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Arithmetical functions associated with arbitrary sets of integers

by

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1. Introduction. In two previous papers [2], [3], also to be denoted by I and II, respectively, we considered properties of arithmetical functions associated with direct factor sets. It is the purpose of the present paper to extend to arbitrary sets of (positive) integers several of the results proved in I and II in the special case of direct factor sets. In this paper we shall use an asterisk (*) to denote reference numbers appearing in the bibliography of I and a double asterisk (**) to denote those occurring in II. For the convenience of the reader, these auxiliary references are briefly indicated at the end of the paper.

The method employed in I and II was based on the concept of arithmetical inversion with respect to conjugate direct factor sets. This method ceases to be applicable, however, in the case of arbitrary sets. In treating the general case we employ the characteristic and inversion functions of a set (see below), in connection with ordinary Möbius inversion and an enumerative principle for residue systems. The remainder of this section is devoted to a brief description of the paper, including a number of definitions.

Let S represent an arbitrary set of positive integers n, and place $\varrho_S(n) = 1$ or 0 according as n is or is not an element of S. The function $\varrho_S(n)$ is the *characteristic function* of S. To define the *zeta-function* $\zeta_S(t)$ of S, place

$$\zeta_S(t) = \sum_{n=1}^{\infty} \frac{\varrho_S(n)}{n^t}. \tag{1.1}$$

With Z denoting the set of positive integers, it is observed that $\zeta_Z(t) = \zeta(t)$, where $\zeta(t)$ is the ordinary zeta-function, $t > 1$. Moreover, it is clear that $\zeta_1(t) = 1$.

Next, the *inversion function* $\mu_S(n)$ of S is defined by the relation

$$\sum_{n=1}^{\infty} \frac{\mu_S(n)}{n^t} = \frac{\zeta_S(t)}{\zeta(t)}, \quad t > 1. \tag{1.2}$$

It is evident that $\mu_1(n) = \mu(n)$ where $\mu(n)$ represents the Möbius function. Forming the Dirichlet product, it follows by (1.2) that

$$\sum_{d|n} \mu_S(d) = \varrho_S(n). \tag{1.3}$$

Hence, by Möbius inversion, one obtains

$$\mu_S(n) = \sum_{d|n} \mu(d)\, \varrho_S\left(\frac{n}{d}\right). \tag{1.4}$$

The idea of inversion function of a set is implicit in Cesàro [1**] and Gegenbauer [4*].

In addition, we define the *enumerative function* of S, denoted by $[x]_S$, $x \geqslant 2$ to be the number of positive integers $\leqslant x$ contained in S, while the *divisor function* $\tau_S(n)$ of S is defined to be the number of (positive) divisors d of n contained in S. The *totient function* $\varphi_S(n)$ of S denotes the number of integers $a \pmod n$ such that $(a, n) \in S$. Obviously, $\varphi_1(n) = \varphi(n)$ where $\varphi(n)$ is the ordinary Euler totient. Similarly, $\tau_Z(n) = \tau(n)$ and $[x]_Z = [x]$, $\tau(n)$ denoting the total number of divisors of n and $[x]$ the integral part of x.

Sections 2 and 3 are concerned principally with the function $N_k(x, S)$, defined to be the number of integral, k-dimensional vectors $\{n_1, \ldots, n_k\}$ such that $1 \leqslant n_i \leqslant x$, $i = 1, \ldots, k$, and $(n_1, \ldots, n_k) \in S$. In § 2 (Theorem 2.2) we obtain an evaluation of $N_k(x, S)$, which in case $k = 1$ (Corollary 2.1), reduces to a formula for $[x]_S$ proved in II for direct factor sets. The proof of Theorem 2.2 is based on evaluations obtained in Theorem 2.1 for the function,

$$T_k(x, S) = \sum_{\substack{n_i \leqslant x \\ (i=1,\ldots,k)}} \tau_S\big((n_1, \ldots, n_k)\big). \tag{1.5}$$

As special cases it is possible to deduce classical results of Bougaïeff and Gram for the exact number of primes beneath a given limit ([4], p. 429-430, [1], [6]).

