1$ rows if and only if $n \geqslant 4$ and n is even. That is, there is a frieze of type Λ_4 if and only if there is an odd number of rows.

The easier part of the proof is to show that our recipe turns a p-angulated polygon into a frieze of type Λ_p. To go back from a frieze to a p-angulated polygon, we had to use the theory of symmetries of the hyperbolic plane. This also turns out to explain the appearance of the numbers λ_p.

Embedding the hyperbolic plane

We can embed the Euclidean plane $\mathscr{E}$ isometrically into three-dimensional space $\mathbb{R}^3$ as a flat surface. It follows from a famous theorem of Nash that the hyperbolic plane $\mathscr{H}$ can also be embedded isometrically into three-dimensional space, and Figure 8 shows part of such an embedding.

Figure 8. Part of an isometric embedding of the hyperbolic plane into three-dimensional space

In a sense, distances grow faster in $\mathscr{H}$ than in $\mathscr{E}$, and that is why the embedding in Figure 8 looks flat near the centre but develops dramatic folds further out.

The hyperbolic plane

The hyperbolic plane $\mathscr{H}$ has not otherwise played a role in my research, but I learned about its symmetries from Ian Short at the Open University. Having managed not to get lost in Milton Keynes' many roundabouts, I gave a seminar and Ian showed me his beautiful paper [7] with Mairi Walker.

To warm up, let us first look at the Euclidean plane, which for present purposes can be defined as the metric space $(\mathscr{E}, d_{\mathscr{E}})$ where $\mathscr{E} = \mathbb{R}^2$ and the distance function is given by

$$d_{\mathscr{E}}\big((x,y),(x',y')\big) = \sqrt{(x'-x)^2 + (y'-y)^2}\,.$$

Geodesics in $\mathscr{E}$, that is, curves locally minimising the distance between points, are straight lines. Each orientation-preserving isometry of $\mathscr{E}$ has the form

$$(x,y) \mapsto (x,y)A + (a,b)$$

where $(a,b) \in \mathscr{E}$ and A is a 2×2-matrix over $\mathbb{R}$ with determinant 1 which is orthogonal, that is, has inverse equal to its transpose. An orientation-preserving isometry is a translation or a rotation, and the set of all orientation-preserving isometries is a group acting on $\mathscr{E}$.

Similarly, the hyperbolic plane can be defined as the metric space $(\mathscr{H}, d_{\mathscr{H}})$ where

$$\mathscr{H} = \{x + iy \mid x, y \in \mathbb{R},\, y > 0\}$$

is the open upper half-plane in $\mathbb{C}$ and the distance function is given by

$$d_{\mathscr{H}}(x+iy, x'+iy') = \operatorname{arcosh}\left(1+\frac{(x'-x)^2+(y'-y)^2}{2yy'}\right)$$

with arcosh being the inverse hyperbolic cosine function defined by $\operatorname{arcosh}(u) = \ln(u + \sqrt{u^2-1})$. Geodesics in $\mathscr{H}$ are either half-circles with centre on the real axis or half-lines of the form $\{\, x + iy \mid y \in \mathbb{R}, y > 0 \,\}$ for a fixed $x \in \mathbb{R}$; see Figure 9. The significance of the blue geodesics in the figure will be explained later. Each orientation-preserving isometry has the form

$$z \mapsto A \cdot z := \frac{az+b}{cz+d}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2-matrix over $\mathbb{R}$ with determinant 1. The multiplicative group of such matrices is denoted $\mathrm{SL}_2(\mathbb{R})$, and $(A, z) \mapsto A \cdot z$ defines an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathscr{H}$ by orientation-preserving isometries.

The metric space $(\mathscr{H}, d_{\mathscr{H}})$ is known as the *Poincaré half-plane model*. An alternative would have been to define the hyperbolic plane as another metric space $(\mathscr{H}', d_{\mathscr{H}'})$ permitting an isometric bijection $\mathscr{H} \to \mathscr{H}'$. A popular choice would have been the *Poincaré disc model*, where $\mathscr{H}'$ is the unit disc. The Poincaré disc model is known from the beautiful "Kreislimit" images by Escher, who communicated extensively with Coxeter about the mathematical aspects of his art. However, the Poincaré half-plane model is more suitable for our purposes.

If a geodesic L in $\mathscr{H}$ is a half-circle, then it determines two "points at infinity": the numbers $\alpha, \beta \in \mathbb{R}$ shown in Figure 7.

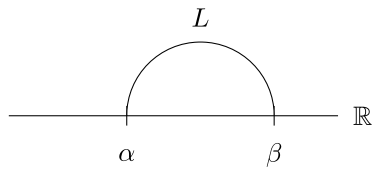


Figure 7. The geodesic L in the hyperbolic plane determines the numbers $\alpha, \beta \in \mathbb{R}$, which we think of as "points at infinity"

If a geodesic L in $\mathscr{H}$ is a half-line $\{\, x + iy \mid y \in \mathbb{R}, y > 0 \,\}$, then it also determines two points at infinity: $x \in \mathbb{R}$ and ∞. Points at infinity are called *ideal points*. They are not themselves points of $\mathscr{H}$ but represent directions of geodesics in $\mathscr{H}$. Geodesics can converge at ideal points, permitting the construction of *ideal p-angles* in $\mathscr{H}$ whose edges and vertices are geodesics and ideal points. Figure 9 shows several ideal 4-angles; three of them are labelled P_1, P_2, P_3.

From Coxeter friezes to p-angulated polygons

Here is how we used the hyperbolic plane $\mathscr{H}$ to go back from a frieze of type Λ_p to a p-angulated polygon.

For each integer $p \geqslant 3$, there is a discrete subgroup G_p of $\mathrm{SL}_2(\mathbb{R})$, which we can think of as a symmetry group of $\mathscr{H}$. The group G_p, known as a *Hecke group*, is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_p = \begin{pmatrix} 1 & \lambda_p \\ 0 & 1 \end{pmatrix}.$$

If we start with the geodesic

$$L = \{\, iy \mid y \in \mathbb{R}, y > 0 \,\}$$

in $\mathscr{H}$ and act with all elements of G_p, then we get an infinite set of geodesics because G_p is an infinite group of isometries. These geodesics and the ideal points they determine are the edges and vertices of a graph $\mathscr{F}_p$, the so-called *p-Farey graph*, which is the skeleton of a tiling of $\mathscr{H}$ by ideal p-angles.

For instance, Figure 9 shows part of $\mathscr{F}_4$ and the corresponding tiling. Three of the 4-angles are labelled P_1, P_2, P_3. Note that the full tiling consists of infinitely many ideal 4-angles: not only does it continue left and right of what is shown in the figure but also down towards the real line, where infinitely many ideal 4-angles are missing from the figure because they would appear too small.

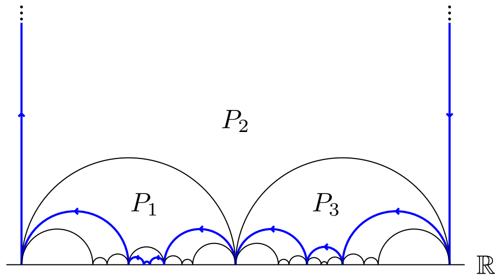


Figure 9. A tiling of the hyperbolic plane by ideal 4-angles. The half-circles and half-lines are geodesics

Using the action of G_p, an arithmetical trick turns a frieze of type Λ_p into a closed walk on $\mathcal{F}_p$. The trick works for friezes of type Λ_p because their quiddity rows contain integral multiples of λ_p, matching the generator T_p. The walk starts and ends at ∞ and encloses a number of ideal p-angles, which provide the p-angulated polygon corresponding to the frieze.

$$
\begin{array}{cccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\sqrt{2} & & \sqrt{2} & & 3\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & \\
 & 1 & & 5 & & 11 & & 3 & & 1 & & 3 & & 7 & & 3 & & 3 \\
\sqrt{2} & & 2\sqrt{2} & & 9\sqrt{2} & & 8\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 5\sqrt{2} & & 5\sqrt{2} & & 4\sqrt{2} & \\
 & 3 & & 7 & & 13 & & 5 & & 1 & & 3 & & 7 & & 13 & & 5 \\
\sqrt{2} & & 5\sqrt{2} & & 5\sqrt{2} & & 4\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & 9\sqrt{2} & & 8\sqrt{2} & \\
 & 3 & & 7 & & 3 & & 3 & & 3 & & 1 & & 5 & & 11 & & 3 \\
\sqrt{2} & & 2\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 3\sqrt{2} & & 2\sqrt{2} & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

Figure 10. A Coxeter frieze of type Λ_4

For instance, the frieze in Figure 10 gives the closed walk shown in blue in Figure 9. It encloses the ideal 4-angles P_1, P_2, P_3 and another 4-angle P_4, which is below P_1 but too small for a label. They provide the 4-angulated 10-gon in Figure 11, corresponding to the frieze in Figure 10.

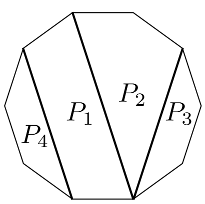


Figure 11. The blue walk in Figure 9 provides this 4-angulated 10-gon, which corresponds to the Coxeter frieze in Figure 10

From dissected polygons to Coxeter friezes

We do not need a p-angulated polygon to get a frieze: any polygon dissected by pairwise non-crossing diagonals will do.

Given a dissected polygon, at each vertex form a sum with a term λ_p for each incident p-angle in the dissection. Cycling clockwise around the polygon gives a doubly infinite sequence of real numbers, which is the quiddity row of a frieze.

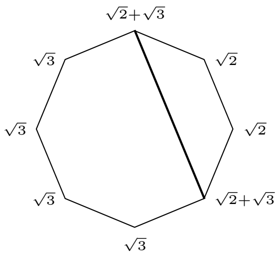


Figure 12. This dissected octagon corresponds to the Coxeter frieze in Figure 13

For instance, the octagon in Figure 12, dissected into a 4-angle and a 6-angle, gives the sequence

$$\ldots, \sqrt{2}, \sqrt{2}+\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{2}+\sqrt{3}, \sqrt{2}, \ldots,$$

hence the frieze in Figure 13. However, unlike friezes of type Λ_p, the friezes obtained from dissected polygons are presently a mysterious class, which we do not understand well.

$$
\begin{array}{ccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\sqrt{2} & & \sqrt{2}+\sqrt{3} & & \sqrt{3} & & \sqrt{3} & & \sqrt{3} & & \sqrt{3} & & \sqrt{2}+\sqrt{3} & & \sqrt{2} \\
 & 1+\sqrt{6} & & 2+\sqrt{6} & & 2 & & 2 & & 2 & & 2+\sqrt{6} & & 1+\sqrt{6} & \\
\sqrt{3} & & 2\sqrt{2}+\sqrt{3} & & 2\sqrt{2}+\sqrt{3} & & \sqrt{3} & & \sqrt{3} & & 2\sqrt{2}+\sqrt{3} & & 2\sqrt{2}+\sqrt{3} & & \sqrt{3} \\
 & 2 & & 2+\sqrt{6} & & 1+\sqrt{6} & & 1 & & 1+\sqrt{6} & & 2+\sqrt{6} & & 2 & \\
\sqrt{3} & & \sqrt{3} & & \sqrt{2}+\sqrt{3} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2}+\sqrt{3} & & \sqrt{3} & & \sqrt{3} \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

Figure 13. This Coxeter frieze corresponds to the dissected 10-gon in Figure 12

Acknowledgements

I thank Stephen Huggett for encouraging me to write this feature for the LMS Newsletter and for comments on a preliminary version, which led to a significantly improved exposition. The model of the hyperbolic plane in Figure 8 was crocheted by Kris

Beicher using a pattern by Daina Taimina [4]. The paper [5] was supported by EPSRC grant EP/P016014/1 “Higher Dimensional Homological Algebra” and is the original source of Figures 9, 10, and 11, which have been amended slightly for this feature.

