## ON THE BLOWUP FOR THE  $L^2$ -CRITICAL FOCUSING NONLINEAR SCHRÖDINGER EQUATION IN HIGHER DIMENSIONS BELOW THE ENERGY CLASS

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ABSTRACT. We consider the focusing mass-critical nonlinear Schrödinger equation and prove that blowup solutions to this equation with initial data in  $H^{s}(\mathbb{R}^{d}), s > s_{0}(d)$  and  $d \geq 3$ , concentrate at least the mass of the ground state at the blowup time. This extends recent work by J. Colliander, S. Raynor, C. Sulem, and J. D. Wright, [\[13\]](#page-20-0), T. Hmidi and S. Keraani, [\[21\]](#page-20-1), and N. Tzirakis, [\[36\]](#page-20-2), on the blowup of the two-dimensional and one-dimensional masscritical focusing NLS below the energy space to all dimensions  $d \geq 3$ .

#### 1. Introduction

<span id="page-0-0"></span>We consider the initial value problem for the focusing  $L_x^2$ -critical nonlinear Schrödinger equation

(1.1) 
$$
\begin{cases} iu_t + \Delta u = -|u|^{\frac{4}{d}}u \\ u(0, x) = u_0(x) \in H^s(\mathbb{R}^d), s \ge 0, \end{cases}
$$

where  $u(t, x)$  is a complex-valued function in spacetime  $\mathbb{R} \times \mathbb{R}^d$ ,  $d \geq 3$ .

It is well known (see, for example, [\[6\]](#page-19-0)) that the Cauchy problem [\(1.1\)](#page-0-0) is locally wellposed in  $H^s(\mathbb{R}^d)$  for  $s \geq 0$ . Moreover, the unique solution obeys conservation of mass:

<span id="page-0-1"></span>
$$
M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M(u_0).
$$

If  $s \geq 1$ , the *energy* is also finite and conserved:

$$
E(u(t)) := \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u(t,x)|^2 - \frac{d}{2(d+2)} |u(t,x)|^{2+\frac{4}{d}}\right) dx = E(u_0).
$$

This equation has a natural scaling. More precisely, the map

(1.2) 
$$
u(t,x) \mapsto u_{\lambda}(t,x) := \lambda^{\frac{d}{2}} u(\lambda^2 t, \lambda x)
$$

maps a solution to  $(1.1)$  to another solution to  $(1.1)$ . The reason why this equation is called  $L_x^2$ -critical (or mass-critical) is because the scaling [\(1.2\)](#page-0-1) also leaves the mass invariant.

Equation [\(1.1\)](#page-0-0) is subcritical for  $s > 0$ . In this case, (1.1) is wellposed in  $H^s(\mathbb{R}^d)$ and the lifespan of the local solution depends only on the  $H^s_x$ -norm of the initial data (see [\[6\]](#page-19-0)). Denote by  $T^* > 0$  the maximal forward time of existence. As

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a consequence of the local well-posedness theory, we have the following blowup criterion:

either 
$$
T^* = \infty
$$
 or  $T^* < \infty$  and  $\lim_{t \to T^*} ||u(t)||_{H^s_x} = \infty$ .

The blowup behavior for solutions from  $H_x^1$  initial data has received a lot of attention. The results are closely related to the ground state Q which is the unique positive radial solution to the elliptic equation

<span id="page-1-0"></span>
$$
\Delta Q - Q + |Q|^{\frac{4}{d}}Q = 0.
$$

,

Using the sharp Gagliardo-Nirenberg inequality (see [39]),

(1.3) 
$$
||u||_{L_x^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_d ||u||_{L_x^2}^{\frac{4}{d}} ||\nabla u||_{L_x^2}^2 \quad \text{with} \quad C_d := \frac{d+2}{d ||Q||_{L_x^2}^{\frac{4}{d}}}
$$

it is not hard to see that the mass of the ground state is the minimal mass required for the solution to develop a singularity. Indeed, in the case  $||u_0||_{L_x^2} < ||Q||_{L_x^2}$ , [\(1.3\)](#page-1-0) combined with the conservation of energy imply that the solution to [\(1.1\)](#page-0-0) is global. This is sharp since the pseudoconformal invariance of the equation [\(1.1\)](#page-0-0) allows us to built a solution with mass equal to that of the ground state that blows up at time  $T^*$ :

$$
u(t,x) := |T^* - t|^{-\frac{d}{2}} e^{[i(T^* - t)^{-1} - i|x|^2(T^* - t)^{-1}]} Q\left(\frac{x}{T^* - t}\right).
$$

Moreover, F. Merle,  $[28]$ , showed that up to the symmetries of  $(1.1)$ , this is the only blowup solution with minimal mass. Furthermore, any blowup solution must concentrate at least the mass of the ground state at the blowup time; more precisely, as shown in [\[31\]](#page-20-4), there exists  $x(t) \in \mathbb{R}^d$  such that

<span id="page-1-2"></span>(1.4) 
$$
\forall R > 0, \ \lim_{t \to T^*} \int_{|x - x(t)| \le R} |u(t,x)|^2 dx \ge \int_{\mathbb{R}^d} Q^2 dx.
$$

Of course, the goal is to establish all these properties for blowup solutions from data in  $L_x^2$  rather than  $H_x^1$ . Unfortunately, all the methods used in the  $H_x^1$  setting break down at the  $L_x^2$  level. Moreover, as [\(1.1\)](#page-0-0) is  $L_x^2$ -critical, even the local well-posedness theory in  $L_x^2$  is substantially different from that in  $H_x^s$  for  $s > 0$ . Specifically, the lifespan of the local solution depends on the profile of the initial data, rather than on its  $L_x^2$ -norm (see [\[6\]](#page-19-0)). In particular, this leads to the following blowup criterion:

either 
$$
T^* = \infty
$$
 or  $T^* < \infty$  and  $||u||_{L^{2+\frac{4}{d}}_{t,x}([0,T^*)\times \mathbb{R}^d)} = \infty$ .

From the global theory for small data (see [\[6\]](#page-19-0)), we know that if the mass of the initial data is sufficiently small, then there exists a unique global solution to [\(1.1\)](#page-0-0). However, for large (but finite) mass initial data, that is also sufficiently smooth and decaying, the viriel identity guarantees that finite time blowup occurs; see, [\[18,](#page-20-5) [42\]](#page-21-0).

<span id="page-1-1"></span>The first blowup result for general  $L_x^2$  initial data belongs to J. Bourgain, [\[2\]](#page-19-1), who proved the following parabolic concentration of mass at the blowup time:

(1.5) 
$$
\lim_{t \nearrow T^*} \sup_{\text{cubes } I \subset \mathbb{R}^2 \atop \text{side}(I) < (T^* - t)^{\frac{1}{2}}} \left( \int_I |u(t, x)|^2 \, dx \right)^{\frac{1}{2}} \ge c(||u_0||_{L_x^2}) > 0,
$$

where  $c(\|u_0\|_{L_x^2})$  is a small constant depending on the mass of the initial data. This result was extended to dimension  $d = 1$  by S. Keraani, [\[25\]](#page-20-6), and to dimensions  $d \geq 3$ by P. Begout and A. Vargas, [\[1\]](#page-19-2). The conjecture is that rather that  $c(\|u_0\|_{L_x^2})$  on the right-hand side of [\(1.5\)](#page-1-1), one should have the mass of the ground state as in [\(1.4\)](#page-1-2); however, this appears to be a very difficult problem.

The goal of this paper is to reproduce as much of the  $H_x^1$  theory as we can at lower regularity (but unfortunately, well above  $L_x^2$ ) in dimensions three and higher. In two dimensions, this was pursued by J. Colliander, S. Raynor, C. Sulem, and J. D. Wright, [\[13\]](#page-20-0); using the I-method (introduced by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, cf. [\[10\]](#page-20-7)), they established a mass concentration property (at the blowup time) for radial blowup solutions from initial data in  $H^s(\mathbb{R}^2)$  with  $s > \frac{1+\sqrt{11}}{5}$ :

<span id="page-2-0"></span>(1.6) 
$$
\limsup_{t \nearrow T^*} \|u\|_{L^2(|x| < (T^* - t)^{\frac{s}{2}} \gamma(T^* - t))} \ge \|Q\|_{L^2_x},
$$

where  $\gamma(s) \to \infty$  arbitrarily slowly as  $s \to 0$ . While the right-hand side in [\(1.6\)](#page-2-0) is the conjectured one, we should remark that the width of concentration, that is,  $(T^* - t)^{\frac{s}{2}+}$  is larger than the expected concentration width,  $(T^* - t)^{\frac{1}{2}+}$ . Moreover, the lim sup in [\(1.6\)](#page-2-0) is not present in the  $H_x^1$  theory (see [\(1.4\)](#page-1-2)); the reason for its appearance is basically the lack of information on the blowup rate of the  $H_x^s$ -norm.