In § 3 we are concerned with asymptotic estimates for $N_k(x, S)$ and $T_k(x, S)$, $k \geqslant 2$, valid for arbitrary sets S. The results contained in (3.4) and (3.6) were obtained for $N_k(x, S)$ in II, in the case of direct factor sets.

In the last section (§ 4) we obtain an evaluation (4.6) of $\varphi_S(n)$, proved for direct factor sets in I. More generally, we evaluate a function $c_S(n, r)$ generalizing Ramanujan's sum (Theorem 4.4) and a second function $\varphi_S(x, n)$, generalizing Legendre's totient $\varphi(x, n)$ (Theorem 4.2). The evaluation formulas obtained for $c_S(n, r)$ and $\varphi_S(x, n)$ were proved for direct factor sets in I and II, respectively.

2. Evaluation of $T_k(x, S)$ and $N_k(x, S)$. We first consider $T_k(x, S)$, $k \geqslant 1$.

THEOREM 2.1.

$$T_k(x, S) = \sum_{n \leqslant x} \varrho_S(n) \left[\frac{x}{n}\right]^k; \tag{2.1}$$

alternatively,

$$T_k(x, S) = \sum_{n \leqslant x} N_k\left(\frac{x}{n}, S\right). \tag{2.2}$$

Proof. By definition of $T_k(x, S)$ one obtains

$$T_k(x, S) = \sum_{\substack{n_i \leqslant x \\ (i=1,\ldots,k)}} \sum_{d|(n_1,\ldots,n_k)} \varrho_S(d). \tag{2.3}$$

This formula may be restated in the form,

$$T_k(x, S) = \sum_{\substack{n_i \leqslant x \\ (i=1,\ldots,k)}} \sum_{\substack{d|n_i \\ (d\delta_i = n_i)}} \varrho_S(d) = \sum_{d \leqslant x} \varrho_S(d) \sum_{\substack{\delta_i \leqslant x/d \\ (i=1,\ldots,k)}} 1,$$

from which (2.1) follows immediately. (2.3) may also be reformulated as

$$T_k(x, S) = \sum_{\substack{n_i \leqslant x \\ (i=1,\ldots,k)}} \sum_{\substack{d\delta = (n_1,\ldots,n_k) \\ (\delta\delta_i = n_i)}} \varrho_S(d) = \sum_{\delta \leqslant x} \sum_{\substack{d = (\delta_1,\ldots,\delta_k) \\ \delta_i \leqslant x/d \\ (i=1,\ldots,k)}} \varrho_S(d),$$

so that (2.2) follows by definition of $N_k(x, S)$.

We next prove the following analogue of (2.1).

THEOREM 2.2.

$$N_k(x, S) = \sum_{n \leqslant x} \mu_S(n) \left[\frac{x}{n}\right]^k. \tag{2.4}$$

Proof. By (2.1) and (2.2) it follows that

$$\sum_{n \leqslant x} N_k\left(\frac{x}{n}, S\right) = \sum_{n \leqslant x} \varrho_S(n) \left[\frac{x}{n}\right]^k. \tag{2.5}$$

Applying the second Möbius inversion formula ([6**], Theorem 268) to (2.5) one obtains

$$N_k(x, S) = \sum_{\delta\leqslant x}\mu(\delta)\sum_{\substack{d\leqslant x/\delta\\(a=d\delta)}}\left[\frac{x/\delta}{d}\right]^k\varrho_S(d) = \sum_{a\leqslant x}\left[\frac{x}{a}\right]^k\sum_{a=d\delta}\mu(\delta)\varrho_S(d),$$

and (2.5) results on the basis of (1.4).

Remark. Theorem 2.2 can also be proved on the basis of (1.3) independently of the function $T_k(x, S)$.