FURTHER READING

[1] J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns, The Mathematical Gazette 57 (1973), 87–94.

[2] J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns (continued from p. 94), The Mathematical Gazette 57 (1973), 175–183.

[3] H. S. M. Coxeter, Frieze patterns, Acta Arith. 18 (1971), 297–310.

[4] D. W. Henderson and D. Taimina, Crocheting the hyperbolic plane, Math. Intelligencer 23 (2001), 17–28.

[5] T. Holm and P. Jørgensen, A p-angulated generalisation of Conway and Coxeter's theorem on frieze patterns, Internat. Math. Res. Notices 2020 (2020), 71–90.

[6] S. Morier-Genoud, Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics, Bull. London Math. Soc. 47 (2015), 895–938.

[7] I. Short and M. Walker, *Geodesic Rosen continued fractions*, Quart. J. Math. **67** (2016), 519–549.

Peter Jørgensen

Peter Jørgensen is Professor of Mathematics at Aarhus University. He works on algebra and combinatorics and recently moved to Denmark after 17 years in the UK. He tried for years to improve his English accent and managed once to be mistaken for a person from York.

The Enduring Appeal of the Probabilist's Urn

CÉCILE MAILLER

Pólya urns are classic objects in probability theory and have been extensively studied for more than 100 years due to their wide range of applications. We present classic results on these processes, their applications to large networks, and recent theoretical developments.

"Is it a contractual obligation for all probabilists to have a drawing of an urn of coloured balls on their blackboards?". This was the question of a non-mathematician friend of mine when they entered my office a few years ago. Drawing balls from urns appears in many classic exercises in a first course in probability (e.g. an urn contains two black balls and one red ball, I pick a ball uniformly at random from the urn, what is the probability that it is black?). The aim of this article is to explain why balls-in-urns is still an object of interest for probabilists (and hence why they may end up on their blackboards!) and what questions on this object remain open.

A Pólya urn is a stochastic process that depends on two parameters: the initial composition of the urn (how many balls of which colours there are in the urn at time zero), and a replacement rule, which is encoded by a $d \times d$ matrix $R = (R_{i,j})_{1 \le i,j \le d}$, where d is the number of different colours a ball can be. At every integer time step, a ball is picked uniformly at random from the urn and, if it is of colour i, it is put back in the urn together with $R_{i,j}$ additional balls of colour j, for all $1 \le j \le d$.

This article focuses on the following question: "What is the composition of the urn in the limit when time goes to infinity?". The difficulty in answering this question comes from the fact that the simple drawing-and-replacing procedure is repeated infinitely many times (at every integer time step). A classic example illustrating how complicated phenomena can arise from repeating a simple procedure infinitely many times is the Brownian motion, which can be defined through an infinite sequence of coin flips.

There exist two canonical cases from which the behaviour of most Pólya urns can be inferred: (a) the case when the replacement matrix is the identity, which was first studied by Markov in 1906, and (b) the case when it is "irreducible", for which landmark results were proved by Athreya and Karlin in 1968. These two cases lead to very different outcomes. This can be seen in the simulations of Figure 1: although in both cases, the composition of the urn seems to converge to a limiting value, this value seems to be random in the case of an identity replacement matrix, and deterministic in the irreducible case.

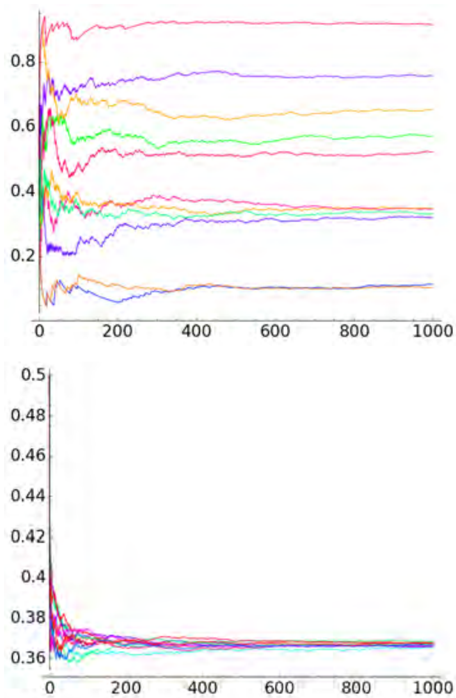


Figure 1. Ten realisations of a 2-colour Pólya urn whose initial composition is one ball of each colour, with identity replacement matrix (top) and with replacement matrix $\binom{2\ 1}{3\ 2}$ (bottom); the vertical axis is the proportion of balls of the first colour in the urn (equal to $1/2$ at time 0), the horizontal axis is time

The reason why Pólya urns have been of interest over the last century is because of their numerous applications. In fact, I often joke that any probability problem can be seen as an urn problem, which is, in essence, what Pólya wrote in 1957:

> "Any problem of probability appears comparable to a suitable problem about bags containing balls and any random mass phenomenon appears as similar in certain essential respects to successive drawings of balls from a system of suitably combined bags."

One of the reasons for the wide applicability Pólya writes about is that Pólya urns are the simplest model for *reinforcement*, i.e. for the rich-get-richer phenomenon. Indeed, imagine a Pólya urn with black and red balls, with replacement matrix the identity (at every time step, we add one ball of the same colour as the drawn ball). If we start with many red balls and few black balls, then we are more likely to add even more red balls to the urn, thus *reinforcing* the higher proportion of red balls.

In this article, we focus on one of these many applications of Pólya urn results: namely, their application to the analysis of the degree distribution of two models of random "trees" that were originally introduced as models for large complex networks (such as the internet, social networks, etc.).

This application to random trees highlights the need for a generalisation of Pólya urns to infinitely many colours (i.e. $d = \infty$). The end of the article focuses on this generalisation, which, in the "irreducible" case, has only been achieved in the last decade, and for which important questions still remain open.

The classic Pólya urn

We start by looking at the classic Pólya urn, i.e. the urn with replacement matrix equal to the identity. We show that, in this case, the composition of the urn converges to a random variable (see the top of Figure 1) whose distribution depends on the initial composition of the urn. We start by looking at the 2-colour case with initial composition one ball of each colour, before looking at (a) different initial compositions, and (b) the d-colour generalisation.

The 2-colour case starting with one ball of each colour can be studied by explicit calculations: let $U_1(n)$ denote the number of red balls in the urn at time n. If $U_1(n) = k$ $(1 \le k \le n+1)$, then we have drawn $k-1$ times a red ball, and $n-k+1$ times a black ball from the urn. The probability that the first $k-1$ balls drawn from the urn are red and the next $n-k+1$ are black is

$$\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{k-1}{2+k}\cdot\frac{1}{3+k}\cdot\frac{2}{4+k}\cdots\frac{n-k+1}{n+1} = \frac{(k-1)!(n-k+1)!}{(n+1)!}.$$

The same formula gives the probability of drawing the same number of red and black balls in any other order. Since there are $\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$ different orders, we get

$$\mathbb{P}(U_1(n) = k) = \frac{1}{n+1}.$$

In other words, for all integer times n, $U_1(n)$ is uniformly distributed among all its possible values $\{1,\dots,n+1\}$, and thus, $U_1(n)/n$ converges in distribution to a uniform random variable on $[0,1]$ when n tends to infinity; this can be seen in the simulations of Figure 2.

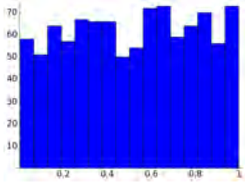


Figure 2. Histogram of the proportion of red balls at time 500 in 1000 realisations of a classic 2-colour Pólya urn

In fact, using "martingale theory", one can prove that the proportion of red balls converges *almost surely* to this uniform limit, which is a stronger statement than convergence in distribution. Almost-sure convergence means that the probability of convergence of the proportion of red balls to a random uniformly-distributed value is equal to one, while convergence in distribution means that the distribution of the proportion of red balls converges to the uniform distribution. The almost-sure convergence of $U_1(n)/n$ is the reason why, on the top of Figure 1, each of the trajectories converges.

We now look at different initial compositions: let $U_1(0)$ and $U_2(0)$ be the number of red and black balls in the urn at time 0. A similar calculation to that above (using Stirling's formula) implies that the proportion of red balls converges to a Beta distribution of parameter $(U_1(0), U_2(0))$. (A Beta distribution is a two-parameter distribution on $[0,1]$: Figure 3 shows how its density depends on the values of the parameters. The Beta distribution of parameter $(1,1)$ is the uniform distribution on $[0,1]$.) Consequently, the long-term behaviour of the urn depends on the initial composition of the urn.

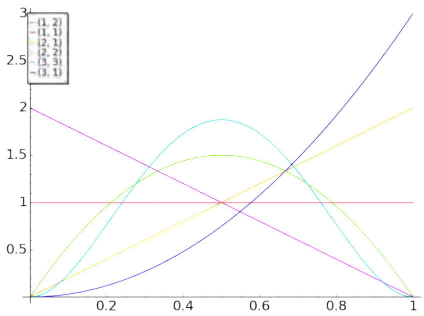


Figure 3. Different Beta distribution densities

Similar results hold when there are $d > 2$ colours in the urn. To study this case, we define $Z(n)$ as the vector whose d coordinates are the proportions of balls of each colour in the urn at time n, i.e.

$$Z(n) := \Big(\frac{U_1(n)}{n+d}, \dots, \frac{U_d(n)}{n+d}\Big), \tag{1}$$

where $U_i(n)$ is the number of balls of colour i in the urn at time n (note that $n+d$ is the total number of balls in the urn at time n since we start with d balls and add one ball at each time step). Using similar methods as in the 2-colour case, one can show that $Z(n)$ converges almost surely (i.e. with probability one) to a limiting value Z_∞, and this limiting value is uniformly distributed on the simplex, i.e. on the set

$$\Big\{(x_1, \dots, x_d) \in [0,1]^d : \sum_{i=1}^{d} x_i = 1\Big\},$$

where the vector $Z(n)$ lives.

The uniform distribution on the simplex is called a Dirichlet distribution of parameter $(1, \dots, 1)$. If we change the initial composition of the urn, then the limiting distribution is a Dirichlet distribution of parameter the initial composition vector:

Theorem 1. *Let $U_i(n)$ be the number of balls of colour i in the urn at time n, in the case when the replacement matrix equals the identity. Almost surely,*

$$Z(n) \to \mathrm{Dirichlet}(U_1(0), \dots, U_d(0)), \text{ as } n \to \infty.$$

The irreducible case

In this section, we look at the case of an irreducible replacement matrix. In contrast to the identity case, the composition of the urn in the irreducible case has a deterministic limit (see the bottom of Figure 1), which does not depend on the initial composition.

A $d \times d$ matrix R is irreducible if for all $1 \leq i, j \leq d$, there exists n such that $(R^n)_{i,j} \neq 0$; in particular, a matrix with all entries positive is irreducible. In our urn context, we also assume that all coefficients of the replacement matrix are non-negative (allowing -1s on the diagonal, meaning that we discard the ball that was drawn, is a straightforward generalisation).

To state the main result of this section, we need a linear algebra theorem, due to Perron and Frobenius:

Theorem 2. *If R is an irreducible matrix with non-negative coefficients, then the spectral radius Λ of R is also a simple eigenvalue of R. (That is, Λ is a real eigenvalue of R, has multiplicity one, and all other eigenvalues of R have absolute value less than Λ.)*

Moreover, there is a unit left-eigenvector v associated to eigenvalue Λ whose coefficients are all non-negative.

Sampling a classic Pólya urn via biased coin flips

A classic exercise in probability theory (see Williams's book Probability with Martingales, Exercise E10.8) is to prove the following result: Start with an urn containing one red and one black ball, sample a uniform random variable Θ on $[0,1]$, and then, at every time step, add to the urn either a red ball, with probability Θ, or a black ball, with probability $1-\Theta$.