T. Himidi and S. Keraani, [\[21\]](#page-20-1), removed the radial assumption used in [\[13\]](#page-20-0). Moreover, they showed that, up to symmetries of the equation, the ground state Q is the profile for blowup solutions with minimal mass and initial data in  $H^s(\mathbb{R}^2)$ with  $s > \frac{1+\sqrt{11}}{5}$ . We will make use of the key innovation of their work, namely Lemma [2.11](#page-11-0) below.

In one dimension, recent work by N. Tzirakis, [\[36\]](#page-20-2), established the analogue of [\(1.6\)](#page-2-0) for arbitrary initial data in  $H^s(\mathbb{R}), s > \frac{10}{11}$ .

Our contribution is to treat (non-radial) data in  $H<sup>s</sup>(\mathbb{R}^d)$  for  $s > s_0(d)$  where

(1.7) 
$$
s_0(d) := \begin{cases} \frac{1+\sqrt{13}}{5}, & \text{for } d = 3, \\ \frac{8-d+\sqrt{9d^2+64d+64}}{2(d+10)}, & \text{for } d \ge 4. \end{cases}
$$

d

<span id="page-2-1"></span>We prove

**Theorem 1.1.** Assume  $d \geq 3$  and  $s > s_0(d)$ . Let  $u_0 \in H^s(\mathbb{R}^d)$  such that the corresponding solution u to [\(1.1\)](#page-0-0) blows up at time  $0 < T^* < \infty$ . Then, there exists a function  $V \in H_x^1$  such that  $||V||_2 \geq ||Q||_2$  and there exist sequences  $\{t_n, \rho_n, x_n\}_{n\geq 1} \subset \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}^d$  satisfying

 $t_n \nearrow T^*$  as  $n \to \infty$  and  $\rho_n \lesssim (T^* - t_n)^{\frac{s}{2}}, \quad \forall n \geq 1$ 

such that

$$
\rho_n^{\frac{\alpha}{2}}u(t_n,\rho_n\cdot+x_n)\rightharpoonup V\quad\text{weakly}\quad\text{as }n\to\infty.
$$

<span id="page-2-2"></span>As a consequence of Theorem [1.1,](#page-2-1) we establish the following mass concentration property for blowup solutions:

**Theorem 1.2.** Assume  $d \geq 3$  and  $s > s_0(d)$ . Let  $u_0 \in H^s(\mathbb{R}^d)$  such that the corresponding solution u to [\(1.1\)](#page-0-0) blows up at time  $0 < T^* < \infty$ . Let  $\alpha(t) > 0$  be such that s

$$
\lim_{t \nearrow T^*} \frac{(T^* - t)^{\frac{s}{2}}}{\alpha(t)} = 0.
$$

Then, there exists  $x(t) \in \mathbb{R}^d$  such that

$$
\limsup_{t\nearrow T^*}\int_{|x-x(t)|\leq \alpha(t)}|u(x,t)|^2dx\geq \int_{\mathbb{R}^d}Q^2dx.
$$

<span id="page-3-0"></span>Under the additional hypothesis that the mass of the initial data equals the mass of the ground state, we may upgrade Theorem [1.1](#page-2-1) to the following:

**Theorem 1.3.** Assume  $d \geq 3$  and  $s > s_0(d)$ . Let  $u_0 \in H^s(\mathbb{R}^d)$  with  $||u_0||_2 = ||Q||_2$ such that the corresponding solution u to [\(1.1\)](#page-0-0) blows up at time  $0 < T^* < \infty$ . Then, there exist sequences  $\{t_n, \theta_n, \rho_n, x_n\}_{n\geq 1} \subset \mathbb{R}_+ \times S^1 \times \mathbb{R}_+^* \times \mathbb{R}^d$  satisfying

$$
t_n \nearrow T^*
$$
 as  $n \to \infty$  and  $\rho_n \lesssim (T^* - t_n)^{\frac{s}{2}}, \forall n \geq 1$ 

such that

$$
\rho_n^{\frac{d}{2}} e^{i\theta_n} u(t_n, \rho_n x + x_n) \to Q \quad strongly \ in \ H_x^{\tilde{s}^-},
$$

where

$$
\tilde{s} := \frac{2d + 8s + s^2d(2 - \min\{1, \frac{4}{d}\})}{4d + 16s - s^2(d\min\{1, \frac{4}{d}\} + 8)}.
$$

In a nutshell, Theorem [1.3](#page-3-0) says that up to the symmetries for [\(1.1\)](#page-0-0), the ground state is the profile for blowup solutions with minimal mass and initial data in  $H_x^s$ ,  $s > s_0(d)$ . Alas, we only show this is true along a sequence of times.

To prove Theorems [1.1](#page-2-1) through [1.3,](#page-3-0) we will rely on the I-method and Lemma [2.11.](#page-11-0) The idea behind the *I-method* is to smooth out the initial data in order to access the theory available at  $H_x^1$  regularity. To this end, one introduces the Fourier multiplier  $I$ , which is the identity on low frequencies and behaves like a fractional integral operator of order  $1 - s$  on high frequencies. Thus, the operator I maps  $H_x^s$  to  $H_x^1$ . However, even though we do have energy conservation for [\(1.1\)](#page-0-0), Iu is not a solution to [\(1.1\)](#page-0-0) and hence, we expect an energy increment. The key is to prove that on intervals of local well-posedness, the modified energy  $E(Iu)$  is an 'almost conserved' quantity and grows much slower than the modified kinetic energy  $\|\nabla Iu\|_{L_x^2}^2$ . This requires delicate estimates on the commutator between I and the nonlinearity. In dimensions one and two, the nonlinearity is algebraic and one can write the commutator explicitly using the Fourier transform and control it by multilinear analysis and bilinear estimates (see [\[13,](#page-20-0) [36\]](#page-20-2)). However, in dimensions  $d \geq 3$  this method fails. Instead, we will have to rely on more rudimentary tools such as Strichartz and fractional chain rule estimates in order to control the commutator.

The remainder of this paper is organized as follows: In Section 2, we introduce notation and prove some lemmas that will be useful. In Section 3, we revisit the  $H_x^s$ local well-posedness theory for [\(1.1\)](#page-0-0). Section 4 is devoted to controlling the modified energy increment. In Sections 5 through 7 we prove Theorems [1.1](#page-2-1) through [1.3.](#page-3-0)

After this work was submitted, we were informed of an independent paper attacking the same problem, [\[14\]](#page-20-8). However, there appear to be several gaps in their argument, for example, in estimating (4.19). While it seems possible to remedy their errors, this would result in a larger value for  $s_0(d)$  than that claimed in their paper, which is already inferior to that given here. We contend that all overlap between this paper and [\[14\]](#page-20-8) can be attributed to the precursors, [\[13\]](#page-20-0) and [\[21\]](#page-20-1).

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### 2. Preliminaries

We will often use the notation  $X \leq Y$  whenever there exists some constant C so that  $X \le CY$ . Similarly, we will use  $X \sim Y$  if  $X \le Y \le X$ . We use  $X \ll Y$  if  $X \leq cY$  for some small constant c. The derivative operator  $\nabla$  refers to the space variable only. We use  $A \pm \infty$  denote  $A \pm \varepsilon$  for any sufficiently small  $\varepsilon > 0$ ; the implicit constant in an inequality involving this notation is permitted to depend on ε.

We use  $L_x^r(\mathbb{R}^d)$  to denote the Banach space of functions  $f : \mathbb{R}^d \to \mathbb{C}$  whose norm

$$
||f||_r := \left(\int_{\mathbb{R}^d} |f(x)|^r dx\right)^{\frac{1}{r}}
$$

is finite, with the usual modifications when  $r = \infty$ .