Since $N_1(x, S) = [x]_S$, it follows from (2.4) that

COROLLARY 2.1 ($k = 1$).

$$[x]_S = \sum_{n\leqslant x}\mu_S(n)\left[\frac{x}{n}\right]. \tag{2.6}$$

We consider some special cases of Corollary 2.1. Let P denote the set of the positive primes, Q the set of all positive powers of the primes, and R the set of all proper prime-powers (powers of primes with the primes themselves deleted). Denote the enumerative functions of P, Q, R by $\pi(x)$, $X(x)$, and $\Psi(x)$, and the inversions functions by $a(n)$, $\beta(n)$, and $\nu(n)$, respectively. It is convenient in the following to term a number n *primitive* if n contains no square factor > 1, *properly primitive* if n is primitive and $\neq 1$, and *semi-primitive* if n has a prime-square factor p^2 such that n/p^2 is primitive and $(p, n/p^2) = 1$. If $\omega(n)$ is used to designate the number of distinct prime factors of n, it is easily verified by (1.4) that ([4*], p. 423)

$$a(n) = \begin{cases}(-1)^{\omega(n)+1}\omega(n) & (n \text{ properly primitive}),\\ (-1)^{\omega(n)} & (n \text{ semi-primitive}),\\ 0 & (\text{otherwise});\end{cases} \tag{2.7}$$

in addition,

$$\beta(n) = \begin{cases}(-1)^{\omega(n)+1}\omega(n) & (n \text{ properly primitive}),\\ 0 & (\text{otherwise});\end{cases} \tag{2.8}$$

therefore

$$\nu(n) = \beta(n)-a(n) = \begin{cases}(-1)^{\omega(n)+1} & (n \text{ semi-primitive}),\\ 0 & (\text{otherwise}).\end{cases} \tag{2.9}$$

We obtain the following results on the basis of (2.6), in connection with (2.7), (2.8), and (2.9).

COROLLARY 2.2 ([1], (6), p. 16).

$$\pi(x) = \sum_{n\leqslant x}{}'(-1)^{\omega(n)}\left[\frac{x}{n}\right] - \sum_{n\leqslant x}{}''(-1)^{\omega(n)}\omega(n)\left[\frac{x}{n}\right], \tag{2.10}$$

where the first summation is restricted to semi-primitive values of n and the second to properly primitive n.

COROLLARY 2.3 ([6], (125), p. 297).

$$X(x) = -\sum_{n\leqslant x}{}''(-1)^{\omega(n)}\omega(n)\left[\frac{x}{n}\right], \tag{2.11}$$

where the summation if over properly primitive values of n.

COROLLARY 2.4.

$$\Psi(x) = -\sum_{n\leqslant x}{}'(-1)^{\omega(n)}\left[\frac{x}{n}\right], \tag{2.12}$$

the summation being over semi-primitive values of n.

3. Asymptotic estimates for $T_k(x, S)$ and $N_k(x, S)$. We obtain first the asymptotic value of $T_k(x, S)$, or equivalently, the average order of $\tau_S((n_1, \ldots, n_k))$ as a function of k integral variables, $k\geqslant 2$.

THEOREM 3.1 *If $k\geqslant 2$, then for all sets S,*

$$T_k(x, S) = x^k\zeta_S(k)+\begin{cases}O(x^{k-1}) & \text{if } k>2,\\ O(x\log x) & \text{if } k=2.\end{cases} \tag{3.1}$$

Proof. By (2.1)

$$T_k(x, S) = \sum_{n\leqslant x}\varrho_S(n)\left[\frac{x}{n}\right]^k = \sum_{n\leqslant x}\varrho_S(n)\left(\frac{x}{n}+O(1)\right)^k$$

$$= \sum_{n\leqslant x}\varrho_S(n)\left(\left(\frac{x}{n}\right)^k+O\left(\frac{x^{k-1}}{n^{k-1}}\right)\right) = x^k\sum_{n\leqslant x}\frac{\varrho_S(n)}{n^k}+O\left(x^{k-1}\sum_{n\leqslant x}\frac{\varrho_S(n)}{n^{k-1}}\right)$$

$$= x^k\sum_{n=1}^{\infty}\frac{\varrho_S(n)}{n^k}+O\left(x^k\sum_{n>x}\frac{\varrho_S(n)}{n^k}\right)+O\left(x^{k-1}\sum_{n\leqslant x}\frac{\varrho_S(n)}{n^{k-1}}\right).$$

Since $\varrho_S(n)$ is obviously bounded, the first O-term in the last expression is $O(x)$, while the second is $O(x\log x)$ or $O(x^{k-1})$ according as $k=2$ or $k>2$ ([3], (4.1), (4.3)). This proves the theorem.