Lemma 1. *If $\hat{U}_1(n)$ is the number of red balls in this urn at time n, then the process $(\hat{U}_1(n))_{n\geq 0}$ has the same distribution as $(U_1(n))_{n\geq 0}$ in the classic Pólya urn case with initial composition $(1,1)$.*

Conditionally on the random variable Θ, $\hat{U}_1(n)/n \to \Theta$ almost surely when n tends to infinity. Therefore, we can rephrase Lemma 1 as follows: one way to sample a Pólya urn with identity replacement matrix and initial composition one ball of each colour is to first sample its uniform limit Θ, and then perform coin flips with bias Θ to decide what ball to add to the urn at every time step.

To prove this result, you need basic notions of conditional expectation and martingale theory; if you have this background, I strongly encourage you to try!

As in Equation (1), we let $Z(n)$ denote the vector whose coordinates are the proportions of balls of each colour in the urn at time n, i.e.,

$$Z(n) = \Big(\frac{U_1(n)}{\|U(n)\|_1}, \dots, \frac{U_d(n)}{\|U(n)\|_1}\Big),$$

where $\|U(n)\|_1 = U_1(n) + \cdots + U_d(n)$ is the total number of balls in the urn at time n. The following result is due to Athreya and Karlin in 1968:

Theorem 3. *If the replacement matrix R is irreducible, then, for all non-empty initial compositions, $Z(n) \to v$ almost surely when n tends to infinity, where v is the unit left-eigenvector with non-negative coordinates associated to the spectral radius of R.*

Note that this behaviour is drastically different from the behaviour of the classic Pólya urn (see Theorem 1) in the following two ways: (a) the limit in the irreducible case is deterministic and not random as in the identity case, and (b) the limit in the irreducible case does not depend on the initial composition of the urn, while it does in the identity case. I personally find point (b) surprising: say you have an urn with red and black balls, and replacement matrix $\left(\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right)$, the renormalised composition vector converges to $v = \left(1/2, \sqrt{3}/2\right)$ (see the right-hand side of Figure 1: the proportion of red balls indeed converges to $1/(\sqrt{3}+1) \approx 0.37$) whether we started with one black and one red ball or with a million red balls and one black ball!

Fluctuations

From Theorems 1 and 3, we know that the renormalised composition vector converges almost surely to a limit (both in the identity and the irreducible cases). It is natural to ask about its speed of convergence to this limit. Theorems about these "fluctuations" can be proved using martingale theory.

Interestingly, in the irreducible case the outcome depends on the spectral gap of the matrix R, i.e. the ratio σ between the largest of all other real parts of the eigenvalues of R and its spectral radius. If $\sigma < 1/2$, then the fluctuations of $U(n)$ around nv are Gaussian and of order $n^{1/2}$, while if $\sigma > 1/2$, they are non-Gaussian and of order n^{σ} (see, e.g, [3]).

Application to the random recursive tree

As mentioned in the introduction, results on Pólya urns can be applied to the study of more complicated stochastic processes, and among them, several models for complex networks. In this section, we apply Theorem 3 to prove convergence of the "degree distribution" of a model called the random recursive tree (RRT).

Embedding into continuous time

To prove Theorem 3, Athreya and Karlin embed the urn process into continuous time: a technique that is now standard for the analysis of Pólya urns. In particular, Janson uses it in [3] to generalise Athreya and Karlin's result to a much wider class of Pólya urns: he allows balls of different colours to have different weights, the replacement matrix to be random, and relaxes the irreducibility assumption.

The idea is the following. Imagine that each ball in the urn is equipped with a clock that rings after a random time of exponential distribution with parameter 1, independently from all other clocks. When a clock rings, the corresponding ball, if it is of colour i, splits into $R_{i,j} + \mathbf{1}_{i=j}$ balls of colour j (for all $1 \le j \le d$).

Now imagine that every time a clock rings, we take a picture of the urn: using the standard properties of the exponential distribution, one can check that the ordered sequence of these pictures (which is now a discrete time process) is distributed like the Pólya urn of replacement matrix R.

The advantage of the continuous time process is that each ball is now independent from the other balls; the drawback is that we added some randomness by making the split times random. The continuous time process we get by embedding a Pólya urn into continuous time is called a multi-type Galton–Watson process.

The RRT is defined recursively as follows. At time zero, the tree is two nodes linked by one edge, and at every time step, we add a node to the tree and create an edge between this new node and a node chosen uniformly at random among the nodes that are already in the tree.

This model was introduced as a model for networks by Na and Rapaport in 1970. Two typical features of real-life complex networks are the “small-world” and the “scale-free” properties (see [2] where typical properties of complex networks are discussed). It is thus natural to ask whether the RRT has these two properties.

A graph has the small-world property if the distance between two nodes chosen uniformly at random among all n nodes of the graph is of order at most $\log n$ when n tends to infinity; the RRT has the small-world property (see, e.g., Dobrow 1996).

A graph is called scale-free if its degree distribution is a power-law, i.e. if there exists $\tau > 0$ such that the proportion of nodes of degree i in the graph is of order $i^{-\tau}$ when $i \to \infty$. The degree of a node is the number of other nodes it is linked to by edges (e.g. in a friendship network, it is the number of friends of a node). Most real-life networks are scale-free with τ usually between 2 and 3 (e.g. for the internet, it is estimated that $\tau \approx 2.5$).

In the rest of this section, we prove that the RRT is *not* scale-free; to do so, we use Theorem 3, as originally done by Mahmoud and Smythe in 1992. For all $i \geq 1$, we let $U_i(n)$ denote the number of nodes of degree i in the n-node RRT.

The idea is to view the process $U(n) = (U_1(n), U_2(n), \dots)$ as a Pólya urn: the nodes of the tree are the balls in the urn, and their colour is their degree. At time zero, the urn contains two nodes of colour 1 (because the tree contains two nodes of degree 1), and at every time step, we pick a ball in the urn, say of colour i, remove it from the urn and add instead a ball of colour 1 and a ball of colour $i+1$. In other words, the process $U(n)$ is a Pólya urn of initial composition (2, 0, 0, …) and replacement matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots \\ 1 & -1 & 1 & 0 & 0 & \cdots & \cdots \\ 1 & 0 & -1 & 1 & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

At this point, we could be worried for two reasons: (a) the matrix has negative coefficients and (b) it is infinite, i.e. there are infinitely many colours! As explained above, the -1s on the diagonal are not a problem and Theorem 3 still holds if the matrix is irreducible. But (b) is more worrying since Theorem 3 only holds for urns with finitely-many colours.

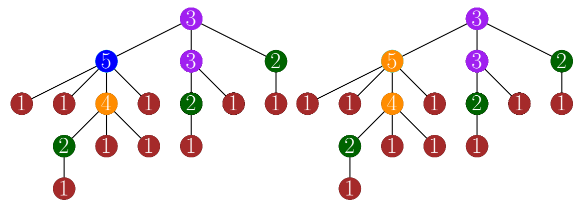


Figure 4. The degree distribution in the RRT seen as an urn, and its truncated version at $m = 4$ (in the truncated version, the node of degree 5 is considered as being of the same colour as the node of degree 4)

Luckily, there exists a trick to reduce to a finite number of colours: we decide to consider all colours above a threshold $m \geq 2$ as one colour. Remarkably, the m-colour process is also a Pólya urn process (this is not true for all infinitely-many-colour urns). For example, take $m = 4$ (see Figure 4), and look at the vector $(U_1(n), U_2(n), U_3(n), \sum_{i\geq 4} U_i(n))$: it is a Pólya urn with replacement matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is irreducible (its spectral radius is 1 and left-eigenvector $v = (1/2, 1/4, 1/8, 1/8)$); thus Theorem 3 applies and gives that, almost surely when n tends to infinity,

$$\frac{1}{n}(U_1(n), U_2(n), U_3(n)) \to \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right).$$

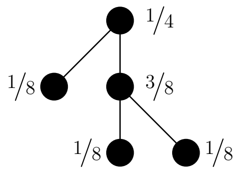
Here we chose $m = 4$ as a threshold, but one can choose an arbitrarily large threshold m, which gives the following result:

Theorem 4. *Let $U_i(n)$ be the number of nodes of degree i in the n-node RRT. For all $i \geq 1$, almost surely when n tends to infinity, $U_i(n)/n \to 2^{-i}$. Consequently, the RRT is not scale-free.*

Application to the preferential attachment tree

Although the RRT does not have the scale-free property (and is therefore not a realistic model for complex networks), the idea of Na and Rapaport to define a dynamical structure (i.e. nodes arriving one by one in the tree) led to the definition of more realistic models for networks. One of these models is the preferential attachment tree (PAT), which was originally defined by Yule in 1923, and popularised by Barabási and Albert in 1999. In this section, we show how one can prove that the PAT is scale-free (see Figure 5 where the typical shapes of the RRT and of the PAT are compared).

The PAT is defined as follows. Start with two nodes linked by an edge, and at every time step, add a node to the tree, and link it to a node chosen at random among existing nodes with probability proportional to their degrees: this is the "preferential attachment". For example, on the picture, the next node to enter the tree will connect with one of the 5 existing nodes with the probabilities displayed.



Theorem 5. *Let $U_i(n)$ denote the number of nodes of degree $i \geq 1$ in the n-node PAT. Almost surely when n tends to infinity,*

$$\frac{U_i(n)}{n} \to \frac{2}{(i+1)(i+2)},$$

and thus the PAT is scale-free with index $\tau = 2$.

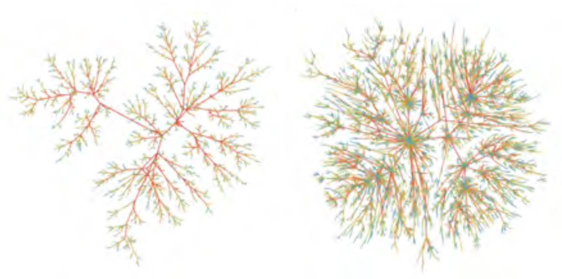


Figure 5. A realisation of the RRT (left) and the PAT (right) at large time (courtesy of Igor Kortchemski); the scale-free property of the PAT can be "seen" on this simulation: heuristically speaking, the PAT has large hubs while degrees in the RRT are much more homogeneous

This result was proved by Mahmoud, Smythe and Szymański in 1993; they use similar methods as for the RRT. The only additional subtlety is that the vector $(U_1(n), U_2(n), \ldots)$ is not a Pólya urn, one has to consider the vector $(U_1(n), 2U_2(n), 3U_3(n), \ldots)$ instead. As in the RRT case, one can truncate the number of colours to finitely-many, and apply Theorem 3 to conclude the proof of Theorem 5.

Infinitely-many colours

The Pólya urns arising from the application to the RRT and the PAT both have infinitely many colours. In these two examples, we considered all colours above a threshold to be the same, and used the remarkable fact that the finitely-many-colour urn obtained after this reduction was still a Pólya urn. However, this property does not hold for all infinitely-many-colour Pólya urns and consequently, as Janson wrote in [3],

> "These examples suggest the possibility of (and desire for) an extension of the results in this paper to infinite sets of types."

For the case when the replacement matrix equals the identity, the generalisation to infinitely many colours dates back to Blackwell and McQueen in 1973. Its equivalent for the irreducible case, however, only dates back to 2017 (with a preliminary particular case in a 2013 paper by Bandyopadhyay and Thacker) and is still the object of some open problems.

The main idea is to look at the composition of the urn, not as a vector as in the finitely-many colour case, but as a measure on a set $\mathscr{C}$ of colours. As in the finitely-many-colour case, we define a discrete time process that depends on two parameters: the initial composition, which is now a finite measure m_0 on $\mathscr{C}$, and the replacement kernel $(K_x)_{x\in\mathscr{C}}$, which is a set (indexed by $\mathscr{C}$) of finite measures on $\mathscr{C}$.