We use  $L_t^q L_x^r$  to denote the spacetime norm

$$
||u||_{q,r} := ||u||_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} := \Bigl(\int_{\mathbb{R}} \Bigl(\int_{\mathbb{R}^d} |u(t,x)|^r dx\Bigr)^{q/r} dt\Bigr)^{1/q},
$$

with the usual modifications when either  $q$  or  $r$  are infinity, or when the domain  $\mathbb{R} \times \mathbb{R}^d$  is replaced by some smaller spacetime region. When  $q = r$  we abbreviate  $L_t^q L_x^r$  by  $L_{t,x}^q$ .

We define the Fourier transform on  $\mathbb{R}^d$  to be

$$
\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi ix \cdot \xi} f(x) dx.
$$

We will make use of the fractional differentiation operators  $|\nabla|^s$  defined by

$$
\widehat{|\nabla|^s f}(\xi) := |\xi|^s \widehat{f}(\xi).
$$

These define the homogeneous Sobolev norms

$$
\|f\|_{\dot H^s_x}:=\||\nabla|^s f\|_{L^2_x}
$$

and the more common inhomogeneous Sobolev norms

$$
||f||_{H_x^{s,p}} := ||\langle \nabla \rangle^s f||_{L_x^p}, \quad \text{where} \quad \langle \nabla \rangle := (1 + |\nabla|^2)^{\frac{1}{2}}.
$$

We will often denote  $H_x^{s,2}$  by  $H_x^s$ .

Let  $F(z) := -|z|^{\frac{4}{d}} z$  be the function that defines the nonlinearity in [\(1.1\)](#page-0-0). Then,

$$
F_z(z) := \frac{\partial F}{\partial z}(z) = -\frac{2+d}{d}|z|^{\frac{4}{d}} \quad \text{and} \quad F_{\bar{z}}(z) := \frac{\partial F}{\partial \bar{z}}(z) = -\frac{2}{d}|z|^{\frac{4}{d}}\frac{z}{\bar{z}}.
$$

We write F' for the vector  $(F_z, F_{\bar{z}})$  and adopt the notation

$$
w \cdot F'(z) := w F_z(z) + \overline{w} F_{\overline{z}}(z).
$$

In particular, we observe the chain rule

$$
\nabla F(u) = \nabla u \cdot F'(u).
$$

<span id="page-5-1"></span>Clearly  $F'(z) = O(|z|^{\frac{4}{d}})$  and we have the Hölder continuity estimate

$$
(2.1) \t\t\t |F'(z) - F'(w)| \lesssim |z - w|^{\min\{1, \frac{4}{d}\}} (|z| + |w|)^{\frac{4}{d} - \min\{1, \frac{4}{d}\}}
$$

for all  $z, w \in \mathbb{C}$ . By the Fundamental Theorem of Calculus,

$$
F(z + w) - F(z) = \int_0^1 w \cdot F'(z + \theta w) d\theta
$$

and hence

$$
F(z + w) = F(z) + O(|w||z|^{\frac{4}{d}}) + O(|w|^{\frac{4+d}{d}})
$$

for all complex values z and w.

Let  $e^{it\Delta}$  be the free Schrödinger propagator. In physical space this is given by the formula

$$
e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) dy
$$

for  $t \neq 0$  (using a suitable branch cut to define  $(4\pi i t)^{d/2}$ ), while in frequency space one can write this as

(2.2) 
$$
\widehat{e^{it\Delta}f}(\xi) = e^{-4\pi^2it|\xi|^2}\widehat{f}(\xi).
$$

In particular, the propagator obeys the *dispersive inequality* 

(2.3) 
$$
\|e^{it\Delta} f\|_{L_x^{\infty}} \lesssim |t|^{-\frac{d}{2}} \|f\|_{L_x^1}
$$

for all times  $t \neq 0$ .

We also recall Duhamel's formula

(2.4) 
$$
u(t) = e^{i(t-t_0)\Delta}u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} (iu_t + \Delta u)(s) ds.
$$

**Definition 2.1.** A pair of exponents  $(q, r)$  is called Schrödinger-admissible if

$$
\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \le q, r \le \infty, \quad and \quad (q, r, d) \ne (2, \infty, 2).
$$

Throughout this paper we will use the following admissible pairs:

$$
(2, \frac{2d}{d-2}) \quad \text{and} \quad (\gamma, \rho) := \left(\frac{2(d+2)}{d-2s}, \frac{2d(d+2)}{d^2+4s}\right) \quad \text{with } 0 < s < 1.
$$

<span id="page-5-0"></span>Let  $\rho^* := \frac{2(d+2)}{d-2s}$  $\frac{(a+2)}{d-2s}$ . Using Hölder and Sobolev embedding, on any spacetime slab  $I \times \mathbb{R}^d$  we estimate

$$
(2.5) \t\t\t ||F(u)||_{\gamma',\rho'} \lesssim |I|^{\frac{2s}{d}} \|u\|_{\gamma,\rho} \|u\|_{\gamma,\rho^*}^{\frac{4}{d}} \lesssim |I|^{\frac{2s}{d}} \| \langle \nabla \rangle^s u\|_{\gamma,\rho}^{1+\frac{4}{d}}.
$$

Let  $I \times \mathbb{R}^d$  be a spacetime slab; we define the Strichartz norm

$$
||u||_{S^0(I)} := \sup_{(q,r) \text{ admissible}} ||u||_{L_t^q L_x^r(I \times \mathbb{R}^d)}.
$$

We define the Strichartz space  $S^0(I)$  to be the closure of all test functions under the Strichartz norm  $\|\cdot\|_{S^0(I)}$ . We use  $N^0(I)$  to denote the dual space of  $S^0(I)$ .

We record the standard Strichartz estimates which we will invoke repeatedly throughout this paper (see [\[33,](#page-20-9) [17\]](#page-20-10) for  $(q, r)$  admissible with  $q > 2$  and [\[24\]](#page-20-11) for the endpoint  $(2, \frac{2d}{d-2})$ :

**Lemma 2.2.** Let I be a compact time interval,  $t_0 \in I$ ,  $s \geq 0$ , and let u be a solution to the forced Schrödinger equation

$$
iu_t + \Delta u = \sum_{i=1}^{m} F_i
$$

for some functions  $F_1, \cdots, F_m$ . Then,

$$
(2.6) \t\t\t\t||\nabla|^s u\|_{S^0(I)} \lesssim \|u(t_0)\|_{\dot{H}^s(\mathbb{R}^d)} + \sum_{i=1}^m \||\nabla|^s F_i\|_{L_t^{q_i'} L_x^{r_i'}(I \times \mathbb{R}^d)}
$$

for any admissible pairs  $(q_i, r_i)$ ,  $1 \leq i \leq m$ .

We will also need some Littlewood-Paley theory. Specifically, let  $\varphi(\xi)$  be a smooth bump supported in the ball  $|\xi| \leq 2$  and equalling one on the ball  $|\xi| \leq 1$ . For each dyadic number  $N \in 2^{\mathbb{Z}}$  we define the Littlewood-Paley operators

$$
\widehat{P_{\leq N}f}(\xi) := \varphi(\xi/N)\widehat{f}(\xi),
$$
  

$$
\widehat{P_{>N}f}(\xi) := [1 - \varphi(\xi/N)]\widehat{f}(\xi),
$$
  

$$
\widehat{P_Nf}(\xi) := [\varphi(\xi/N) - \varphi(2\xi/N)]\widehat{f}(\xi).
$$

Similarly we can define  $P_{\leq N}$ ,  $P_{\geq N}$ , and  $P_{M \leq \leq N} := P_{\leq N} - P_{\leq M}$ , whenever M and N are dyadic numbers. We will frequently write  $f_{\leq N}$  for  $P_{\leq N}f$  and similarly for the other operators. We recall the following standard Bernstein and Sobolev type inequalities:

**Lemma 2.3.** For any  $1 \leq p \leq q \leq \infty$  and  $s > 0$ , we have

$$
||P_{\geq N}f||_{L_x^p} \lesssim N^{-s}|||\nabla|^s P_{\geq N}f||_{L_x^p}
$$
  
\n
$$
|||\nabla|^s P_{\leq N}f||_{L_x^p} \lesssim N^s||P_{\leq N}f||_{L_x^p}
$$
  
\n
$$
|||\nabla|^{\pm s} P_Nf||_{L_x^p} \sim N^{\pm s}||P_Nf||_{L_x^p}
$$
  
\n
$$
||P_{\leq N}f||_{L_x^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}}||P_{\leq N}f||_{L_x^p}
$$
  
\n
$$
||P_Nf||_{L_x^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}}||P_Nf||_{L_x^p}.
$$

