A problem in “Factorisatio Numerorum“.

By

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1. **Introduction.** This note is devoted to a study of the number theoretic function $f(n)$ which gives the number of representations of the natural number n as a product of factors greater than one. Here two representations are considered identical if and only if they contain the same factors written in the same order. We define $f(1)=1$. Some generalizations are indicated at the end of the note.

This function $f(n)$ does not seem to have attracted much attention. It is intimately connected with the algorithm of Möbius

$$\left\{\begin{array}{l} a_1 b_1 = 1, \\ \sum_{d \mid n} a_d b_{n/d} = 0, \qquad n = 2, 3, \dots, \end{array}\right. \tag{1.1}$$

which arises in a number of analytical problems, for instance, in the expansion of the reciprocal of an ordinary Dirichlet series into a series of the same type. The relationship is simply

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \frac{1}{D(s)} = \sum_{n=1}^{\infty} b_n n^{-s}, \quad a_1 \neq 0, \tag{1.2}$$

where the coefficients are connected by (1.1). Taking $a_1 = 1$, $a_n = -1$, $n = 2, 3, \dots$, we get $b_n = f(n)$ as will be shown below.

The only papers on this function which are known to the author

are those of L. Kalmár.[1] These contain a study of the summatory function

$$F(n) = \sum_{m=1}^{n} f(m). \tag{1.3}$$

Denoting by ρ the positive root of the equation

$$\zeta(s) = 2, \tag{1.4}$$

Kalmár proved that

$$F(n) = -\frac{n^{\rho}}{\rho\, \zeta'(\rho)} \{1 + o(1)\}, \tag{1.5}$$

and gave various estimates of the remainder.

It is obvious that $f(n)$ itself is a very irregular function, and next to nothing seems to be known about its behavior under different assumptions regarding the number of prime factors in n. A study of this problem does not call for particularly complicated machinery, and the results are not quite devoid of interest. They were found as a by-product in an investigation of the algorithm of Möbius which will be published elsewhere.[2]

2. **Elementary properties.** Let

$$n = p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \dots p_{i_\nu}^{\alpha_\nu} \tag{2.1}$$

be the representation of n as a product of prime factors. It is clear that $f(n)$ depends only upon the divisibility properties of n and not upon the actual numerical values of the prime factors. It follows that $f(n)$ is a symmetric function of the ν variables $\alpha_1, \alpha_2, \dots, \alpha_\nu$.

Let $d \mid n$ and put $n = m\,d$, $m > 1$. All the factorizations of n which contain m as first factor are obtained by considering all factorizations of d. They are $f(d)$ in number. It is clear that these particular factorizations can be obtained in no other way. Hence we have

$$f(n) = \sum_{d \mid n}' f(d), \tag{2.2}$$

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[1]) A „factorisatio numerorum“ problémájáról, Matematikai és Fizikai Lapok, **38** (1931) 1—15, and Über die mittlere Anzahl der Produktdarstellungen der Zahlen. (Erste Mitteilung) Acta Litterarum ac Scientiarum, Szeged, **5** (1931) 95—107. The second part of the latter paper does not seem to have appeared. I am indebted to Prof. O. Szász for this reference.

[2]) See also E. Hille and O. Szász, On the completeness of Lambert functions, Part I, Bulletin Amer. Math. Soc., **42** (1936), and Part II, Annals of Math., (2) **37** (1936)

where the summation extends over all divisors d of n which are $< n$. This functional equation together with the initial conditions

$$f(1) = f(p) = 1 \tag{2.3}$$

determine $f(n)$ completely.

Formula (2.2) recalls Dedekind's inversion formula. If

$$h(n) = \sum_{d \mid n} g(d)$$

for all n, then

$$g(n) = h(n) - \sum h\left(\frac{n}{p_{i_1}}\right) + \sum h\left(\frac{n}{p_{i_1} p_{i_2}}\right) - \ldots,$$

where the summations extend over all the combinations 1, 2, 3, ... at a time of the distinct prime factors $p_{i_1}, p_{i_2}, \ldots, p_{i_\nu}$ of n. Putting $h(n) = 2f(n)$, $g(n) = f(n)$ we get after simplification

$$f(n) = 2\left\{\sum f\left(\frac{n}{p_{i_1}}\right) - \sum f\left(\frac{n}{p_{i_1} p_{i_2}}\right) + \ldots\right\}, \tag{2.4}$$

which is more suitable for numerical computation than (2.2). In particular [3])

$$f(p^\alpha) = 2f(p^{\alpha-1}) = 2^{\alpha-1}, \tag{2.51}$$

$$f(p^\alpha q^\beta) = 2\,[f(p^\alpha q^{\beta-1}) + f(p^{\alpha-1} q^\beta) - f(p^{\alpha-1} q^{\beta-1})]. \tag{2.52}$$