For all integers $n \geq 0$, we set

$$m_{n+1} = m_n + K_{\xi(n+1)},$$

where $\xi(n+1)$ is a $\mathscr{C}$-valued random variable of distribution $m_n/m_n(\mathscr{C})$ (the normalisation is so that the total mass of $m_n/m_n(\mathscr{C})$ equals 1). The random variable $\xi(n+1)$ can be interpreted as the colour of the ball drawn in the urn at time $n+1$.

The finitely-many-colour case fits in this framework: it corresponds to $\mathscr{C} = \{1, \ldots, d\}$, $m_0 = \sum_{i=1}^d U_i(0)\delta_i$ and, for all $1 \leq i \leq d$, $K_i = \sum_{j=1}^d R_{i,j}\delta_j$, where δ_x is the Dirac mass at x.

However, this new model of "measure-valued Pólya processes" is much more general than the finitely-many colour case: the set of colours can be any measurable set. From our applications to random trees, it is natural to consider $\mathscr{C}$ to be $\mathbb{N}$ or $\mathbb{Z}$, but we can take $\mathscr{C}$ to be $\mathbb{R}$ or any other measurable set.

In these continuous cases, for all Borel sets $B \subseteq \mathscr{C}$, we interpret $m_n(B)$ as the mass of all balls in the urn at time n whose colour belongs to B, and $K_x(B)$ as the mass of balls with colour in B that we add in the urn when drawing a ball of colour x.

Note that $m_n(B)$ and $K_x(B)$ do not have to be integer valued, and in fact m_n and K_x can be absolutely continuous measures, in which case we should think of coloured dust in an urn instead of coloured balls (see Figure 6).

Figure 6. Composition measure of an urn with colour set $\mathbb{R}$

We have now defined a model that allows for infinitely many colours, but can we prove anything about it? Yes: Blackwell and McQueen in 1973 proved an analogue to Theorem 1 for the identity case ($K_x = \delta_x$ for all $x \in \mathscr{C}$). In the appropriate analogue of the irreducible case for infinitely many colours, Bandyopadhyay and Thacker, and Mailler and Marckert, simultaneously in 2017, proved an equivalent of Theorem 3.

More recent results on the infinitely-many-colour irreducible case induce a stronger statement than Theorem 4 for the degree distribution of the RRT (see Mailler and Villemonais 2020):

Theorem 6. *Let $U_i(n)$ be the number of nodes of degree i in the n-node RRT. For all $\varepsilon > 0$, for all functions $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = o((2-\varepsilon)^n)$ when $n \to \infty$, we have, almost surely when $n \to \infty$,*

$$\frac{1}{n}\sum_{i\geq 1} f(i)U_i(n) \to \sum_{i\geq 1} f(i)2^{-i}.$$

The difficulty of the infinitely-many-colour case comes from two main factors: (a) the embedding into continuous-time method, which was very successful in the finitely-many colour case, seems not to be applicable to the infinitely-many-colour case, and (b) all the proofs that rely on linear algebra in the finitely-many colour case need to be adapted to the infinite-dimensional setting using operator theory. For example, a question that remains open at the time of writing this article, and on which I am currently working, is to prove theorems for the fluctuations of these processes around their limits.

Summary

In this article, we focused on the convergence of the composition of Pólya urns in the case when the replacement matrix is the identity or when it is irreducible. We then looked at applications of these results to the analysis of random tree models for complex networks. Motivated by these applications, we looked at the definition a new framework for Pólya urns with infinitely many colours, a topic which is the object of ongoing research.

In the introduction, we stated a quote of Pólya from 1957 highlighting the wide applicability of Pólya urns and this wide applicability continues today. Indeed, in parallel universes, articles with the same title as this one could have told stories ending with applications of Pólya urns to adaptive clinical trials (e.g. Laruelle and Pagès 2013, Zhang 2016), or tissue growth modelling (Borovkov 2019), or Monte-Carlo approximation methods (Wang, Roberts and Steinsaltz 2018, Mailler and Villemonais 2020), or reinforcement learning (ants finding shortest paths in networks: Kious, Mailler and Schapira 2020), all of which are still active areas of research at the time of writing.

FURTHER READING

[1] K.B. Athreya, P.E. Ney. Branching Processes (Chapter V). Springer Berlin, 1972.
[2] R. van der Hofstad. Random Graphs and Complex Networks. Cambridge Series in Statistical and Probabilistic Mathematics, 2017.
[3] S. Janson. Functional limit theorems for multitype branching processes and generalized Pólya urns. Stoch. Proc. App. 110(2) (2004) 177–245.
[4] N.L. Johnson, S. Kotz. Urn Models and their Application; an Approach to Modern Discrete Probability Theory. Wiley, 1977.

Cécile Mailler

Cécile is a lecturer in probability and an EPSRC postdoctoral fellow at the University of Bath. Her main research interests are on branching processes and reinforcement. She particularly enjoys working on problems that arise from applications, and even more so if the model is named after an animal (she has so far managed to work on the “monkey walk” and the “ant walk”!).

The Royal Institution Christmas Lecturer 2019

Mathematician, broadcaster, writer, researcher, Christopher Zeeman Medallist 2018 and Royal Institution Christmas Lecturer 2019 Hannah Fry is interviewed by Eleanor Lingham.

When you meet someone new, how do you describe your work?
I haven't worked out how to do it yet — I have about five jobs — but mathematician and broadcaster just about sums it up.

What is your maths research area?
In layman's terms, it's patterns in human behaviour. In more academic terms, it's complex social systems. Essentially, I'm interested in patterns that change in space and time. In a lot of ways it looks like the areas that I've published in seem really disconnected from each other. I've done work on infectious diseases, terrorism and shopping, but the techniques behind them are the same. You are studying a pattern that is embedded in space, that changes over time.

You're based at the Centre for Advanced Spatial Analysis (CASA) at University College London — can you tell me more about this?
CASA is a totally interdisciplinary department. It specialises in the application and visualisation of spatial analytic techniques and simulation models to urban areas. There are a few of these institutions that have sprung up on the discipline boundaries. If you think about it, the disciplines that were created in the 1800s are not necessarily that relevant anymore. When you use urban space as an umbrella, it includes all kinds of things that people are studying: crime, transport, architecture etc. At CASA people come from a range of backgrounds: geography, physics, mathematics, architecture and even graphic design.

It must be a very exciting place to work...
I really like it. You end up having access to a much wider range of expertise than I have found in other departments. The number of ideas that you're exposed to is much greater. It is a collaborative open environment where everyone simultaneously has their own area of expertise. You don't have distinct silos in the same way as at other institutions.

When you were a child, what did you want to be when you grew up?
I wanted to be in Formula One. Not as a driver — I always liked the more scientific side of things — the engineering side. There is a bit of me that is still very jealous of people who are proper engineers. For example, those UCL engineers who developed a prototype to remove the need for ventilators. I like to think creatively around a problem to solve it, which is what we do as mathematicians.

Hannah Fry

When you were growing up, was there someone who influenced what you chose to study at university?
I was really lucky in that I had the same maths teacher the whole way through secondary school, just by chance. She was called Mrs Andrews and she was the

sort of teacher who just let you make mistakes without ever feeling like a failure. She set up an environment that meant the struggle of looking at a new page of maths, and not knowing how to do it, didn't feel like an insurmountable challenge. I think that's a key thing, as at no point did I feel like I was done with maths, because I was still enjoying it. At every stage, I had this feeling that I was not finished with it yet.

The Michael Faraday Royal Institution Christmas Lecture

How did the move into broadcasting come about it?

It all started during my PhD with a colleague of mine who was doing public outreach through comedy. He convinced me to sign up to *Bright Club*, where academics do stand-up comedy about their work. So I did this comedy night and it went really well. Actually, it was quite terrible, but the standard was so low – you know what academics are like – that relatively speaking it went quite well. There was someone in the audience who asked me to give another talk, and so on – until eventually I was asked to do a talk at the Royal Institution, and then a Radio 4 show which led to a YouTube talk. This led to my TED talk on *The Mathematics of Love*. It wasn't like I was sitting in my office one day and wrote an email to someone and asked them to put me on telly. It was years and years of giving talks and learning my craft that got me there.

Last year you were asked to do the 2019 Royal Institution Christmas Lectures. How did it feel to get that invitation?

The truth is that I was a bit thrown by it because I had always wanted to do them — but when they asked me I was about 12 weeks pregnant. It was fine — I was going to have my baby in May, and the talks are in December, but this was my second baby, and I had worked so hard on my book in my first maternity leave, that I really wanted be able to properly sit back and enjoy it this time round. Ideally I wanted to put the Lectures off a few months, but obviously you can't. It was an amazing experience. Seeing your name on that list is quite a big thing. I'm only the fourth mathematician, and there are only a handful of women — and I think I was the youngest ever Christmas Lecturer — I feel very proud of myself about that.

Are you going to make your children watch them in the future?

Yes — I'll tie them to a chair.

If you could have proved any one maths result what would it be?

I like the ones that are easy to understand but difficult to prove, like the Sphere-packing Problem or the Moving Sofa Problem. Fermat's Last Theorem comes under this too.

That would have been mine, but then again I am Sir Andrew Wiles' number one fan...

Yes well get in the queue! I got to interview him once for *Oxford Mathematics*. He is such a kind man.

Hannah Fry interviewing Sir Andrew Wiles

If you could have met any mathematician from history, who would you like to have had a chat with?

Sophie Germain. It's a tragedy that her name isn't inscribed on the Eiffel Tower with the names of other famous French mathematicians and scientists. Her work on tensors and stresses was fundamental to the construction of the tower and yet her name was not included because she was a woman. I find women through history fascinating — not just because they are women — but because you have a person who wants to be something, and innately is something, but society has put a wall up to stop them — and what they have to do is find a crack and squeeze a way through.

Do you think that any walls remain in mathematics today?
I think they are certainly more porous now.

Do you find that female mathematicians or female broadcasters in scientific areas have different expectations on them than male?
Yes I think so. There are two things that male broadcasters/scientists can do that females can't. First, they can be critical of people. For example, my Radio 4 co-host Adam Rutherford who has recently written the book *How to Argue with a Racist*, can go on Twitter and be very outspoken, and people accept it. I'm not saying he doesn't get any nonsense on Twitter, but when he does get criticism, people are coming from the basis of accepting his expertise. Whereas if I ever say anything on Twitter, the criticisms that I get are essentially undermining my credibility — they are never about the content of what I'm saying. Second, men can make mistakes without having their credibility questioned. For example, I did a 4,000 word article for the *New Yorker* on the p-hacking issue, and there was a comma missing. But as a result of that typing error, I got email after email and lots of tweets from other professional mathematicians and statisticians that were undermining my credibility rather than just saying 'there's a comma missing'. I think that men are allowed to make mistakes, and are allowed to be outspoken against other people or other ideas, without having their credibility questioned.

What advice would you give mathematicians who are thinking of joining social media?
My experience on social media has been overwhelming positive. The number of connections I have been able to forge, and the number of projects I have been able to join, is huge. Twitter is good for academics as a way to network — especially nowadays with everything that's happening in the world. For example, Adam Kucharski of London School of Tropical Medicine's Twitter following grew hugely, as people wanted to hear an expert voice on infectious diseases. As a result, he is making a genuine difference, and the public is more informed because of his tweets.

The Eiffel Tower under construction

We have a lot of PhD students who read the *Newsletter*. Given that you finished your PhD in 2011, do you have any advice for someone who is finishing their thesis?
You poor things! PhDs are horrible and the mistake I made, which I see a lot of my PhD students making — and I try and stop them doing it — is that it's easy to fall into the trap of thinking that this is your magnus opus — that this document needs to be perfect. You do not have to do that because it's just a stepping stone towards the next stage. I spent way too long finishing it, when I should have just submitted it. I ended up with no corrections — which sounds amazing — but actually just means I spent four months too long on it.