For  $N > 1$ , we define the Fourier multiplier  $I<sub>N</sub>$  (cf. [\[10\]](#page-20-7)) by

$$
\tilde{I}_N \tilde{u}(\xi) := m_N(\xi) \hat{u}(\xi),
$$

where  $m_N$  is a smooth radial decreasing function such that

$$
m_N(\xi) = \begin{cases} 1, & \text{if} \quad |\xi| \le N \\ \left(\frac{|\xi|}{N}\right)^{s-1}, & \text{if} \quad |\xi| \ge 2N. \end{cases}
$$

Thus,  $I_N$  is the identity operator on frequencies  $|\xi| \leq N$  and behaves like a fractional integral operator of order 1-s on higher frequencies. In particular,  $I_N$  maps  $H^s_x$  to  $H_x^1$ ; this allows us to access the theory available for  $H_x^1$  data. We collect the basic properties of  $I_N$  into the following:

<span id="page-6-3"></span><span id="page-6-1"></span><span id="page-6-0"></span>**Lemma 2.4.** Let  $1 < p < \infty$  and  $0 \le \sigma \le s < 1$ . Then,

$$
||I_N f||_p \lesssim ||f||_p
$$

<span id="page-6-2"></span>(2.8)  $\|\nabla^{\sigma}P_{>N}f\|_{p} \lesssim N^{\sigma-1} \|\nabla I_N f\|_{p}$ 

(2.9)  $||f||_{H_x^s} \lesssim ||I_N f||_{H_x^1} \lesssim N^{1-s} ||f||_{H_x^s}.$ 

Proof. The estimate [\(2.7\)](#page-6-0) is a direct consequence of the multiplier theorem. To prove [\(2.8\)](#page-6-1), we write

$$
\||\nabla|^{\sigma} P_{>N} f\|_p = \|P_{>N} |\nabla|^{\sigma} (\nabla I_N)^{-1} \nabla I_N f\|_p.
$$

The claim follows again from the multiplier theorem.

Now we turn to [\(2.9\)](#page-6-2). By the definition of the operator  $I<sub>N</sub>$  and [\(2.8\)](#page-6-1),

$$
||f||_{H_x^s} \lesssim ||P_{\leq N}f||_{H_x^s} + ||P_{>N}f||_2 + |||\nabla|^s P_{>N}f||_2
$$
  
\n
$$
\lesssim ||P_{\leq N}I_Nf||_{H_x^1} + N^{-1}||\nabla I_Nf||_2 + N^{s-1}||\nabla I_Nf||_2
$$
  
\n
$$
\lesssim ||I_Nf||_{H_x^1}.
$$

On the other hand, since the operator  $I_N$  commutes with  $\langle \nabla \rangle^s$ , we get

$$
||I_N f||_{H^1_x} = ||\langle \nabla \rangle^{1-s} I_N \langle \nabla \rangle^s f||_2 \lesssim N^{1-s} ||\langle \nabla \rangle^s f||_2 \lesssim N^{1-s} ||f||_{H^s_x},
$$

which proves the last inequality in  $(2.9)$ . Note that a similar argument also yields (2.10)  $\|I_N f\|_{\dot{H}^1_x} \lesssim N^{1-s} \|f\|_{\dot{H}^s_x}.$ 

 $\Box$ 

The estimate [\(2.8\)](#page-6-1) shows that we can control the high frequencies of a function f in the Sobolev space  $H_x^{\sigma,p}$  by the smoother function  $I_N f$  in a space with a loss of derivative but a gain of negative power of  $N$ . This fact is crucial in extracting the negative power of N when estimating the increment of the modified Hamiltonian.

In dimensions one and two, one can use multilinear analysis to understand commutator expressions like  $F(I_N u) - I_N F(u)$ ; on the Fourier side, one can expand this commutator into a product of Fourier transforms of  $u$  and  $I_N u$  and carefully measure the frequency interactions to derive an estimate (see for example [\[13,](#page-20-0) [36\]](#page-20-2)). However, this is not possible in dimensions  $d \geq 3$ . Instead, we will have to rely on the following rougher (weaker, but more robust) lemma:

<span id="page-7-2"></span><span id="page-7-1"></span>**Lemma 2.5.** Let  $1 < r, r_1, r_2 < \infty$  be such that  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$  and let  $0 < \nu < s$ . Then,

$$
(2.11) \t\t\t ||I_N(fg) - (I_N f)g||_r \lesssim N^{-(1-s+\nu)}\|I_N f\|_{r_1}\|\langle \nabla \rangle^{1-s+\nu}g\|_{r_2}.
$$

*Proof.* Applying a Littlewood-Paley decomposition to  $f$  and  $g$ , we write  $I_N(fg)–(I_N f)g$ 

$$
= I_N(fg_{\leq 1}) - (I_N f)g_{\leq 1} + \sum_{1 < M \in 2^{\mathbb{Z}}} \left[ I_N(f_{\leq M}g_M) - (I_N f_{\leq M})g_M \right] \\
+ \sum_{1 < M \in 2^{\mathbb{Z}}} \left[ I_N(f_{\geq M}g_M) - (I_N f_{\geq M})g_M \right] \\
= I_N(f_{\geq N}g_{\leq 1}) - (I_N f_{\geq N})g_{\leq 1} + \sum_{N \leq M \in 2^{\mathbb{Z}}} \left[ I_N(f_{\leq M}g_M) - (I_N f_{\leq M})g_M \right] \\
+ \sum_{1 < M \in 2^{\mathbb{Z}}} \left[ I_N(f_{\geq M}g_M) - (I_N f_{\geq M})g_M \right] \\
= I + II + III.
$$

<span id="page-7-0"></span>The second equality above follows from the fact that the operator  $I_N$  is the identity operator on frequencies  $|\xi| \leq N$ ; thus,

$$
I_N(f_{\ll N}g_{\leq 1}) = (I_Nf_{\ll N})g_{\leq 1} \quad \text{and} \quad I_N(f_{\leq M}g_M) = (I_Nf_{\leq M})g_M \quad \text{for all } M \ll N.
$$

We first consider II. Dropping the operator  $I_N$ , by Hölder and Bernstein we estimate

$$
||I_N(f_{\leq M}g_M) - (I_N f_{\leq M})g_M||_r \lesssim ||f_{\leq M}||_{r_1} ||g_M||_{r_2}
$$
  

$$
\lesssim \left(\frac{M}{N}\right)^{1-s} ||I_N f||_{r_1} ||g_M||_{r_2}
$$
  

$$
\lesssim M^{-\nu} N^{-(1-s)} ||I_N f||_{r_1} |||\nabla|^{1-s+\nu} g||_{r_2}.
$$

Summing over all M such that  $N \lesssim M \in 2^{\mathbb{Z}}$ , we get

(2.13) 
$$
II \lesssim N^{-(1-s+\nu)} \|I_N f\|_{r_1} \||\nabla|^{1-s+\nu} g\|_{r_2}.
$$

We turn now towards  $III$ . Applying a Littlewood-Paley decomposition to  $f$ , we write each term in III as

$$
I_N(f_{\gg M}g_M) - (I_N f_{\gg M})g_M = \sum_{1 \ll k \in \mathbb{N}} [I_N(f_{2^k M}g_M) - (I_N f_{2^k M})g_M] = \sum_{\substack{1 \ll k \in \mathbb{N} \\ N \lesssim 2^k M}} [I_N(f_{2^k M}g_M) - (I_N f_{2^k M})g_M].
$$

To derive the second inequality, we used again the fact that the operator  $I_N$  is the identity on frequencies  $|\xi| \leq N$ .