Let us return to (2.2). Sum both sides of this equation with respect to all values of n for which $\alpha_1 + \alpha_2 + \ldots + \alpha_\nu = k$, a given integer. Here $\alpha_\mu \geqq 0$ and the basis $p_{i_1}, p_{i_2}, \ldots, p_{i_\nu}$ is kept fixed. Then

$$S_{k,\nu} \equiv \sum_{(n)} f(n) = \sum_{(n)} \sum_{d \mid n}{}' f(d). \tag{2.6}$$

The values of d which occur in the last member are all of the form $d = p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \ldots p_{i_\nu}^{\beta_\nu}$ where $\beta_1 + \beta_2 + \ldots + \beta_\nu = \varkappa$, $0 \leqq \varkappa \leqq k-1$. Moreover, every such integer occurs in the sum. Let us collect all terms which have the same value of $\varkappa$. A divisor d with $\varkappa = k-1$ divides exactly ν different values of n. It follows that every such d occurs ν times, and $\Sigma f(d)$ extended over these values equals $\nu S_{k-1,\nu}$. A d with $\varkappa = k-2$ is a divisor of exactly $\binom{\nu}{1} + \binom{\nu}{2} = \binom{\nu+1}{2}$ different values of n and these

[3]) Formula (2.52) was communicated to me by Dr. Marshall Hall who had found it by a different method.

terms contribute $\binom{\nu+1}{2} S_{k-2,\nu}$. We can prove by complete induction that

$$S_{k,\nu} = \binom{\nu}{1} S_{k-1,\nu} + \binom{\nu+1}{2} S_{k-2,\nu} + \ldots + \binom{\nu+k-1}{k} S_{0,\nu}. \tag{2.7}$$

Let us now put

$$S_\nu(z) = \sum_{k=0}^{\infty} S_{k,\nu} z^k, \quad \nu = 1, 2, 3, \ldots \tag{2.8}$$

Using (2.7) we see that

$$1 + (1-z)^{-\nu} S_\nu(z) = 2\,S_\nu(z),$$

whence

$$S_\nu(z) = \frac{(1-z)^\nu}{2(1-z)^\nu - 1}. \tag{2.9}$$

$S_\nu(z)$ has ν simple poles at $z = z_{\mu,\nu} = 1 - 2^{-1/\nu}\omega^\mu$ where $\omega = e^{2\pi i/\nu}$, $\mu = 0, 1, 2, \ldots, \nu-1$. Expanding the corresponding principal parts in geometric series and adding, we get

$$\begin{cases} S_{0,\nu} = 1, \\ S_{k,\nu} = \dfrac{1}{\nu} 2^{-1-1/\nu} \displaystyle\sum_{\mu=0}^{\nu-1} \omega^\mu (1 - 2^{-1/\nu}\omega^\mu)^{-k-1}, \quad k \geqq 1. \end{cases} \tag{2.10}$$

In this sum the term corresponding to $\mu = 0$ dominates all the rest. Indeed, putting $2^{-1/\nu} = R$, we have for $\nu > 1$, $0 < \mu < \frac{\nu}{2}$,

$$|1 - R\omega^\mu| > R\,|1 - \omega^\mu| = 2R \sin\frac{\mu\pi}{\nu} > 4R\frac{\mu}{\nu} > 2\sqrt{2}\,\frac{\mu}{\nu},$$

$$1 - R < \frac{\log 2}{\nu},$$

whence

$$\begin{cases} S_{k,\nu} = \dfrac{1}{\nu} 2^{-1-1/\nu} (1 - 2^{-1/\nu})^{-k-1} [1 + \eta_k], \\ |\eta_k| < 4^{-k}. \end{cases} \qquad (k \geqq 1,\ \nu > 1) \tag{2.11}$$

It follows in particular that

$$\begin{cases} f(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \ldots p_{i_\nu}^{\alpha_\nu}) < \dfrac{1}{\nu} (1 - 2^{-1/\nu})^{-k-1}, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_\nu = k, \quad k = 0, 1, 2, \ldots, \quad \nu = 1, 2, 3, \ldots \end{cases} \tag{2.12}$$

It seems likely that this estimate is not a very close one for large values of ν.