COROLLARY 3.1. *The mean value of* $\tau_S((n_1, \dots, n_k))$ *as a function of* $n_1, \dots, n_k$ $(k \geqslant 2)$ *is* $\zeta_S(k)$; *that is,*

$$\lim_{x\to\infty} \frac{T_k(x, S)}{x^k} = \zeta_S(k). \tag{3.2}$$

This result was proved by Gegenbauer [5**] for various special sets S.

We now consider the function $N_k(x, S)$.

THEOREM 3.2. *For* $k \geqslant 2$ *and an arbitrary set* S,

$$N_k(x, S) = \frac{x^k \zeta_S(k)}{\zeta(k)} + \begin{cases} O(x^{k-1}) & \text{if} \quad k > 2, \\ O(x\log^2 x) & \text{if} \quad k = 2; \end{cases} \tag{3.3}$$

moreover, if $\mu_S(n)$ *is bounded (and in particular if* S *is a directfacto r set* ([3], Lemma 4.1)), *then*

$$N_2(x, S) = \frac{x^2 \zeta_S(2)}{\zeta(2)} + O(x\log x). \tag{3.4}$$

Proof. Applying the second Möbius inversion formula to (2.2) one obtains

$$N_k(x, S) = \sum_{n\leqslant x} \mu(n) T_k\left(\frac{x}{n}, S\right). \tag{3.5}$$

It follows then by (3.1) that

$$N_k(x, S) = \frac{x^k \zeta_S(k)}{\zeta(k)} + O\left(x^k \sum_{n>x} \frac{\mu(n)}{n^k}\right) + \begin{cases} O\left(x^{k-1} \sum_{n\leqslant x} \frac{1}{n^k}\right) & \text{if} \quad k > 2, \\ O\left(x \sum_{n\leqslant x} \frac{1}{n} \log\frac{x}{n}\right) & \text{if} \quad k = 2. \end{cases}$$

The first O-term is $O(x)$ because $|\mu(n)| \leqslant 1$; the second O-term is $O(x^{k-1})$ if $k > 2$, while in case $k = 2$, it is

$$O\left(x\log x \sum_{n\leqslant x} \frac{1}{n}\right) = O(x\log^2 x).$$

This proves (3.3).

Assume now that $\mu_S(n)$ is bounded. Then by (2.4), as in the argument of Theorem 3.1,

$$N_2(x, S) = x^2 \sum_{n=1}^{\infty} \frac{\mu_S(n)}{n^2} + O\left(x^2 \sum_{n>x} \frac{\mu_S(n)}{n^2}\right) + O\left(x \sum_{n\leqslant x} \frac{|\mu_S(n)|}{n}\right).$$

In view of the boundedness of $\mu_S(n)$, the first O-term is $O(x)$ and the second $O(x\log x)$. Thus (3.4) follows by (1.2).

COROLLARY 3.2. *If* $k \geqslant 2$, S *arbitrary, then the asymptotic density* $\delta_k(S)$ *of the* k*-dimensional, positive integral vectors with greatest common divisor in* S *is given by*

$$\delta_k(S) = \lim_{x\to\infty} \frac{N_k(x, S)}{x^k} = \frac{\zeta_S(k)}{\zeta(k)}. \tag{3.6}$$

The result in (3.6) is due to Cesàro [1**], [2**].

Let $S \equiv r$ consist of the single positive integer r. Then $\zeta_r(k) = r^{-k}$ and we have by (3.6) the well-known result ([3], (5.9)),

COROLLARY 3.3. *The asymptotic density of the* k*-dimensional vectors* $(k \geqslant 2)$ *with greatest common divisor equal to* r *is* $1/r^k\zeta(k)$.

4. Evaluation of $\varphi_S(n)$ **and generalizations.** We first define two generalizations of $\varphi_S(n)$. Let $\varphi_S(x, n)$ denote the number of positive integers $a \leqslant x$ such that $(a, n) \in S$. In case $S = 1$, $\varphi_S(x, n)$ reduces to Legendre's function $\varphi(x, n)$. In addition, for a positive integer r, we place $e(n, r) = \exp(2\pi i n/r)$ and define

$$c_S(n, r) = \sum_{(a,r)\in S} e(na, r), \tag{4.1}$$

the summation being over an S-reduced residue system (mod$r$), that is, over the integers a(mod$r$) such that $(a, r) \in S$. Note that $c_S(n, r)$ reduces to Ramanujan's sum in case $S \equiv 1$.