We write

$$
[I_N(f_{2^kM}g_M)-(I_Nf_{2^kM})g_M]\hat{ }(\xi)=\int_{\xi=\xi_1+\xi_2}(m_N(\xi_1+\xi_2)-m_N(\xi_1))\widehat{f_{2^kM}(\xi_1)}\widehat{g_M}(\xi_2).
$$

For  $|\xi_1| \sim 2^k M$ ,  $k \gg 1$ , and  $|\xi_2| \sim M$ , the Fundamental Theorem of Calculus implies

$$
|m_N(\xi_1+\xi_2)-m_N(\xi_1)|\lesssim 2^{-k}\big(\frac{2^kM}{N}\big)^{s-1}.
$$

By the Coifman-Meyer multilinear multiplier theorem, [\[8,](#page-19-3) [9\]](#page-20-12), and Bernstein, we get

$$
||I_N(f_{2^kM}g_M) - (I_Nf_{2^kM})g_M||_r \lesssim 2^{-k} \left(\frac{2^kM}{N}\right)^{s-1} ||f_{2^kM}||_{r_1} ||g_M||_{r_2}
$$
  

$$
\lesssim 2^{-k} M^{-(1-s+\nu)} ||I_Nf||_{r_1} |||\nabla|^{1-s+\nu}g||_{r_2}.
$$

Summing over M and k such that  $N \lesssim 2^k M$ , and recalling that  $0 < \nu < s$ , we get

(2.14) 
$$
III \lesssim N^{-(1-s+\nu)} \|I_N f\|_{r_1} \||\nabla|^{1-s+\nu} g\|_{r_2}.
$$

To estimate I, we apply the same argument as for III. We get

$$
I = \|I_N(f_{\geq N}g_{\leq 1}) - (I_N f_{\geq N})g_{\leq 1}\|_r \lesssim \sum_{k \in \mathbb{N}, 2^k \geq N} \|I_N(f_{2^k}g_{\leq 1}) - (I_N f_{2^k})g_{\leq 1}\|_r
$$
  

$$
\lesssim \sum_{k \in \mathbb{N}, 2^k \geq N} 2^{-k} \|I_N f\|_{r_1} \|g\|_{r_2}
$$
  
(2.15)  

$$
\lesssim N^{-1} \|I_N f\|_{r_1} \|g\|_{r_2}.
$$

<span id="page-8-1"></span><span id="page-8-0"></span>Putting  $(2.12)$  through  $(2.15)$  together, we derive  $(2.11)$ .

As an application of Lemma [2.5](#page-7-2) we have the following commutator estimate:

<span id="page-9-0"></span>**Lemma 2.6.** Let  $1 < r, r_1, r_2 < \infty$  be such that  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ . Then, for any  $0 < \nu < s$  we have

<span id="page-9-1"></span>
$$
(2.16) \qquad \|\nabla I_N F(u) - (I_N \nabla u) F'(u)\|_r \lesssim N^{-1+s-\nu} \|\nabla I_N u\|_{r_1} \|\langle \nabla \rangle^{1-s+\nu} F'(u)\|_{r_2}
$$
\n
$$
(2.17) \qquad \|\nabla I_N F(u)\|_r \lesssim \|\nabla I_N u\|_{r_1} \|F'(u)\|_{r_2} + N^{-1+s-\nu} \|\nabla I_N u\|_{r_1} \|\langle \nabla \rangle^{1-s+\nu} F'(u)\|_{r_2}.
$$
\n
$$
B_{\text{res}} f \quad \text{for}
$$

Proof. As

$$
\nabla F(u) = \nabla u \cdot F'(u),
$$

the estimate [\(2.16\)](#page-9-0) follows immediately from Lemma [2.5](#page-7-2) with  $f := \nabla u$  and  $g :=$  $F'(u)$ . The estimate [\(2.17\)](#page-9-1) is a consequence of [\(2.16\)](#page-9-0) and the triangle inequality.  $\Box$ 

<span id="page-9-3"></span>Since we work at regularity  $0 < s < 1$ , we will need the following fractional chain rules to estimate our nonlinearity in  $H_x^s$ .

**Lemma 2.7** (Fractional chain rule for a  $C^1$  function). Suppose that  $F \in C^1(\mathbb{C})$ ,  $\sigma \in (0,1)$ , and  $1 < r, r_1, r_2 < \infty$  such that  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ . Then,

$$
\||\nabla|^{\sigma} F(u)\|_{r} \lesssim \|F'(u)\|_{r_1} \||\nabla|^{\sigma} u\|_{r_2}.
$$

<span id="page-9-4"></span>**Lemma 2.8** (Fractional chain rule for a Lipschitz function). Let  $F$  be a Lipschitz function,  $\sigma \in (0,1)$ , and  $1 < r < \infty$ . Then,

$$
\||\nabla|^{\sigma} F(u)\|_{r} \lesssim \|F'\|_{\infty} \||\nabla|^{\sigma} u\|_{r}.
$$

<span id="page-9-5"></span>**Lemma 2.9** (Fractional derivatives for fractional powers). Let F be a Hölder continuous function of order  $0 < \alpha < 1$ . Then, for every  $0 < \sigma < \alpha$ ,  $1 < r < \infty$ , and  $\frac{\sigma}{\alpha} < \delta < 1$  we have

(2.18) 
$$
\| |\nabla|^{\sigma} F(u) \|_{r} \lesssim \| |u|^{\alpha - \frac{\sigma}{\delta}} \|_{r_{1}} \| |\nabla|^{\delta} u \|_{\frac{\sigma}{\delta} r_{2}}^{\frac{\sigma}{\delta}},
$$

provided  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$  and  $(1 - \frac{\sigma}{\alpha \delta})r_1 > 1$ .

The first two results originate in [\[7\]](#page-19-4) and [\[32\]](#page-20-13); for a textbook treatment see [\[35\]](#page-20-14). The third result can be found in Appendix A of [\[38\]](#page-21-1). Using the chain rule estimates in these lemmas, we can upgrade the pointwise in time commutator estimate in Lemma [2.6](#page-8-1) to a spacetime estimate:

<span id="page-9-7"></span><span id="page-9-2"></span>**Lemma 2.10.** Let I be a compact time interval and let  $\frac{1}{1+\min\{1,\frac{4}{d}\}} < s < 1$ . Then,

<span id="page-9-6"></span>
$$
(2.19)\quad \| (I_N \nabla u) F'(u) - \nabla I_N F(u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(I \times \mathbb{R}^d)} \lesssim N^{-\min\{1, \frac{4}{d}\} s +} \| \langle \nabla \rangle I_N u \|_{S^0(I)}^{1+\frac{4}{d}}
$$

$$
(2.20) \quad \|\langle \nabla \rangle I_N F(u)\|_{N^0(I)} \lesssim |I|^{\frac{2s}{d}} \|\langle \nabla \rangle I_N u\|_{S^0(I)}^{1+\frac{4}{d}} + N^{-\min\{1,\frac{4}{d}\}s +} \|\langle \nabla \rangle I_N u\|_{S^0(I)}^{1+\frac{4}{d}}.
$$

*Proof.* Throughout the proof, all spacetime norms will be taken on the slab  $I \times \mathbb{R}^d$ .

As by hypothesis  $s > \frac{1}{1 + \min\{1, \frac{4}{d}\}}$ , there exists  $\varepsilon_0 > 0$  such that for any  $0 < \varepsilon < \varepsilon_0$ we have  $s > \frac{1+\varepsilon}{1+\min\{1,\frac{4}{d}\}}$ . Let  $\nu := \min\{1,\frac{4}{d}\}s - (1-s) - \varepsilon$  with  $0 < \varepsilon < \varepsilon_0$ . It is easy to check that  $0 < \nu < s$ . Applying Lemma [2.6](#page-8-1) with this value of  $\nu$ , we get

$$
\begin{split} \| (\nabla I_N u) F'(u) - \nabla I_N F(u) \|_{2, \frac{2d}{d+2}} \\ &\lesssim N^{-\min\{1, \frac{4}{d}\} s + \varepsilon} \| \nabla I_N u \|_{2, \frac{2d}{d-2}} \| \langle \nabla \rangle^{\min\{1, \frac{4}{d}\} s - \varepsilon} F'(u) \|_{\infty, \frac{d}{2}}. \end{split}
$$

<span id="page-10-1"></span>The claim [\(2.19\)](#page-9-2) will follow immediately from the estimate above, provided we show

(2.21) 
$$
\|\langle \nabla \rangle^{\min\{1,\frac{4}{d}\}s-\varepsilon}F'(u)\|_{\infty,\frac{d}{2}} \lesssim \|\langle \nabla \rangle I_N u\|_{\infty,2}^{\frac{4}{d}}.
$$

We start by observing that for any  $(q, r)$  admissible pair,

<span id="page-10-0"></span>(2.22) 
$$
\|\langle \nabla \rangle^s u\|_{q,r} \lesssim \|\langle \nabla \rangle I_N u\|_{q,r}.
$$

Indeed, decomposing  $u := u \le N + u > N$  and using Lemma [2.4](#page-6-3) and the fact that  $I_N$ is the identity operator on frequencies  $|\xi| \leq N$ , we get

$$
||u||_{q,r} \le ||u_{\le N}||_{q,r} + ||u_{>N}||_{q,r}
$$
  
\n
$$
\lesssim ||I_N u_{\le N}||_{q,r} + N^{-1} ||\nabla I_N u_{>N}||_{q,r}
$$
  
\n
$$
\lesssim ||\langle \nabla \rangle I_N u||_{q,r}.
$$

Similarly, we estimate

$$
\begin{aligned} |||\nabla|^s u||_{q,r} &\le |||\nabla|^s u_{\le N}||_{q,r} + |||\nabla|^s u_{>N}||_{q,r} \\ &\lesssim |||\nabla|^s I_N u_{\le N}||_{q,r} + N^{s-1} ||\nabla I_N u||_{q,r} \\ &\lesssim ||\langle \nabla \rangle I_N u||_{q,r}, \end{aligned}
$$

and the estimate [\(2.22\)](#page-10-0) follows.