3. Generating Dirichlet series. Let $f_k(n)$ denote the number of representations of n as the product of k factors, each greater than one when $n>1$, the order of the factors being essential. It is well known that

$$\sum_{n=2}^{\infty} f_k(n)\, n^{-s} = [\zeta(s)-1]^k,\quad k=1,\,2,\,3,\ldots$$

for $\Re(s)=\sigma>1$. Since

$$f(n)=\sum_{k=1}^{\infty} f_k(n),\ f(1)=1,$$

we have

$$(3.1)\qquad \sum_{n=1}^{\infty} f(n)\, n^{-s} = \{2-\zeta(s)\}^{-1}$$

for $\Re(s)>\rho$, where ρ is the positive root the equation (1.4).

Let us now consider ν distinct primes $p_{i_1}<p_{i_2}<\ldots<p_{i_\nu}$, and let P denote the multiplicative system of all integers of the form $p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2}\ldots p_{i_\nu}^{\alpha_\nu}$, where the exponents are non-negative integers. We refer to $p_{i_1}, p_{i_2},\ldots, p_{i_\nu}$ as the basis of P. Put

$$(3.2)\qquad \zeta(s;\ P)=\prod_{\mu=1}^{\nu}[1-p_{i_\mu}^{-s}]^{-1},\ \Re(s)>0.$$

Then

$$(3.3)\qquad \sum_{(P)} f(n)\, n^{-s} = \{2-\zeta(s;P)\}^{-1}\equiv F(s;P),$$

where on the left the summation extends over all integers in P By a well-known theorem of Landau on Dirichlet series with positive coefficients, the series converges for $\Re(s)>\rho(P)$, where $\rho(P)$ is the positive root of the equation

$$(3.4)\qquad \zeta(s,P)=2.$$

Since $\zeta(\sigma;P)$ is monotone decreasing from $+\infty$ to 1 as σ goes from 0 to $+\infty$, this equation has one and only one positive root. We have clearly $\zeta(\sigma;P)<\zeta(\sigma)$ for $\sigma>1$. Hence the former function reaches the value 2 before the latter does, i. e.,

$$(3.5)\qquad 0<\rho(P)<\rho.$$

If $p_{i_1}=2$ and $\nu>1$, $\zeta(1;P)>2$, so that the lower bound 0 can be replaced by 1 in (3.5).

Taking a sequence of multiplicative systems of the type described above, such that $P_1\subset P_2\subset\ldots\subset P_\mu$, where $\subset$ indicates a proper subset, we have obviously

$$\zeta(\sigma;\,P_1)<\zeta(\sigma;\,P_2)<\ldots<\zeta(\sigma;\,P_\mu),\qquad \sigma>0,$$

so that

$$(3.6)\qquad \rho(P_1)<\rho(P_2)<\ldots<\rho(P_\mu)<\rho.$$

If the sequence is infinite and $P_\mu\subset P_{\mu+1}$ for all μ, there exists a unique limiting system P_∞ which contains all systems P_μ and is the smallest multiplicative system of positive integers having this property. Let N be the system of all natural numbers, i. e., the multiplicative system based on all primes. Then $P_\infty\subseteqq N$. Put

$$(3.7)\qquad \zeta(s;P_\infty)=\prod_{\nu=1}^{\infty}[1-p_{i_\nu}^{-s}]^{-1},$$

where the product is extended over the basis of P_∞. Let the abscissa of (absolute) convergence of the product be σ_0, $0\leqq\sigma_0\leqq1$. For $\sigma_0+\varepsilon\leqq\sigma\leqq1/\varepsilon$, $\zeta(\sigma;P_\mu)$ converges uniformly to $\zeta(\sigma;P_\infty)$ as $\mu\to\infty$. Moreover, $\zeta(\sigma;P_\mu)<\zeta(\sigma;P_\infty)$ and for $\sigma>1$ the latter is $\leqq\zeta(\sigma;N)=\zeta(\sigma)$ where the first sign of equality holds if and only if it holds identically. Consequently

$$(3.8)\qquad \rho(P_\mu)\uparrow\rho(P_\infty)\leqq\rho,$$

with obvious notation.

In (3.6) it was assumed that the systems involved had finite bases. But it is clear that $P_1\subset P_2\subset N$ implies

$$(3.9)\qquad \rho(P_1)<\rho(P_2)<\rho(N)=\rho,$$

whether or not the bases are finite.