We first express the function $\varphi_S(x, r)$ in terms of $\varphi(x, r)$ and then evaluate $\varphi_S(x, r)$ on the basis of Legendre's evaluation of $\varphi(x, r)$. The basis of this procedure is the following enumerative principle:

LEMMA 4.1. *Let* d *range over the divisors of* n *and for each* d *let* X *range over the positive integers* $\leqslant x/d$ *such that* $(X, n/d) = 1$. *Then the set* $\{y\}$, $y = dX$, *forms a least positive residue system* (mod $[x]$).

The proof, which is similar to that of the familiar special case in which $x = n$, will be omitted.

THEOREM 4.1. *For arbitrary sets* S,

$$\varphi_S(x, n) = \sum_{d|n} \varrho_S(d) \varphi\left(\frac{x}{d}, \frac{n}{d}\right). \tag{4.2}$$

Proof. In Lemma 4.1, $(y, n) = d$; hence it follows that the number of $y \leqslant x$ with $(y, n) = d$ is $\varphi(x/d, n/d)$. Summing over all $d \in S$ one obtains (4.2).

The following result, in case S is the set of h-free integers or the set of k-th powers, is due to Gegenbauer ([5], § 3, (8), (12)).

THEOREM 4.2.

$$\varphi_S(x, n) = \sum_{d|n} \mu_S(d)\left[\frac{x}{d}\right]. \tag{4.3}$$

Proof. The case $S = 1$ of (4.3) is Legendre's classical formula ([4], Chapter V, (5)), on the basis of which, by (4.2),

$$\varphi_S(x, n) = \sum_{d|n} \varrho_S(d) \sum_{\substack{D|n/d \\ (Dd=\delta)}} \mu(D)\left[\frac{x/d}{D}\right] = \sum_{\delta|n}\left[\frac{x}{\delta}\right]\sum_{Dd=\delta} \varrho_S(d)\,\mu(D).$$

The theorem results by virtue of (1.4).

The function $c_S(n, r)$ can be treated in a similar manner.

THEOREM 4.3. *For arbitrary* S,

$$c_S(n, r) = \sum_{d|r} \varrho_S(d)\, c\left(n, \frac{r}{d}\right). \tag{4.4}$$

Proof. By Lemma 4.1 in the case $x = n = r$, it follows that an S-reduced residue system $(\bmod r)$ is furnished by the set $a = dX$, where d ranges over the divisors of r contained in S and for each such d, X ranges over a reduced residue system $(\bmod r/d)$. Therefore, by (4.1),

$$c_S(n, r) = \sum_{\substack{d|r \\ d\in S}} \sum_{(X, r/d)=1} e(dXn, r) = \sum_{d|r} \varrho_S(d) \sum_{(X, r/d)=1} e\left(Xn, \frac{r}{d}\right),$$

and (4.4) results by definition of $c(n, r) = c_1(n, r)$.

THEOREM 4.4.

$$c_S(n, r) = \sum_{d|(n,r)} d\mu_S\left(\frac{r}{d}\right). \tag{4.5}$$

Proof. The case $S \equiv 1$ of (4.5) is Ramanujan's evaluation of $c(n, r)$. Using this familiar fact, in connection with (4.4), it follows that

$$c_S(n, r) = \sum_{d|r} \varrho_S(d) \sum_{\delta|(n, r/d)} \delta\mu\left(\frac{r/d}{\delta}\right) = \sum_{\delta|(n,r)} \delta \sum_{d|r/\delta} \varrho_S(d)\,\mu\left(\frac{r/\delta}{d}\right),$$

and (4.5) is a consequence of (1.4).

We note that $\varphi_S(n, n) = c_S(n, n) = \varphi_S(n)$. Hence by either (4.3) or (4.5) we have the evaluation of $\varphi_S(n)$,

COROLLARY 4.1.

$$\varphi_S(n) = \sum_{d|n} d\mu_S\left(\frac{n}{d}\right). \tag{4.6}$$

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