The African Institute for Mathematical Sciences

The African Institute for Mathematical Sciences (AIMS) was the brainchild of Neil Turok, a South African physicist. Born to parents who were activists in the anti-apartheid movement, Neil nursed an ambition of raising standards and consciousness across Africa. AIMS South Africa came to life in Muizenberg in 2003, a manifestation of Neil's dream of creating, in his own words, "a pan-African centre of excellence for training and research which would show the world that the stereotypes about Africa are wrong".

Neil Turok

With a vision to lead the transformation of Africa through innovative scientific training, technical advances and breakthrough discoveries which benefit the whole of society, AIMS, with the support of its partners, launched the Next Einstein Initiative, establishing centres in Senegal (2011), Ghana (2012), Cameroon (2013), Tanzania (2014) and Rwanda (2016).

An ecosystem of transformation and innovation comprising Centres of Excellence, Research Centres, the AIMS Industry Initiative and gender-responsive Teacher Training, AIMS equally created two critical initiatives: Quantum Leap Africa, a think tank looking into the coming quantum revolution, and the Next Einstein Forum to propel Africa on to the global scientific stage.

AIMS Centres of Excellence are feeding the STEM pipeline by training African students to tackle the continent's challenges via scientific innovation in big data, cybersecurity, financial mathematics, climate science and related fields. With lecturers from top universities and research labs from around the world, AIMS centres offer a fully-funded Master's in mathematical sciences, including a co-operative option with a direct link to industry, a Master's in machine intelligence and various short training programs. Each centre offers annual fellowships to teaching assistants, which permits postdoc and doctorate researchers to work as dedicated tutors, providing capacity building for the students. The institute's unique residential character allows for optimum interaction between students, lecturers, and tutors in a harmonious environment at all hours.

Africa's youth are at the heart of the AIMS ecosystem, with an alumni community now totalling over 2200, from 43 countries. 32% of our alumni are women and 70% remain on the African continent. To date, 302 graduates have obtained their PhDs, with 372 currently pursuing a PhD. The three main areas of specialisation chosen by these doctoral students are applied mathematics, physics and theoretical pure mathematics. Meanwhile, around 862 are pursuing careers in higher education, research, ICT, financial services, trade, engineering, statistics, energy and data management, among others.

Angela Tabiri, AIMS Ghana & University of Glasgow graduate, Founder of FEMAfricMaths, and Martial L. Ndeffo Mbah, AIMS South Africa & University of Cambridge graduate

With over 600 papers published by our researchers in leading journals, AIMS' output is one of the highest per-capita of any institution in Africa.

In response to the covid-19 pandemic, researchers from the University of Rwanda, the country's joint covid-19 taskforce, AIMS, and the Perimeter Institute for Theoretical Physics in Canada have laid out an efficient and cost-effective method to optimize testing in Rwanda. This approach to pooled testing was developed by Wilfred Ndifon, a mathematical epidemiologist and Chief Scientific Officer at AIMS.

Wilfred Ndifon

AIMS' commitment to scientific excellence goes hand in hand with its commitment to promoting gender equity and inclusion. AIMS applies a hard quota of at least 30% in recruiting top female students, to level the playing field for women. The AIMS Women in STEM Initiative is accelerating progress for African women in STEM through advocacy, evidence-based reporting, and public engagement. In 2018, AIMS announced the first cohort of the Women in Climate Change Science Fellowship. With three fellows awarded up to USD 30,000 each over one year, the funding included support for up to three dependents. The program will support up to 20 outstanding women through to 2022 working on the impact of climate patterns on humanity. AIMS students and staff benefit from regular gender, inclusion and diversity training, mentoring and access to strong women role models to encourage paradigm shifts. AIMS believes female researchers must equally be in the driver's seat to contribute to a more sustainable approach to Africa's and the world's challenges.

For more information visit: nexteinstein.org.

Dr Layih Butake
Senior Outreach Manager and Acting Director of Communications
AIMS Global Network

2021 HEILBRONN FOCUSED RESEARCH GRANTS

Heilbronn Institute for Mathematical Research

Call for proposals

The Heilbronn Institute for Mathematical Research is offering a number of grants of up to £7.5K to fund focused research groups to work on adventurous and challenging mathematical problems, or to discuss important new developments in mathematics.

These grants will support travel and local expenses for groups to come together to focus intensively on a problem or to discuss a significant new development in mathematics. We expect these groups to be normally 8 or fewer people. Groups are encouraged to include international participants, but should also involve a substantial UK-based component.

Open to all mathematicians and to any department in the UK.

Proposals from these areas of research, interpreted broadly, will be given priority: Pure Mathematics, Probability and Statistics, and Quantum Information.

Proposals of no more than one page of A4 should be sent by **9am, Thursday 14th January 2021** to: heilbronn-manager@bristol.ac.uk. For further particulars and additional information, please visit our website: http://heilbronn.ac.uk/opportunities/

Microtheses provide space for current and recent research students to communicate their findings with the community. We welcome submissions (see newsletter.lms.ac.uk for guidance). Authors are rewarded with LMS associate membership for one year.

Microthesis: Single-Cylinder Square-Tiled Surfaces

LUKE JEFFREYS

Square-tiled surfaces arise naturally in a variety of settings in low-dimensional topology. Cylinders in square-tiled surfaces decompose the surface into simpler pieces. Here, we consider the construction of square-tiled surfaces having simultaneously a single cylinder in the horizontal direction and a single cylinder in the vertical direction.

Square-tiled surfaces

A surface is an object that is intrinsically two-dimensional. Examples are given by a sphere (the surface of a ball) and a torus (the surface of a ring doughnut). In fact, all surfaces of this type (closed and orientable) are classified by the number of 'doughnut holes'. We call this number g the *genus* of the surface. As such, the sphere is a surface of genus zero and the torus is a surface of genus one. Surfaces of genus zero, one, and two are shown in Figure 1.

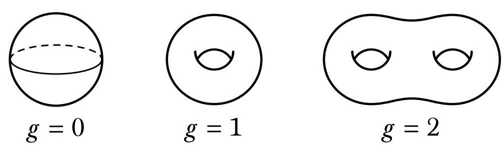


Figure 1. Surfaces of genus zero, one and two

A torus can be obtained from a square by gluing the top and bottom sides of the square to form a cylinder and then gluing the ends of the cylinder to form a torus. Alternatively, this process can be viewed as taking the unit square in $\mathbb{C}$, shown in Figure 2, and then identifying the bottom of the square with the top under the map $z \mapsto z + i$ and the left side with the right under the map $z \mapsto z + 1$.

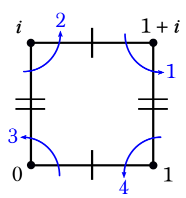


Figure 2. A square torus

The resulting surface inherits the geometry of the plane. Indeed, it can be checked that the four vertices of the square are identified to form a single point on the torus and that, by following the blue arrows around these vertices, there is an angle 2π around the resulting point on the surface. Hence, there is an angle 2π around every point on the surface.

Generalising the construction of the square torus, a *square-tiled surface* is a surface realised by identifying by translation ($z \mapsto z + c$, $c \in \mathbb{C}$) the sides of a collection of unit squares in $\mathbb{C}$. See for example the surfaces shown in Figure 3. Sides with the same label are identified by translation. Visualising the construction is more challenging here, but it can be shown that both of the surfaces in Figure 3 have genus two.

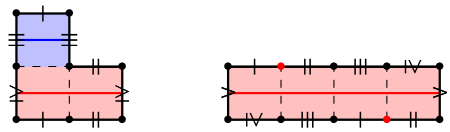


Figure 3. Two square-tiled surfaces of genus two

Square-tiled surfaces are a special case of a more general object called a *translation surface*. These are surfaces that can be realised by identifying, again by translation, the sides of a collection of polygons, not necessarily just squares, in $\mathbb{C}$. Such surfaces arise in the study of a number of dynamical systems including billiards in rational polygons, electron transport in Fermi surfaces, and certain self-maps of the interval.

Similar to the case of the torus in Figure 2, the black vertices of the polygons in Figure 3 are identified with each other to form a single point on each surface. Here, however, the angle around these points is 6π. In fact, for $g \geq 2$, translation surfaces inherit the geometry of the plane except at a finite number of isolated singularities called *cone-points*. The angle around these cone-points, called the *cone-angle*, will be greater than 2π, and is always an integer multiple of 2π. We say that a cone-point has order $k \geq 1$ if it has cone-angle $2\pi(k+1)$. By the Gauss-Bonnet theorem, if a translation surface has cone-points of orders $k_1, \ldots, k_n$, then we must have $\sum_{i=1}^{n} k_i = 2g-2$. From the above discussion, the surfaces in Figure 3 have a cone-point with cone-angle $6\pi = 2\pi(2+1)$, and hence order 2. Since $2g-2=2$ in this case, these are the only cone-points on these surfaces.

For $g \geq 2$, we define the moduli space $\mathcal{H}_g$ to be the set of translation surfaces of genus g, up to an equivalence relation arising from 'cutting and pasting' the polygons. Given a partition $(k_1, \ldots, k_n)$ of $2g-2$, we define the *stratum* $\mathcal{H}_g(k_1, \ldots, k_n)$ to be the subset of $\mathcal{H}_g$ consisting of those translation surfaces having n distinct cone-points of orders $k_1, \ldots, k_n$. For example, the surfaces in Figure 3 lie in the stratum $\mathcal{H}_2(2)$, since each has a single cone-point of order 2.

A stratum need not be connected and may have up to three connected components. For the strata $\mathcal{H}_g(2g-2)$ and $\mathcal{H}_g(g-1, g-1)$, two connected components of special interest are the *hyperelliptic components*. These components consist entirely of translation surfaces with extra symmetry (they are double-covers of spheres), and are denoted by $\mathcal{H}_g^{hyp}(2g-2)$ and $\mathcal{H}_g^{hyp}(g-1, g-1)$, respectively.

Dynamical questions about strata depend on calculations of their volume, and counting square-tiled surfaces has played a crucial role in these calculations. It can be helpful to consider the combinatorial properties of the square-tiled surfaces being counted. One such property is related to the notion of cylinders.

Cylinders

Consider a curve, an embedded circle, in a square-tiled surface like those shown in dark red and dark blue in Figure 3. An *annular neighbourhood* of a curve can be thought of as a 'fattening up' of the curve. A *cylinder* in a square-tiled surface is a maximal annular neighbourhood — that is, a widest possible 'fattening' — not containing any cone-points in its interior. The surface on the left of Figure 3 has two horizontal cylinders shown in light blue and light red while the surface on the right has a single horizontal cylinder. It can be checked that the surface on the right also has a single vertical cylinder. Cylinders allow us to decompose a surface into topologically simpler pieces.

A square-tiled surface in a stratum $\mathcal{H}_g(k_1, \ldots, k_n)$ can have between 1 and $g+n-1$ cylinders in its horizontal or vertical directions. If a square-tiled surface has simultaneously a single vertical cylinder and a single horizontal cylinder then we shall call it a *1,1-square-tiled surface*. As such, 1,1-square-tiled surfaces have the simplest possible decomposition into horizontal and vertical cylinders. Moreover, any square-tiled surface in this stratum must be built from at least $2g+n-2$ squares, and so we ask:

Question 1. Can a 1,1-square-tiled surface in $\mathcal{H}_g(k_1, \ldots, k_n)$ be built using only $2g+n-2$ squares?