4. The summatory functions. Let P be a multiplicative system of the type described above, the basis being finite or infinite. The well known relation in the theory of Dirichlet series betwen the partial sums of the coefficients and the abscissa of convergence shows that

$$(4.1)\qquad \sum_{(P)}^{m\leqq n} f(m)=O[n^{\rho(P)+\varepsilon}]$$

for every positive ε. Here the summation is extended over all integers m in P which are $\leqq n$. We can get much more precise results, however.

Suppose first that the basis consists of a single prime p. Then by (2.51)

$$\sum_{p^\alpha \leqq n} f(p^\alpha) = B(n)\, n^{\rho(P)}, \quad \rho(P) = \frac{\log 2}{\log p}, \tag{4.2}$$

where $\frac{1}{2} \leqq B(n) \leqq 1$. We note that the corresponding generating Dirichlet series $F(s; P)$ has infinitely many poles on the line $\sigma = \rho(P)$.

If $1 < \nu \leqq \infty$, the situation is different. In order that $\zeta(\sigma + it; P) = 2$ for $\sigma = \rho(P)$ it is necessary as well as sufficient that $p_{i_\mu}^{-s}$ is positive for all values of μ. For a $t \neq 0$, this condition contradicts the linear independence of the logarithms of the prime numbers. Hence $F(s, P)$ has a single pole on the line $\sigma = \rho(P)$ at $s = \rho(P)$. This pole is simple and the residue obviously equals $-[\zeta'(\rho(P); P)]^{-1}$. There are of course infinitely many poles in any strip $\rho(P) - \delta \leqq \sigma \leqq \rho(P)$ for every $\delta > 0$ owing to the almost periodic character of $\zeta(s; P)$ in such a strip, but the only thing we need to know about these poles is the fact that there is only one of them on the right boundary of the strip. It follows that the hypotheses of the Ikehara-Wiener theorem[4]) are satisfied so that

$$\sum_{(P)}^{m \leqq n} f(m) = -\frac{n^{\rho(P)}}{\rho(P)\, \zeta'(\rho(P); P)} \{1 + o(1)\}. \tag{4.3}$$

Here P is any multiplicative system whose basis contains at least two primes. In particular, we might take $P = N$ in which case we obtain Kalmár's formula (1.5).

5. Estimates of $f(n)$ in terms of n. It follows from (4.3) that there exists a positive $C_1(P)$ such that

$$f(n) < C_1(P)\, n^{\rho(P)}, \quad n \varepsilon P. \tag{5.1}$$

If ν is finite we can also get a converse inequality. The number of terms on the left hand side of (4.3) is then equal to the number of solutions of the inequality

$$\alpha_1 \log p_{i_1} + \alpha_2 \log p_{i_2} + \ldots + \alpha_\nu \log p_{i_\nu} \leqq \log n \tag{5.2}$$

in non-negative integers. This number is obviously $O[(\log n)^\nu]$. Suppose that n is so chosen that $f(n)$ is the largest term of the sum in (4.3). This will happen for infinitely many values of n. Hence there exists a positive $C_2(P)$ such that

[4]) See, e. g., N. Wiener, The Fourier integral, Cambridge, 1933, pp. 127—130.

$$f(n) > C_2(P)\, (\log n)^{-\nu}\, n^{\rho(P)}, \quad n \varepsilon P, \tag{5.3}$$

for infinitely many values of n in P. But here we can replace ν by $\nu - 1$ and if $\nu > 1$ we can take $C_2(P)$ as large as we please. The case $\nu = 1$ is already settled by formula (2.51) so we can take $\nu > 1$. Suppose that it has been shown that for some choice of μ and τ

$$f(n) = o[(\log n)^{-\mu}\, n^\tau]$$

for n in P, $n \to \infty$. We have then also

$$\sum_{(P)}^{m \leqq n}{}' f(m) = o\left\{ \sum_{(P)}^{m \leqq n}{}' (\log m)^{-\mu}\, m^\tau \right\}$$

$$= o\left\{ \sum [\alpha_1 \log p_{i_1} + \ldots + \alpha_\nu \log p_{i_\nu}]^{-\mu} [p_{i_1}^{\alpha_1} \ldots p_{i_\nu}^{\alpha_\nu}]^\tau \right\},$$

where the summation here and below extends over all positive α's satisfying (5.2). If μ is taken to be positive, the relation between the arithmetic and the geometric means shows that the last sum is bounded above by some multiple of

$$\sum (\alpha_1 \ldots \alpha_\nu)^{-\frac{\mu}{\nu}} [p_{i_1}^{\alpha_1} \ldots p_{i_\nu}^{\alpha_\nu}]^\tau.$$