As  $F'(u) = O(|u|^{\frac{4}{d}})$ , by [\(2.22\)](#page-10-0) we get

<span id="page-10-2"></span>(2.23) 
$$
\|F'(u)\|_{\infty,\frac{d}{2}} \lesssim \|\langle \nabla \rangle I_N u\|_{\infty,2}^{\frac{4}{d}}.
$$

We first prove [\(2.21\)](#page-10-1) for  $d \leq 4$ . By [\(2.23\)](#page-10-2), we estimate

$$
\begin{split} \|\langle \nabla \rangle^{\min\{1,\frac{4}{d}\}s-\varepsilon}F'(u)\|_{\infty,\frac{d}{2}}&=\|\langle \nabla \rangle^{s-\varepsilon}F'(u)\|_{\infty,\frac{d}{2}}\\ &\lesssim \|\langle \nabla \rangle^sF'(u)\|_{\infty,\frac{d}{2}}\\ &\lesssim \|F'(u)\|_{\infty,\frac{d}{2}}+\||\nabla|^sF'(u)\|_{\infty,\frac{d}{2}}\\ &\lesssim \|\langle \nabla \rangle I_N u\|_{\infty,2}^{\frac{4}{d}}+\||\nabla|^sF'(u)\|_{\infty,2}\end{split}
$$

Using Lemmas [2.7](#page-9-3) (for  $d = 3$ ) and [2.8](#page-9-4) (for  $d = 4$ ) together with [\(2.22\)](#page-10-0), we estimate

$$
\||\nabla|^s F'(u)\|_{\infty,\frac{d}{2}} \lesssim \||\nabla|^s u\|_{\infty,2} \|u\|_{\infty,2}^{\frac{d}{d}-1} \lesssim \|\langle \nabla \rangle I_N u\|_{\infty,2}^{\frac{d}{d}}.
$$

Thus, [\(2.21\)](#page-10-1) holds for  $d \leq 4$ .

If  $d > 4$ , using Lemma [2.9](#page-9-5) (with  $\alpha := \frac{4}{d}$ ,  $\sigma := \frac{4s}{d} - \varepsilon$ , and  $\delta := s$ ) and [\(2.22\)](#page-10-0), we get

$$
\|\nabla|^{\min\{1,\frac{4}{d}\}s-\varepsilon}F'(u)\|_{\infty,\frac{d}{2}}=\||\nabla|^{\frac{4s}{d}-\varepsilon}F'(u)\|_{\infty,\frac{d}{2}}\lesssim\|u\|_{\infty,2}^{\frac{\varepsilon}{s}}\|\nabla|^s u\|_{\infty,2}^{\frac{4}{s}-\frac{\varepsilon}{s}}\leq\|\langle\nabla\rangle I_N u\|_{\infty,2}^{\frac{4}{d}-\frac{\varepsilon}{s}}.
$$

From this and  $(2.23)$  we derive  $(2.21)$  in the case  $d > 4$ . The claim  $(2.19)$  follows.

We now consider  $(2.20)$ . Using  $(2.5)$ ,  $(2.7)$ , and  $(2.22)$ , we obtain

$$
||I_N F(u)||_{N^0(I)} \lesssim ||F(u)||_{\gamma',\rho'} \lesssim |I|^{\frac{2s}{d}} ||\langle \nabla \rangle^s u||_{\gamma,\rho}^{1+\frac{4}{d}} \lesssim |I|^{\frac{2s}{d}} ||\langle \nabla \rangle I_N u||_{\gamma,\rho}^{1+\frac{4}{d}}.
$$

 $\frac{d}{2}$ .

Similarly, by Hölder and  $(2.22)$ ,

$$
||(I_N \nabla u)F'(u)||_{N^0(I)} \lesssim ||(I_N \nabla u)F'(u)||_{\gamma',\rho'}
$$
  
\n
$$
\lesssim |I|^{\frac{2s}{d}} ||I_N \nabla u||_{\gamma,\rho} ||u||_{\gamma,\rho^*}^{\frac{4}{d}}
$$
  
\n
$$
\lesssim |I|^{\frac{2s}{d}} ||I_N \nabla u||_{\gamma,\rho} ||\langle \nabla \rangle^s u||_{\gamma,\rho}^{\frac{4}{d}}
$$
  
\n
$$
\lesssim |I|^{\frac{2s}{d}} ||\langle \nabla \rangle I_N u||_{\gamma,\rho}^{1+\frac{4}{d}}.
$$

The estimate  $(2.20)$  follows from the estimates above,  $(2.19)$  and the triangle inequality.  $\Box$ 

We end this section with the following concentration-compactness lemma:

<span id="page-11-0"></span>**Lemma 2.11** (Concentration-compactness, [\[20\]](#page-20-15)). Let  $\{v_n\}_{n\geq 1}$  be a bounded sequence in  $H^1(\mathbb{R}^d)$  such that

$$
\limsup_{n \to \infty} \|\nabla v_n\|_2 \le M < \infty
$$

and

$$
\limsup_{n \to \infty} ||v_n||_{2 + \frac{4}{d}} \ge m > 0.
$$

Then, there exists  $\{x_n\}_{n\geq 1} \subset \mathbb{R}^d$  such that, up to a subsequence,

$$
v_n(\cdot + x_n) \to V
$$
 weakly in  $H_x^1$  as  $n \to \infty$ 

with  $||V||_2 \ge \sqrt{\frac{d}{d+2}} \frac{m^{\frac{2}{d}+1}}{M} ||Q||_2$ .

# 3. LOCAL  $H_x^s$  THEORY

<span id="page-11-1"></span>In this section, we review the local  $H_x^s$  theory for the equation [\(1.1\)](#page-0-0) as described in [\[6\]](#page-19-0).

**Proposition 3.1** (Local well-posedness in  $H_x^s$ , [\[6\]](#page-19-0)). Let  $0 < s < 1$  and  $u_0 \in$  $H<sup>s</sup>(\mathbb{R}^d)$ . Then, the equation [\(1.1\)](#page-0-0) is wellposed on  $[0, T_{lwp}]$  with

$$
T_{lwp} = C_0 \|\langle \nabla \rangle^s u_0\|_2^{-\frac{2}{s}}.
$$

Moreover, the unique solution u enjoys the following estimate:

$$
\|\langle \nabla \rangle^s u\|_{S^0([0,T_{lwp}])} \lesssim \|\langle \nabla \rangle^s u_0\|_2.
$$

Here,  $C_0$  and the implicit constant depend only on the dimension d and the regularity s.