We proved that the answer is 'yes', unless the 1,1-square-tiled surface lies in a hyperelliptic component. It is the extra symmetry of such surfaces that forces the need for additional squares. Specifically, we have:

Theorem 7. *([2, Theorem 1.1]) For a connected component $\mathcal{C} \neq \mathcal{H}_g^{hyp}(2g-2), \mathcal{H}_g^{hyp}(g-1, g-1)$ of a stratum $\mathcal{H}_g(k_1, \ldots, k_n)$, there exist 1,1-square-tiled surfaces in $\mathcal{C}$ with $2g+n-2$ squares. In contrast, 1,1-square-tiled surfaces in $\mathcal{H}_g^{hyp}(2g-2)$ and $\mathcal{H}_g^{hyp}(g-1, g-1)$ require at least $4g-4$ and $4g-2$ squares respectively, and there exist 1,1-square-tiled surfaces in these components that realise these bounds.*

Now available within the `surface_dynamics` SageMath package [1], the 1,1-square-tiled surfaces constructed in Theorem 1 have applications to the study of surface homeomorphisms and Teichmüller space.

FURTHER READING

[1] V. Delecroix, et al., Surface-dynamics 0.4.4, pypi.org/project/surface-dynamics/

[2] L. Jeffreys, Single-cylinder square-tiled surfaces and the ubiquity of ratio-optimising pseudo-Anosovs. Submitted, 2019. arXiv:1906.02016

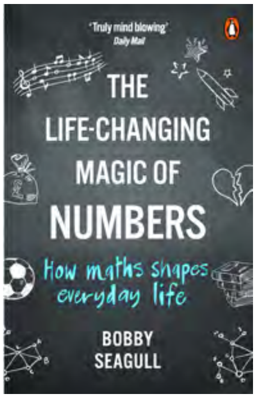
Luke Jeffreys

Luke recently completed a PhD at the University of Glasgow supervised by Vaibhav Gadre and Tara Brendle, and supported by an EPSRC studentship (EPSRC DTG EP/N509668/1 M&S). His main research interests include Teichmüller dynamics and hyperbolic geometry. In his spare time, he enjoys gymnastics, martial arts and music.

The Life-Changing Magic of Numbers

by Bobby Seagull, Penguin, 2019, £9.99, ISBN: 978-0753552803

Review by Francesca Iezzi



During my experience as a Public Engagement professional, I have often clashed with the common perception that Mathematics is irrelevant to our everyday lives — something you would study at school, but never actually use outside specific spheres. I have also been confronted with the opinion that numerical literacy is something reserved only for experts.

In his book *The Life-Changing Magic of numbers*, Bobby Seagull appeals to non-experts, exploring a variety of instances and situations where numeracy can prove useful in managing our everyday lives and in better understanding the world around us: from cooking to fitness, from managing our finances and investments to challenging common assumptions or detecting fraud.

Each of the seventeen chapters is dedicated to a specific topic, in which the important role of Mathematics, and especially Numeracy, is highlighted. The author ends each chapter with a puzzle (solutions are generously provided at the end of the book). In his unique style, Seagull explores, in a very accessible manner, topics such as the Law of Large Numbers (and the consequent "likelihood of the unexpected") or the Black-Scholes equation (and its relevance in the financial world). He highlights the link between Mathematics and the Arts and goes so far as to introduce the Golden Ratio and Escher's work. He explores the connection between Mathematics and the hobby of collecting, presenting an interesting view of how such a simple habit can prompt a child to familiarise themselves with counting and categorising.

The author engages the reader with his colloquial style, his sense of humour and his grasp of the English language. Seagull's use of anecdotes from his personal life, touching upon his childhood, his professional experience as an intern in the finance sector and as a teacher, his hobbies, and various other activities in which he has taken part, lend an intimate feel to the book: I truly felt as if he was speaking to me from across the room. What makes the book so relatable is that he touches on common instances which each of us may have experienced, such as requesting a mortgage, meeting an acquaintance in unexpected circumstances, or making a cake. To give a broader perspective, he also refers to other fields, such as sports, music, fitness, economics, and media. In all of this he provides the readers with interesting general knowledge, and not just the mathematical underpinnings.

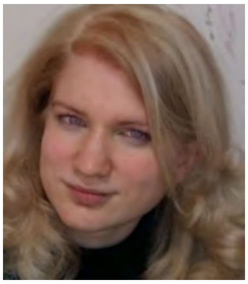
A summary of each chapter is beyond the scope of this review and I would certainly fail to do justice to the author; however, let me share with the reader some of the highlights that struck me.

I found Chapter 4 about the Mathematics behind Magic particularly intriguing and fascinating, as it shows how some of the most amazing "Magic Tricks" are based on nothing more than Mathematics combined with some simple assumptions. As the author claims: "In a world where numbers sometimes feel like dry data, magic — using maths — still has the power to enthral and give us a sense of innocent wonder".

I was equally intrigued by Chapter 11, where the author introduces and demystifies some of the concepts behind Financial Mathematics and gives an account of some of the reasons behind the 2008 financial crisis. The chapter is pervaded by funny anecdotes, as Seagull recalls his experience as a twenty-year-old intern at Lehman Brothers', where, as soon as he sensed the coming crisis, he depleted all vending machines, to make sure the percentage of his salary devoted to buying snacks would not be lost. As the author suggests, "To try to prevent [a crisis] happening again, all of us would benefit from understanding, or at least appreciating, the significance of the underlying mathematics that led to the debacle".

Overall, I found the book very effective in promoting the importance and relevance of mathematics, and in raising awareness that "Mathematics can teach us a lot about understanding our place on a complicated planet, helping to simplify the way in which we can comprehend how the world works".

No mathematical knowledge is assumed, making the book suitable for anyone who is interested in how Mathematics, and in particular Statistics and Numeracy, underpin various aspects of our lives.



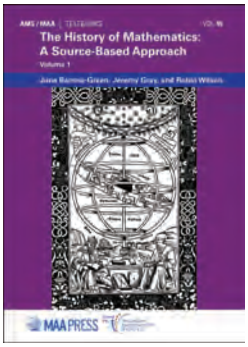
Francesca Iezzi

Francesca Iezzi is the Mathematics Engagement Officer at the University of Edinburgh. Her previous research was in geometric group theory and low dimensional topology. Apart from mathematics, she likes cooking and travelling. As a good Italian, she is a firm advocate of the theory "no pineapple on pizza".

The History of Mathematics: A Source-Based Approach Volume 1

by June Barrow-Green, Jeremy Gray, and Robin Wilson, MAA Press, 2019, £88.50, US$89.00, ISBN: 978-1470443528

Review by Mark McCartney



For too long the teaching of the history of mathematics has been a minority sport within UK universities. There are of course a number of reasons for this. Curricula are already crowded; in some universities final year modules are heavily geared towards the research interests of the department; and in other cases there are even some benighted souls who seem to think that history of mathematics is a 'soft option' that should not been given space on a degree. (To any reader who espouses this latter view I merely give a derisory 'Ha!'.)

The history of mathematics is not only a discipline in its own right, it is also an enormously enriching subject to have as part of any undergraduate mathematics curriculum. To start with the most minor benefit, it forces mathematics students used to writing solely in equations to develop their communication and presentation skills in prose. But that, though useful for producing more rounded graduates, is really just incidental compared to the other benefits. History exposes students to arguments which, like much of life, are not mathematically clear cut, but rather have shades of meaning and ambiguity. It makes it clear that mathematics was not created single-handed by a series of intellectual giants who never made a mistake. It brings out of the shadows the mathematical practice of an enormous cast of people who, though little known, made significant and interesting contributions to mathematics, mathematical communities, and wider society. Some mathematicians lived relatively ordinary lives, some lived lives that were anything but ordinary, but all of them lived in times, places and social contexts filled with colour, interconnection and surprise. Hang it all, the history of mathematics is just so plain *interesting*. To learn about it gives the student a broader and more balanced grasp of the mathematical sciences as a whole. If you don't have a module on the subject on your degree programme then you really need to go and sit in the corner for a while and consider the error of your ways.

Having considered the error of your ways, your next task is to search for an appropriate textbook for your new course. This rather conveniently brings me to the current book under review. June Barrow-Green, Jeremy Gray and Robin Wilson are all British scholars with international reputations and in *The History of Mathematics: A Source-Based Approach* they share with the reader the fruits of three lifetimes of experience of teaching and scholarship in the subject. The book, volume one of a two-volume set, takes the reader from what we know of the beginning of mathematics in Egypt and Mesopotamia up to the early seventeenth century and the very edge of calculus. If volume two is of a comparable size the set will be around 1000 pages.

The current volume is a thorough, very readable, nicely illustrated, and well-balanced coverage of the history of mathematics. It is peppered with footnotes, references to up-to-date scholarship and judicious suggestions for further reading. Finally, as the subtitle indicates, the history is brought to life with numerous excerpts from original sources. This engagement with source texts gives depth and authenticity to the learning process. Original sources and artefacts are the raw material of history. Using them is a wonderful way to teach which enthuses students and challenges them to think more deeply and more critically about the past.

Chapters are relatively self-contained and as such a selection of chapters could be used to form a short course on a more specific topic, say Greek mathematics (Chapters 3–6), or Renaissance mathematics (Chapters 10 and 11).

If volume 2 is of the same calibre as the current volume (and I have no reason to think it won't be!), then taken together Barrow-Green, Gray and Wilson will have produced a *definitive* text on the history of mathematics which will be a standard resource for a generation of students and lecturers.

There is only one downside. At £88.50 this book does not come cheap, and that cost is likely to put off quite a few students, which is a real pity. However, irrespective of whether you are a student or not, I encourage you to think of £88.50 as not so much as a cost, as an investment. If you are interested in the history of mathematics then this book should be on your shelf.

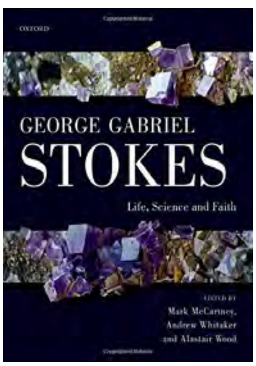
Mark McCartney

Mark McCartney lectures in mathematics at Ulster University. He is also President of the British Society for the History of Mathematics. He, rather inevitably, thinks that you should join this Society immediately.

George Gabriel Stokes: Life, Science and Faith

Edited by Mark McCartney, Andrew Whitaker, and Alastair Wood, Oxford University Press, 2019, £35, ISBN: 978-0198822868

Review by Keith Hannabuss



In this volume marking the bicentenary of George Gabriel Stokes' birth, experts explore various aspects of his life, faith, and pure and applied mathematical, and physical/engineering work. One of several distinguished nineteenth century Irish mathematical physicists, his name is associated with many important concepts and results, such as the Navier–Stokes equation, Stokes phenomenon, Stokes lines, Stokes parameters. (Probably many mathematicians first encounter Stokes' Theorem, to which Stokes' name became attached only because he set it in a Smith's Prize examination question in Cambridge in 1854, having seen it in a letter from William Thomson, later Lord Kelvin. Their lifelong friendship is also described in [1].)

Alastair Wood sets the scene in an introductory chapter on Stokes' life and an overview of his work. Stokes came from a family of academics and Church of Ireland clergymen, and this family background is explored further in a chapter by Michael Sandford. Stokes was the recipient of many honours, including a baronetcy in 1889, and for the last year of his life he was also Master of Pembroke College, Cambridge.

June Barrow-Green describes Stokes' mathematical education, and his move to England for his later schooling at Bristol College, before proceeding to Pembroke College, Cambridge. There he was coached by the legendary William Hopkins, who took on only the highest fliers, and in 1841 Stokes graduated as Senior Wrangler (the examination timetable and two papers appear in an Appendix), and he soon added the Smith's Prize to his honours. Given his undergraduate achievements it is unsurprising that Stokes was then elected to a Fellowship at Pembroke. His scientific research got off to a very swift start, and in 1849 he became Lucasian Professor.

Two chapters on *Optics* (by Olivier Darrigol), and one on *Fluid Dynamics* (by Peter Lynch) explore Stokes' work in those areas, where his exceptional mathematical skill coupled with good physical intuition, enabled him to make progress where others had failed. Stokes first rose to prominence with his work on fluorescence (a name which he invented from the mineral fluorspar by analogy with opalescence). His independent discovery of the Navier–Stokes equations provided more rigorous general arguments than Navier's.