Summing for α_ν we get an expression which is of the order of magnitude of

$$n^\tau \sum (\alpha_1 \ldots \alpha_{\nu-1})^{-\frac{\mu}{\nu}} (\log n - \alpha_1 \log p_{i_1} - \ldots - \alpha_\nu \log p_{i_\nu})^{-\frac{\mu}{\nu}},$$

and if $\mu < \nu$ the sum is of the order of magnitude of the integral

$$\int_S [x_1 \ldots x_{\nu-1} (\log n - x_1 \log p_{i_1} - \ldots - x_{\nu-1} \log p_{i_{\nu-1}})]^{-\frac{\mu}{\nu}}\, dS$$

taken over that portion of the $(\nu - 1)$-dimensional space in which all factors of the bracket are positive. A simple calculation shows that this integral is $O[(\log n)^{\nu - 1 - \mu}]$. Thus, taking $\tau = \rho(P)$ we get

$$\sum_{(P)}^{m \leqq n} f(m) = o[(\log n)^{\nu - 1 - \mu}\, n^{\rho(P)}]. \tag{5.4}$$

Strictly speaking, the summation should extend over only those integers in P which have exactly ν distinct prime factors, but formula

(3.6) shows that the added terms do not disturb the estimate. But (5.4) contradicts (4.3) unless $\mu < \nu - 1$. It follows that

(5 5) $$f(n) > M(\log n)^{-\nu+1}\, n^{\rho(P)}$$

for an arbitrarily large M and for infinitely many values of n in P, provided $\nu > 1$. It is possible that this estimate could be still further improved. But it is good enough to show that if ρ is the root of equation (1.4) and δ is a fixed, arbitrarily small positive number then the inequality

(5.6) $$f(n) > n^{\rho-\delta}$$

holds for infinitely many values of n.

6. Generalizations. The previous discussion admits of very considerable extensions. In § 2 we considered a number theoretical function $f(n)$ satisfying the functional equation (2.2) with the initial conditions (2.3). In the discussion we have used only the divisibility properties of the natural numbers. It is clear that the results will remain unchanged if we replace the natural numbers by any other system having similar divisibility properties. Consider the set of integral ideals $\mathfrak{a}$ in a commutative ring R without divisors of zero. We suppose that the finite chain condition is satisfied, that the prime ideals, except the null ideal, have no proper divisors, and that R is integrally closed with respect to its quotient field. In this case every ideal in R has a unique representation as product of prime ideals $\mathfrak{p}$, the conditions mentioned being necessary and sufficient for unique factorization. We can then define $\mathfrak{f}(\mathfrak{a})$ as the number of representations of $\mathfrak{a}$ as product of ideals, omitting powers of the unit ideal $\mathfrak{o}$, i. e., R considered as an ideal, two representations being considered equal if and only if they involve the same factors written in the same order. We have $\mathfrak{f}(\mathfrak{o}) = \mathfrak{f}(\mathfrak{p}) = 1$. Further (2.2) is satisfied, and it is clear that

(6.1) $$\mathfrak{f}(\mathfrak{p}_{i_1}^{e_1}\mathfrak{p}_{i_2}^{e_2}\ldots\mathfrak{p}_{i_\nu}^{e_\nu}) = f(2^{e_1}3^{e_2}\ldots p_\nu^{e_\nu}).$$

In order to extend the discussion of §§ 3, 4, and 5 we need a definition of the absolute value of an ideal, $|\mathfrak{a}|$, such that

(6.2) $$|\mathfrak{a}\mathfrak{b}| = |\mathfrak{a}|\;|\mathfrak{b}|,$$

and that it is possible to form an analog of the zeta function

(6.3) $$\zeta_R(s) = \sum_{(\mathfrak{a})} |\mathfrak{a}|^{-s}.$$

The convergence of such a series imposes certain restrictions on R and on the choice of the absolute value. Thus, only a finite number of ideals in R can have the same absolute value, and $|\mathfrak{a}| > 1 + \delta$ for some fixed $\delta > 0$ if $\mathfrak{a} \neq \mathfrak{o}$.