<span id="page-11-3"></span>As a direct consequence of the  $H_x^s$  local well-posedness result, we have the following lower bound on the blowup rate of the  $H^s_x$ -norm:

**Corollary 3.2** (Blowup criterion, [\[6\]](#page-19-0)). Let  $0 < s < 1$  and  $u_0 \in H_x^s$ . Assume that the unique solution u to [\(1.1\)](#page-0-0) blows up at time  $0 < T^* < \infty$ . Then, there exists a constant C depending only on d and s such that

$$
||u(t)||_{H_x^s} \geq C(T^*-t)^{-\frac{s}{2}}.
$$

<span id="page-11-2"></span>As a variation of Proposition [3.1,](#page-11-1) we have

**Proposition 3.3** ('Modified' local well-posedness). Let  $\frac{\frac{4}{d}+1}{1+\min\{1,2\}}$  $\frac{\frac{1}{d}+1}{1+\min\{1,\frac{4}{d}\}+\frac{4}{d}} < s < 1$  and  $u_0 \in H_x^s$ . Let also

<span id="page-12-4"></span><span id="page-12-0"></span>(3.1) 
$$
N \gg ||u_0||_{H_x^s}^{\frac{4}{\min\{4,d\}s-(4+d)(1-s)-d\varepsilon}} \quad \text{for any } \varepsilon > 0 \text{ sufficiently small}
$$

(3.2) 
$$
\tilde{T}_{lwp} := c_0 \|\langle \nabla \rangle I_N u_0\|_2^{-\frac{2}{s}} \quad \text{for a small constant } c_0 = c_0(d, s).
$$

<span id="page-12-3"></span>Then, [\(1.1\)](#page-0-0) is wellposed on  $[0, \tilde{T}_{lwp}]$  and moreover

$$
(\mathbf{3.3}) \qquad \qquad \|\langle \nabla \rangle I_N u\|_{S^0([0,\tilde{T}_{lwp}])} \lesssim \|\langle \nabla \rangle I_N u_0\|_2.
$$

Proof. As by Lemma [2.4,](#page-6-3)

 $||u_0||_{H^s_x} \lesssim ||\langle \nabla \rangle I_N u_0||_2,$ 

choosing  $c_0$  sufficiently small, we get

$$
\tilde{T}_{lwp} \leq T_{lwp}.
$$

<span id="page-12-1"></span>Thus, [\(1.1\)](#page-0-0) is wellposed in  $H^s_x$  on  $[0, \tilde{T}_{lwp}]$ . Let u be the unique solution; by Proposition [3.1](#page-11-1) we have

.

(3.4) 
$$
\|\langle \nabla \rangle^s u\|_{S^0([0, \tilde{T}_{lwp}])} \lesssim \|u_0\|_{H^s_x}
$$

On the other hand, by Strichartz, for any  $t \leq \tilde{T}_{lwp}$  we estimate

$$
\|\langle \nabla \rangle I_N u\|_{S^0([0,t])} \lesssim \|\langle \nabla \rangle I_N u_0\|_2 + \|\langle \nabla \rangle I_N F(u)\|_{N^0([0,t])}.
$$

As by hypothesis,  $s > \frac{\frac{4}{d}+1}{1+\min\{1\}}$  $\frac{\frac{1}{d}+1}{1+\min\{1,\frac{4}{d}\}+\frac{4}{d}} > \frac{1}{1+\min\{1,\frac{4}{d}\}}$ , by Lemma [2.10](#page-9-7) we get

<span id="page-12-2"></span>
$$
\begin{aligned} \| \langle \nabla \rangle I_N u \|_{S^0([0,t])} &\lesssim \| \langle \nabla \rangle I_N u_0 \|_2 + t^{\frac{2s}{d}} \| \langle \nabla \rangle I_N u \|_{S^0([0,t])}^{1 + \frac{4}{d}} \\ &+ N^{-\min\{1, \frac{4}{d}\} s + \varepsilon} \| \langle \nabla \rangle I_N u \|_{S^0([0,t])}^{1 + \frac{4}{d}} \end{aligned}
$$

for any  $\varepsilon > 0$  sufficiently small. For N satisfying [\(3.1\)](#page-12-0), we use Lemma [2.4](#page-6-3) and [\(3.4\)](#page-12-1) to estimate

$$
N^{-\min\{1,\frac{4}{d}\}s+\varepsilon} \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}^{1+\frac{4}{d}} \n\lesssim N^{-\min\{1,\frac{4}{d}\}s+\varepsilon} N^{(1+\frac{4}{d})(1-s)} \|\langle \nabla \rangle^s u\|_{S^0([0,t])}^{1+\frac{4}{d}} \n\ll \|u_0\|_{H_x^s}^{\frac{4}{\min\{4,d\}s-(4+d)(1-s)-d\varepsilon}} \times [-\min\{1,\frac{4}{d}\}s+\varepsilon+(1+\frac{4}{d})(1-s)]} \|u_0\|_{H_x^s}^{1+\frac{4}{d}} \n\lesssim \|u_0\|_{H_x^s}^{1+\frac{4}{d}} \|u_0\|_{H_x^s}^{1+\frac{4}{d}} \lesssim \|\langle \nabla \rangle I_N u_0\|_2.
$$

Here, we have used the fact that  $s > \frac{\frac{4}{d}+1}{1+\min\{1\}}$  $\frac{\overline{d}+1}{1+\min\{1,\frac{4}{d}\}+\frac{4}{d}}$  implies that there exists  $\varepsilon_1 > 0$ sufficiently small such that for any  $0 < \varepsilon < \varepsilon_1$  we have  $s > \frac{\frac{4}{d} + 1 + \varepsilon}{1 + \min\{1, \frac{4}{d}\}}$  $\frac{\overline{d}+1+\epsilon}{1+\min\{1,\frac{4}{d}\}+\frac{4}{d}}$ ; thus the power of N in the estimates above is negative for  $0 < \varepsilon < \varepsilon_1$ .

Returning to [\(3.5\)](#page-12-2), we conclude that

$$
\|\langle\nabla\rangle I_N u\|_{S^0([0,t])}\lesssim \|\langle\nabla\rangle I_N u_0\|_2 + t^{\frac{2s}{d}}\|\langle\nabla\rangle I_N u\|_{S^0([0,t])}^{1+\frac{4}{d}}.
$$

Standard arguments yield [\(3.3\)](#page-12-3), provided

$$
t \leq \tilde{T}_{lwp} = c_0(d, s) \|\langle \nabla \rangle I_N u_0\|_2^{-\frac{2}{s}}.
$$

 $\Box$ 

#### 4. Modified energy increment

The main purpose of this section is to prove that the modified energy of  $u$ ,  $E(I_N u)$ , grows much slower than the modified kinetic energy of u,  $\|\nabla I_N u\|_2^2$ . As will be shown later, this result is crucial in establishing the main theorems.

Before stating the result, we need to introduce more notation. We define

$$
\Lambda(t) := \sup_{0 \le \tau \le t} ||u(\tau)||_{H^s_x} \text{ and } \Sigma(t) := \sup_{0 \le \tau \le t} ||I_N u(\tau)||_{H^1_x}.
$$

With this notation we have the following

<span id="page-13-0"></span>**Proposition 4.1** (Increment of the modified energy). Let  $s_0(d) < s < 1$  and let  $u_0 \in H_x^s$  such that the corresponding solution u to [\(1.1\)](#page-0-0) blows up at time  $0 < T^*$  $\infty$ . Let  $0 < T < T^*$ . Then, for

<span id="page-13-3"></span>(4.1) 
$$
N(T) := C\Lambda(T)^{\frac{p(s)}{2(1-s)}},
$$

we have

$$
|E(I_{N(T)}u(T))| \lesssim \Lambda(T)^{p(s)}.
$$

Here, C and the implicit constant depend only on s,  $T^*$ , and  $||u_0||_{H^s_x}$ , and  $p(s)$  is given by

$$
p(s) := \frac{2(2 + \frac{2}{s} + \frac{8}{d})(1 - s)}{\min\{1, \frac{4}{d}\}s - (\frac{2}{s} + \frac{8}{d})(1 - s)}.
$$

Note that by Lemma [2.4,](#page-6-3)  $\Lambda(T) \leq \Sigma(T)$ . Thus, if the solution blows up at time  $T^*$ , the modified energy  $E(I_{N(T)}u(T))$  is at most  $O(\Sigma(T)^{p(s)})$ , which is much smaller than the modified kinetic energy,  $\|\nabla I_{N(T)}u(T)\|_2^2 = O(\Sigma(T)^2)$  for  $s > s_0(d)$ .

We prove Proposition [4.1](#page-13-0) in two steps. The first step is to control the increment of the modified energy of u on intervals of local well-posedness  $[0, \tilde{T}_{lwp}]$ . The second step is to divide the interval  $[0, T]$  into finitely many subintervals of local wellposedness, control the increment of the modified energy of  $u$  on each of these subintervals, and sum these bounds.

We start with the following

<span id="page-13-2"></span>**Lemma 4.2** (Local increment of the modified energy). Let  $\frac{1+\frac{4}{d}}{1+\frac{4}{d}+\min\{1,\frac{4}{d}\}} < s < 1$ and let  $u_0 \in H_x^s$ . Assume that N and  $\tilde{T}_{lwp}$  satisfy [\(3.1\)](#page-12-0) and [\(3.2\)](#page-12-4) respectively, that is,

$$
N \gg ||u_0||_{H_x^s}^{\frac{4}{\min\{4,d\}s-\frac{4}{(4+d)(1-s)\}}}
$$
  
 $\tilde{T}_{lwp} := c_0 ||\langle \nabla \rangle I_N u_0 ||_2^{-\frac{2}{s}}$  for a small constant  $c_0 = c_0(d, s)$ .