In 1851 Stokes became a Fellow of the Royal Society, a theme developed by Sloan Evans Despeaux in the chapter *Stokes and the Royal Society*. Stokes soon became Secretary of the Society and a very conscientious chief editor of their publications, dealing efficiently and diplomatically with difficult cases when an aggrieved author contested a referee's report. Several friends regretted the fact that Stokes was devoting so much time to this work at the expense of his scientific researches, and was failing to keep up the momentum of his first 15 years of research. He continued as Secretary until 1885 when poor health forced T. H. Huxley's resignation and Stokes was elected President of the Society, an office which he held for five years.

Until reading Andrew Whitaker's chapter I was unaware that Stokes made important contributions to the investigations into the Dee Bridge and Tay Bridge disasters (in 1847 and 1879, respectively).

Examples of the importance of Stokes's work to other areas abound, such as his calculation of the drag on a sphere moving through a fluid, which played a crucial role in Millikan's 1916 experiments to determine the charge on an electron using oil droplets. Stokes parameters, introduced to describe general mixtures of polarised light, later provided an equally useful description of mixed electron spin states in quantum mechanics, and now even qubit states in a quantum computer. I particularly liked his one-line derivation of Fresnel's relations for reflection and refraction coefficients from the observation that optical processes are time reversible, so that when the reflected and refracted rays are time reversed to become incident they must combine to produce only the time reversal of the original incident ray.

Less well-known than his work in mathematical physics are Stokes' contributions to pure mathematics, which led G. H. Hardy to described him as 'a most acute pure mathematician'. These are described in a chapter by Richard Paris. An 1847 paper by Stokes on Fourier series introduced ideas of uniform convergence well before Weierstrass, and used the Riemann–Lebesgue Lemma well before Riemann. In another paper, his use of saddle-point arguments pre-dates Riemann's use by a decade, and his studies of the Airy caustic developed asymptotic methods well before Poincaré defined asymptotic expansions. Stokes discovered that there were different asymptotic expansions in regions of the complex plane separated by certain lines, now called the Stokes lines.

Designers tended to extrapolate previous designs to ever larger scales until a disaster showed their limitations, and tight budgets and timetables encouraged the use of cheaper materials and corner-cutting during construction. Most bridge designs catered for a static load in the middle of the bridge, not a heavy train moving across it at speed, and any safe speed limits imposed were often ignored by the drivers. The Dee and Tay Bridge disasters each involved several of these problems.

In each case a mix of experimental results and Stokes' theoretical work led to an understanding of the contributing causes. Whilst designers had considered the vertical motion of the Dee Bridge as it bore the weight of the train, the horizontal motion was actually more critical. The Tay Bridge had collapsed in a high wind with the loss of 75 lives, and Stokes found that the wind pressure on a bridge could be far higher than the conventional wisdom suggested.

The constructors had considered a tolerance of 10 lb/sq. ft to be adequate, whilst Stokes came up with a figure of 56 lb/sq. ft.

The penultimate chapter by Stuart Mathieson covers the topic of *Faith and Thought*. Unlike Faraday and Maxwell, who always refused to discuss questions about science and their Christian beliefs, which they considered a personal matter, Stokes, a churchwarden at St Paul's in Cambridge for thirty years, was quite open about his views.

The final chapter by Andrew Fowler views Stokes' work, particularly in fluid dynamics, from a different perspective, taking several topics of current interest and explaining how Stokes' work feeds into them. The examples are very varied, including the motion of glaciers, continental drift, and dendritic growth of crystals. They provide a fitting conclusion to a wide-ranging survey of the life, work, and faith of one of the foremost nineteenth-century mathematicians of these islands.

This book is a welcome addition to the growing series of studies in mathematical history, and I recommend it to anyone interested the history of mathematics or curious to learn more about the man whose name appears in so many mathematical contexts.

FURTHER READING

[1] D.B. Wilson, Kelvin and Stokes, Adam Hilger, 1987.

Keith Hannabuss

Keith Hannabuss is an Emeritus Lecturer in the Mathematical Institute, Oxford, and Emeritus Fellow of Balliol College. His main research interest is in operator algebras and quantum field theory, but he also has a longstanding interest in the History of Mathematics, particularly during the nineteenth century. His main hobbies are hill-walking and classical music.

Obituaries of Members

Fred H.J. Cornish: 1930 – 2020

Professor Fred H.J. Cornish, who was elected a member of the London Mathematical Society on 19 October 1990, died on 16 May 2020, aged 89.

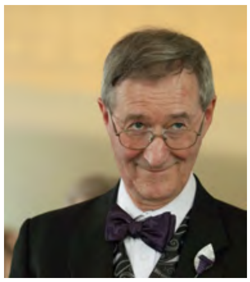
Tony Sudbery writes: Fred was born on 29 June 1930 in Exeter and attended Queen Elizabeth's Grammar School, Crediton. He was an undergraduate at Wadham College, Oxford, and while an undergraduate he co-authored three papers on conductivity in metals. He then completed a DPhil in general relativity. He held a postdoctoral position in the University of British Columbia at Vancouver, where he met his wife Monica, and then served a spell in the Navy. From 1959 he was a lecturer in mathematics at the University of Leeds until 1967, when he was appointed Professor and Head of Department of Mathematics at the recently founded University of York, following the tragic death of Paddy Kennedy, its first head of department. He was Deputy Vice-Chancellor of the University between 1975 and 1981; this included a period as Acting Vice-Chancellor in 1978.

Fred was active and widely respected in university administration and in national policy-making for university mathematics. In the 1990s he was chair of the committee that led to the "Neumann report" of the LMS, which was responsible for the introduction of 4-year undergraduate degrees in mathematics (the MMath). His research activity continued in both general relativity and classical electromagnetism, following his work at Leeds on the definition of energy and momentum of a gravitating system, on incorporating electromagnetism in general relativity, and on the interaction between radiation and point charges in classical electromagnetism. In 1986 he published a proof that according to classical electromagnetism a small electric dipole can undergo self-acceleration, an argument that aroused extended interest. With his research students he produced an explanation of the lack of gravitational radiation in the "photon rocket" solution of Einstein's equations, and other topics in general relativity. But possibly his most influential work lay outside this area, in his analysis of the equivalence between the hydrogen atom and the four-dimensional harmonic oscillator in classical and quantum mechanics.

Fred was admired throughout the University of York for his integrity, selflessness and sense of fair play, and for the welcome and help that he gave to new staff. John Brindley, a colleague in Leeds, comments that "He was a much loved colleague and a real gentleman in every sense of that word". He and Monica extended great hospitality to the mathematicians and to many others in York. Fred was a model of academic life for younger staff, who held him in great esteem and affection.

Monica died in 2019. They are survived by their children Rachel, John and Richard, and by six grandchildren.

Philip P.G. Dyke: 1948 – 2020



Professor Phil Dyke, who was elected a member of the London Mathematical Society on 28 February 2003, died on 4 June 2020 aged 71.

Colin Christopher writes: Phil died after a short battle with pancreatic cancer, diagnosed only months before. His final days were typically marked with more concern for his family and his students than for himself. Throughout his career, he thoroughly enjoyed the roles of lecturer, head of school and author and will be greatly missed by his colleagues and students.

Phil was born in Muswell Hill in London in quite humble circumstances. He attended the local Secondary Modern school where his ability in mathematics was first noticed. After encouragement from his brother, he stayed on to study A levels and then attended Northern Polytechnic in 1966. This was followed by a PhD in Meteorology at Reading University, where he met his wife Heather.

After working as a post-doc at UEA, Phil obtained his first lectureship in 1974 at Heriot Watt University. He moved to Plymouth in 1984, becoming Head of Mathematics in 1985. During his time as head, Phil was always distinguished by a genuine openness and friendship to all around him. Even in quite trying times, he would always see the best side of people. One striking aspect of this people-centredness was the ability to recall the date of birth of all of his staff–and of many others too! As a new lecturer, I can remember well his willingness to make time for junior staff.

Phil was promoted to a Chair of Applied Mathematics in 1993. Publishing work in geophysical fluids, he also took a strong interest in other areas of mathematics. He had a special love for classical mechanics and would often burst through an office door to share his joy in some intricate mechanics problem which he had found. This interest, and his love of teaching, was expressed by a growing number of undergraduate textbooks which he authored. The last of these was occupying him during his final illness. Phil became head of the newly merged School of Computing, Communications and Electronics from 2003 to 2007, where his tact and diplomacy was put to great use in the initial years of the merger. In 2014, Phil moved to a part-time role in order to spend time caring for his wife Heather, who had developed early-onset dementia. His deportment over this trying time was exemplary and earnt the respect of all his colleagues. Heather died in 2019, and not long after this Phil was diagnosed with cancer.

Phil was very much a family man with a great concern and love for his children. He was also a keen musician, playing guitar in several Jazz bands in his earlier years. From 2001 to 2007 he was Chair of Governors for one of the local grammar schools, a role which enabled him to express his concern for the development of young people in a practical way. He leaves his three children: Ottilie, Adrian and Eleanor, and a legacy of much kindness and thoughtfulness.

Kirill Mackenzie: 1951 – 2020

Kirill Mackenzie, who was elected a member of the London Mathematical Society on 21 November 1986, died on 2 May 2020, aged 68.

David Jordan writes: Kirill was born, as Charles Howard Mackenzie, in Melbourne in November 1951, the only child of Hamish and Elizabeth. His first name was his own later addition. He was educated in Melbourne, graduating from Monash University in 1973 with a BSc and in 1979 with a PhD in differential geometry, under the supervision of Juraj Virsik. Between 1979 and 1986 he held several tutorships, at Monash and the University of Melbourne, and a Queen Elizabeth II Fellowship at the ANU in Canberra. In 1982, midway through this Fellowship, he had a visiting position, for six months, in São Paulo, where he gave a series of graduate lectures in Portuguese. His first book *Lie groupoids and Lie algebroids in differential geometry*, which included substantial original research, was completed in 1985 and published in the LMS Lecture Notes series in 1987. He moved to the UK in 1986, as a Senior Research Assistant in Durham, working with Philip Higgins on abstract Lie algebroids. Another fruitful collaboration around this time was with Ronnie Brown on double Lie groupoids. From Durham, he visited two of the pioneers of his subject, Jean Pradines in Toulouse and Alan Weinstein in Berkeley. In Berkeley he also met Ping Xu, then a graduate student, with whom he would subsequently collaborate successfully on Lie bialgebroids and Poisson groupoids. He moved to Sheffield in 1989 to take up a Lectureship and was promoted in 1999 to a Readership, the position that he held at the time of his death.

In the early 1990's, Kirill made several visits to the Institut Henri Poincaré, including five weeks at the Centre Émile Borel during which he worked with Yvette Kosmann-Schwarzbach, with whom he would exchange several further research visits. By this time the theory of Lie groupoids and Lie algebroids had been transformed through the emergence, in the work of Weinstein and others, of new examples arising in Poisson and symplectic geometry, including the Lie algebroid structure on the cotangent bundle of a Poisson manifold. Kirill embraced these developments both in his second book *General theory of Lie groupoids and Lie algebroids* (2005), again in the LMS Lecture Notes series, and in his research on a Lie theory for double and multiple Lie groupoids and Lie algebroids and its application in Poisson geometry. He referred to this theory as 'multiple Lie theory' but Ted Voronov calls it 'Mackenzie theory'.

Kirill was a committed teacher with firm views on the curriculum. At Sheffield he introduced several modules including an MMath option on *Optics and Symplectic Geometry* that was probably unique in the UK. He was keen on project work and from 2001 to 2006 co-ordinated final year MMath projects in pure mathematics in Sheffield. His PhD students, undergraduate students and personal tutees benefited from and appreciated the time and care that they received from him.