These conditions are of course satisfied in the classical case of integral ideals in an algebraic field if we take $|\mathfrak{a}| = N\mathfrak{a}$, i. e., the number of residue classes in R modulo $\mathfrak{a}$. (6.3) is then simply the Dedekind zeta function. It is known, however, that in a ring having unique factorization of ideals it is always possible to introduce an evaluation (= Bewertung), for the elements as well as for the ideals, based upon a norm $\|\alpha\|$ which satisfies the conditions of Kürschák, one of which is (6.2). But the existence of a zeta function is in general not ensured in an evaluation ring. A noteworthy exception is given by the fields obtained by a finite algebraic extension of the field of all rational functions of an indeterminate with coefficients modulo a prime p. The number theory of such fields, which goes back to Dedekind, has been developed in recent yeurs by Artin, Hasse, F. K. Schmidt and others.[5])

In these two cases we can obtain estimates of the summatory function $\Sigma\,\mathfrak{f}(\mathfrak{a})$ where the summation extends over all ideals in the ring the absolute values of which do not exceed a given integer. This follows from the formula

(6.4) $$\sum_{(\mathfrak{a})} \mathfrak{f}(\mathfrak{a})\,|\mathfrak{a}|^{-s} = \{2 - \zeta_R(s)\}^{-1}.$$

In the case of algebraic fields the Ikehara-Wiener theorem applies and gives the analog of Kalmár's estimate; in the case of characteristic p we are dealing with a rational function of p^s. We can also obtain estimates of $\mathfrak{f}(\mathfrak{a})$ itself by restricting the summation to those ideals of absolute value less than a given integer which involve a given sub-set of prime ideals. The necessary generating Dirichlet series can be formed as in § 3 and the discussion of §§ 4 and 5 is easily carried over. In the congruence case we are still dealing with rational functions of an exponential function (at least as long as the basis is finite), and the same is true in the Dedekind case if the norms of the prime ideals in the basis are all powers of the same rational prime. In the general Dedekind case the Ikehara-Wiener theorem still gives the estimates.

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[5]) See, e. g., F. K. Schmidt, Analytische Zahlentheorie in Körpern der Charakteristik p, Math. Zeitschrift, 33 (1931) 1—32. I am indebted to Profs. Ö. Ore and M. Zorn for calling my attention to this possibility and for explications.

It should be noted, however, that the existence of a zeta function for a sub-set of finite basis is independent of the existence of such a function for the whole set. In order to define an analog of $\zeta(s; P)$ we only need to have $|\alpha| > 1$ for $\alpha \neq 0$ in the sub-set. It follows that it is possible to obtain estimates for $f(\alpha)$ in terms of $|\alpha|$ in other cases than those mentioned above.

Finally it is possible to extend some of these considerations to the case of non-commutative rings.

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Verallgemeinerung einer Mordellschen Beweismethode in der Geometrie der Zahlen.

Zweite Mitteilung.

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Satz 7. *Ist M eine im n-dimensionalen Raum liegende Menge vom Volumen $V > k_0 k_1 \dots k_n$, wo $k_0, k_1, \dots, k_n$ positive Zahlen bedeuten, und hat jedes zu M gehörige Punktepaar $(u_1, \dots, u_n)$ und $(u_1', \dots, u_n')$ die Eigenschaft, dass der Punkt $\left(\frac{u_1 - u_1'}{k_1}, \dots, \frac{u_n - u_n'}{k_n}\right)$ einer gewissen Menge N angehört, so enthält N ausser dem Koordinatenursprung mehr als $k_0 - 1$ verschiedene Gitterpunkte $(v_1, \dots, v_n)$, die der Bedingung genügen, dass die erste nicht verschwindende der Zahlen $v_1, \dots, v_n$ positiv ist.*

Der Beweis ist genau dem in der ersten Mitteilung[1]) gegebenen Beweis von Satz 1 analog und verläuft wie folgt: Ist A_l für ganzes $l > 0$ die Anzahl der Punkte $\left(\frac{k_1 u_1}{l}, \dots, \frac{k_n u_n}{l}\right)$ von M mit ganzen $u_1, \dots, u_n$, so strebt $\frac{A_l}{l^n}$ bei unbeschränkt wachsendem l nach $\frac{V}{k_1 k_2 \dots k_n}$. Wegen $V > k_0 k_1 \dots k_n$ ist somit $A_l > k_0 l^n$ bei hinreichend grossem l. Die betrachteten Punkte $(u_1, \dots, u_n)$ gehören zu höchstens l^n verschiedenen Restklassen mod l, so dass wenigstens eine dieser Restklassen mindestens m verschiedene dieser A_l Punkte enthält; hierbei ist m die kleinste

[1]) Acta Arithmetica, **1** (1935), S. 62—66.