Kirill had some health issues over the years, not least with his eyesight, but his sudden death at a time when he was still active in research and teaching came as a shock. Away from mathematics, he enjoyed music, poetry, art and cooking and was interested in classical civilizations and philosophy. He is survived by his wife Margaret, a pharmacist, whom he met in Melbourne and married in Durham in 1989, and Margaret's daughter Michaela and son Dan from an earlier marriage. He will be missed and fondly remembered.

Alan R. Pears: 1938 – 2020

Dr Alan Pears, who was elected a member of the London Mathematical Society on 20 December 1962, died on 4 July 2020, aged 82. Dr Pears was a member of LMS Council (1983–92), Meetings and Membership Secretary (1983–92), Joint Editor of the LMS *Newsletter* (1993–98) and Obituaries Editor of the LMS *Bulletin* (1995–2005).

Peter Saunders writes: Alan Pears was born in Middlesbrough in 1938. He attended Harrogate Grammar School and graduated in mathematics at Cambridge in 1960. He was awarded his PhD by Cambridge in 1964, having worked under the supervision first of R.B. Braithwaite and then of C.H. Dowker. In 1963 he took up a lectureship at Queen Elizabeth College, where he was promoted to senior lecturer in 1984. Alan transferred to King's College London in 1985 when it merged with QEC and Chelsea College and continued at King's until his early retirement in 1995. He also taught for the Open University from 1974 to 2001, first as a part-time tutor and then as an associate lecturer.

Alan's thesis was on topological games, and after he left Cambridge he continued his research into general topology. He wrote several papers on the definitions and properties of several notions of dimension, an established topic in this context, culminating in 1975 in a well-received monograph *Dimension Theory of General Spaces*. It will come as no surprise to those who knew him that the book was described in a review by James Keesling as 'deep and comprehensive, with extensive historical notes and important examples.' Alan was nothing if not thorough.

After the book was finished, Alan's career took a new direction. He became very deeply involved in LMS matters, holding the key post of Meetings and Membership Secretary for nine years as well as serving on the Finance and General Purposes and Programme Committees. Alan also oversaw a significant expansion of the Society's activities, including an increase in funding for a wide range of mathematical activities throughout the UK and the introduction of the Invited Lectures series. These were important changes in their own right, but they also helped prepare the Society for the transformation that was soon to take place with the purchase of De Morgan House.

The high point of Alan's time at the centre of the LMS was the joint AMS-LMS meeting in 1992. Everyone who was there was appreciative of how well the meeting was organised and how smoothly everything ran. What few of them knew was that while the AMS had a large administrative staff to see to their side of the arrangements, the LMS — who as hosts had far more to do — relied on basically two people, Alan Pears and Susan Oakes. Later that year, when Alan finally stood down from his major posts in the LMS, a short note appeared in the *Newsletter* to mark his departure. In it, the President and the Past President (John Kingman and John Coates) wrote: "He has simply been one of the major forces behind the Society over the last decade."

After his time on Council, Alan continued to contribute to UK mathematics through both the LMS and other organisations; he was for six years treasurer of the British Mathematical Olympiad Committee. During a period that was especially difficult for British universities, he was Deputy Head of the King's College mathematics department and a member of the senior academic committees of both the College and the University of London. In 2005 he was co-author with Susan Oakes and Adrian Rice of *The Book of Presidents* 1865–1965.

In the department, Alan was always ready to take on a demanding job and do it well. His lectures were noted for their clarity and he took great pains with his students. Above all, however, he was, as a colleague recently recalled to me, "a nice man". He is survived by his wife Sallie, to whom he was married for over fifty years, and two sons, Jonathan and Simon.

Death Notice

We regret to announce the following death:

- Sir F.R. Vaughan Jones, FRS, who was elected an Honorary Member of the London Mathematical Society on 25 November 2002, died on 6 September 2020.

LMS Meeting

Computer Science Colloquium: *Algorithms, Complexity and Logic*

19 November 2020, 10am–4pm. Event to be held via Zoom, with support from the ICMS

Website: tinyurl.com/cscoll2020

The Computer Science Colloquium is an annual day of themed talks on a topical issue at the interface of mathematics and computer science. The event is aimed at PhD students and post-docs, although others are welcome to attend. The theme of the next colloquium will be 'Algorithms, Complexity and Logic', and will include the following speakers: Nobuko Yoshida (Imperial College London), Kitty Meeks (Glasgow), Anupam Das (Birmingham) and Igor Carboni Oliveira (Warwick).

LMS Meeting

Annual General Meeting

20 November 2020, 3pm–5pm. Meeting to be held via Zoom, with support from the ICMS

Website: tinyurl.com/y7q5dsgk

The meeting will open with Society business, including the announcement of the annual LMS election results, and will be following by the 2020 Naylor Lecture from Professor Nicholas J. Higham (Manchester). The lecture is titled *The Mathematics of Today's Floating-Point Arithmetic*. The Naylor Lecture is aimed at a general mathematical audience. All interested, whether LMS members or not, are most welcome to attend this event.

LMS Invited Lectures Series 2022: Call for Proposals

LONDON MATHEMATICAL SOCIETY EST 1865

The annual LMS Invited Lectures Series consists of meetings held in the UK at which a single speaker gives a course of about ten expository lectures, examining some subject in depth, over a five day period (Monday to Friday) during a University vacation. The meetings are residential and open to all interested.

A £1,250 honorarium is offered directly from the LMS to the Invited Lecturer and £6,000 in funding is given to the host department to cover both the Lecturer's expenses (travel, accommodation and subsistence) and support attendance at the lectures.

Proposals for the Invited Lectures 2022

Any member who would like to suggest a topic and lecturer and be prepared to organise the meeting at their own institution or a suitable conference centre can submit a proposal. For further details, please visit the Society's website: tinyurl.com/y98espkj. The deadline for proposals is 1 February 2021.

LMS Invited Lecturer 2021

The LMS Invited Lecture Series 2021 on equations in groups and complexity will be given by Professor Olga Kharlampovich (CUNY Graduate Center and Hunter College) at the University of Newcastle (dates to be determined).

Recent previous Invited Lecturers:

- 2020: Professor Yulia Mishura (University of Kyiv) *Fractional Calculus and Fractional Stochastic Calculus, including Rough-Paths, with Applications*, Zoom via Brunel University, 15-19 June.
- 2019: Professor Søren Asmussen (Aarhus University) *Advanced Topics in Life Insurance Mathematics*, ICMS in Edinburgh, 20-24 May.
- 2018: Professor Art Owen (Stanford University) *From the Foundations of Simulation to Quasi Monte Carlo*, Warwick University, 9-13 July.

Enquiries about the Invited Lectures may be addressed to Professor Brita Nucinkis, the Chair of the Society Lectures and Meetings Committee: lmsmeetings@lms.ac.uk.

Covid-19: Owing to the coronavirus pandemic, many events may be cancelled, postponed or moved online. Members are advised to check event details with organisers.

Society Meetings and Events

November 2020

19 Computer Science Colloquium, London
20 Society Meeting and AGM, London

June 2021

2-4 Midlands Regional Meeting and Workshop, Lincoln

September 2021

6-10 Northern Regional Meeting, Conference in Celebration of the 60th Birthday of Bill Crawley-Boevey, University of Manchester

January 2022

4-6 South West & South Wales Regional Meeting, Swansea

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

December 2020

11-14 Number Theory and Discrete Mathematics, Ramanujan Mathematical Society (491)

January 2021

8 Burnside Rings for Profinite Groups, Lancaster (online)
20-22 Fry Inaugural Series: Challenges and Recent Advances in Mathematical Physics, Heilbronn Institute, Bristol (490)

March 2021

14 International Day of Mathematics (491)

July 2021

12-16 New Challenges in Operator Semigroups, St John's College, Oxford (490)

August 2021

16-20 IWOTA, Lancaster University (481)

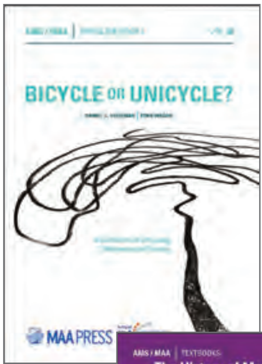
September 2021

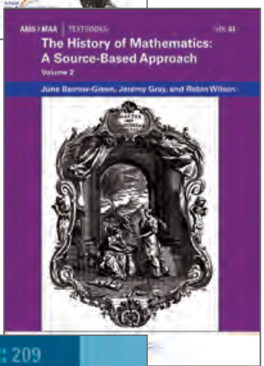
1-3 Scaling Limits: From Statistical Mechanics to Manifolds, Cambridge (489)
19-24 8th Heidelberg Laureate Forum, Heidelberg, Germany
21-23 Conference in Honour of Sir Michael Atiyah, Isaac Newton Institute, Cambridge (487)

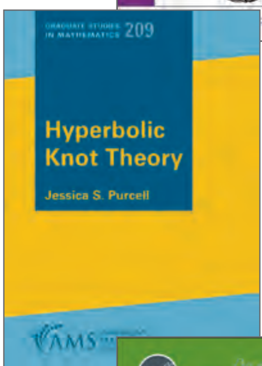
July 2022

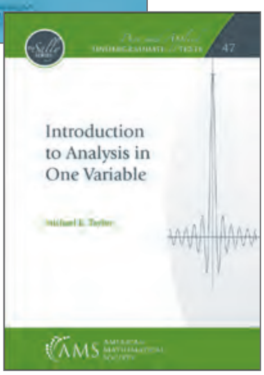
24-26 7th IMA Conference on Numerical Linear Algebra and Optimization, Birmingham (487)

AMERICAN MATHEMATICAL SOCIETY









BICYCLE OR UNICYCLE?

A Collection of Intriguing Mathematical Puzzles

Daniel J. Velleman, Amherst College and University of Vermont & Stan Wagon, Macalester College

Presents a collection of 105 mathematical puzzles whose defining characteristic is the surprise encountered in their solutions. Solvers will be surprised, even occasionally shocked, at those solutions. The problems unfold into levels of depth and generality very unusual in the types of problems seen in contests.

Problem Books, Vol. 36

MAA Press

Sep 2020 286pp 9781470447595 Paperback £52.50

THE HISTORY OF MATHEMATICS

A Source-Based Approach, Volume 2

June Barrow-Green, The Open University, Jeremy Gray, The Open University & Robin Wilson, The Open University

A comprehensive history of the development of mathematics. This second volume takes the reader from the invention of the calculus to the beginning of the twentieth century. In addition to being an innovative and insightful textbook, it is an invaluable resource for students and scholars of the history of mathematics.

AMS/MAA Textbooks, Vol. 61

MAA Press

Jan 2021 724pp 9781470443825 Paperback £84.95

HYPERBOLIC KNOT THEORY

Jessica S. Purcell, Monash University

Provides an introduction to hyperbolic geometry in dimension three, with motivation and applications arising from knot theory. The book was written to be interactive, with many examples and exercises. Some important results are left to guided exercises.

Graduate Studies in Mathematics, Vol. 209

Nov 2020 369pp 9781470454999 Paperback £93.50

INTRODUCTION TO ANALYSIS IN ONE VARIABLE

Michael E. Taylor, University of North Carolina

A text for students who are ready to explore the logical structure of analysis as the backbone of calculus. It begins with a development of the real numbers, building this system from more basic objects, and produces basic algebraic and metric properties of the real number line as propositions, rather than axioms.

Pure and Applied Undergraduate Texts, Vol. 47

Oct 2020 247pp 9781470456689 Paperback £80.95

Free delivery at **eurospanbookstore.com/ams**

AMS is distributed by EUROSPAN

CUSTOMER SERVICES:
Tel: +44 (0)1767 604972
Fax: +44 (0)1767 601640
Email: eurospan@turpin-distribution.com

FURTHER INFORMATION:
Tel: +44 (0)20 7240 0856
Fax: +44 (0)20 7379 0609
Email: info@eurospan.co.uk