Then,

$$
\sup_{t\in[0,\tilde{T}_{lwp}]}|E(I_Nu(t))| \leq |E(I_Nu_0)| + CN^{-\min\{1,\frac{4}{d}\}s} + (\|I_N\langle \nabla \rangle u_0\|_2^{2+\frac{4}{d}} + \|I_N\langle \nabla \rangle u_0\|_2^{2+\frac{8}{d}}).
$$

Here, the constant C depends on s,  $T^*$ , and  $||u_0||_{H_x^s}$ .

<span id="page-13-1"></span>*Proof.* Note that by Proposition [3.3,](#page-11-2) [\(1.1\)](#page-0-0) is wellposed on  $[0, \tilde{T}_{lwp}]$ . Furthermore, the unique solution u to [\(1.1\)](#page-0-0) on  $[0, \tilde{T}_{lwp}]$  satisfies

$$
(\mathbf{4.2}) \qquad \qquad \|\langle \nabla \rangle I_N u\|_{S^0([0,\tilde{T}_{lwp}])} \lesssim \|\langle \nabla \rangle I_N u_0\|_2.
$$

Let  $0 < t \leq \tilde{T}_{lwp}$ ; throughout the rest of the proof, all spacetime norms will be taken on  $[0, t] \times \mathbb{R}^d$ . By the Fundamental Theorem of Calculus, we can write the modified energy increment as

$$
E(I_N u(t)) - E(I_N u_0) = \int_0^t \frac{\partial}{\partial s} E(I_N u(s)) ds
$$
  
= Re  $\int_0^t \int_{\mathbb{R}^d} \overline{I_N u_t} (-\Delta I_N u + F(I_N u)) dx ds.$ 

As  $I_N u_t = i \Delta I_N u - i I_N F(u)$ , we have

$$
Re \int_0^t \int_{\mathbb{R}^d} \overline{I_N u_t} (-\Delta I_N u + I_N F(u)) dx ds = 0.
$$

Thus, after an integration by parts,

<span id="page-14-0"></span>
$$
E(I_N u(t)) - E(I_N u_0) = Re \int_0^t \int_{\mathbb{R}^d} \overline{I_N u_t} [F(I_N u) - I_N F(u)] dx ds
$$
  
(4.3)  

$$
= -Im \int_0^t \int_{\mathbb{R}^d} \overline{\nabla I_N u} \cdot \nabla [F(I_N u) - I_N F(u)] dx ds
$$

<span id="page-14-1"></span>(4.4) 
$$
- Im \int_0^t \int_{\mathbb{R}^d} \overline{I_N F(u)} \cdot [F(I_N u) - I_N F(u)] dx ds.
$$

Consider the contribution from [\(4.3\)](#page-14-0). By the triangle inequality,

$$
\|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}} \lesssim \|(\nabla I_N u)[F'(I_N u) - F'(u)]\|_{2, \frac{2d}{d+2}} + \|(\nabla I_N u)F'(u) - \nabla I_N F(u)\|_{2, \frac{2d}{d+2}}.
$$

By Hölder,  $(2.1)$ ,  $(2.8)$ , and  $(2.22)$ , we estimate

$$
\begin{split} \|\nabla I_N u\| & [F'(I_N u) - F'(u)]\|_{2, \frac{2d}{d+2}} \\ &\lesssim \|\nabla I_N u\|_{2, \frac{2d}{d-2}} \|F'(I_N u) - F'(u)\|_{\infty, \frac{d}{2}} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{S^0([0,t])} \|\, |I_N u - u| \min\{1, \frac{4}{d}\} (\|I_N u| + |u|)^{\frac{4}{d} - \min\{1, \frac{4}{d}\}}\|_{\infty, \frac{d}{2}} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{S^0([0,t])} \|P_{> N} u\|_{\infty, 2}^{\min\{1, \frac{4}{d}\}} \|u\|_{\infty, 2}^{\frac{4}{d} - \min\{1, \frac{4}{d}\}} \\ &\lesssim N^{-\min\{1, \frac{4}{d}\}} \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}^{1+\frac{4}{d}} . \end{split}
$$

<span id="page-14-2"></span>Combining this with [\(2.19\)](#page-9-2), we get

$$
(4.5) \qquad \|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}} \lesssim N^{-\min\{1, \frac{4}{d}\}s +} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1 + \frac{4}{d}}.
$$

Therefore

<span id="page-14-4"></span>
$$
|4.3| \lesssim \|\nabla I_N u\|_{2, \frac{2d}{d-2}} \|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}}
$$
  
(4.6)  

$$
\lesssim N^{-\min\{1, \frac{4}{d}\}s +} \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}^{2+\frac{4}{d}}.
$$

We turn now toward [\(4.4\)](#page-14-1). By [\(4.5\)](#page-14-2) and Sobolev embedding, we estimate

<span id="page-14-3"></span>
$$
|(4.4)| \lesssim \|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}} \|\nabla|^{-1} I_N F(u)\|_{2, \frac{2d}{d-2}}
$$
  

$$
\lesssim N^{-\min\{1, \frac{4}{d}\}s +} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1 + \frac{4}{d}} \|I_N F(u)\|_{2, 2}.
$$

To estimate the last factor in [\(4.7\)](#page-14-3), we drop the operator  $I<sub>N</sub>$  and use Sobolev embedding to obtain

$$
\|I_NF(u)\|_{2,2}\lesssim \|u\|_{\frac{2(d+4)}{d},\frac{2(d+4)}{d}}^{1+\frac{4}{d}}\lesssim \||\nabla|^{\frac{d}{d+4}}u\|_{\frac{2(d+4)}{d},\frac{2(d+4)}{d+2}}^{1+\frac{4}{d}}.
$$

Note that  $\left(\frac{2(d+4)}{d}, \frac{2(d+4)}{d+2}\right)$  is a Schrödinger admissible pair. Decompose  $u := u \le N +$  $u_{>N}$ . To estimate the low frequencies, we use the fact that  $I_N$  is the identity on frequencies  $|\xi| \leq N$ :

$$
\|\nabla|^{\frac{d}{d+4}}u_{\leq N}\|_{\frac{2(d+4)}{d},\frac{2(d+4)}{d+2}}\lesssim \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}.
$$

For the high frequencies, we use Lemma [2.4](#page-6-3) to get

$$
\||\nabla|^{\frac{d}{d+4}}u_{>N}\|_{\frac{2(d+4)}{d},\frac{2(d+4)}{d+2}}\lesssim N^{-\frac{4}{d+4}}\|\nabla I_N u_{>N}\|_{S^0([0,t])},
$$

provided  $s > \frac{d}{d+4}$ ; this condition is satisfied since by assumption,

$$
s > \frac{1 + \frac{4}{d}}{1 + \frac{4}{d} + \min\{1, \frac{4}{d}\}} > \frac{d}{d+4}.
$$

<span id="page-15-0"></span>Therefore,

(4.8)  $||I_N F(u)||_{2,2} \lesssim ||\langle \nabla \rangle I_N u||_{S^0([0,t])}^{1+\frac{4}{d}}.$ 

<span id="page-15-1"></span>By  $(4.7)$  and  $(4.8)$ , we obtain

(4.9) 
$$
|(4.4)| \lesssim N^{-\min\{1,\frac{4}{d}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}^{2+\frac{8}{d}}.
$$

Collecting  $(4.2)$ ,  $(4.6)$ , and  $(4.9)$ , we get

$$
|E(I_N u(t)) - E(I_N u_0)| \lesssim N^{-\min\{1, \frac{4}{d}\} s + \left( \left\| \langle \nabla \rangle I_N u_0 \right\|_2^{2 + \frac{4}{d}} + \left\| \langle \nabla \rangle I_N u_0 \right\|_2^{2 + \frac{8}{d}} \right).
$$
  
is proves Lemma 4.2

This proves Lemma [4.2.](#page-13-2)

<span id="page-15-3"></span>Next, we use Lemma [4.2](#page-13-2) to prove Proposition [4.1.](#page-13-0)

Let  $T < T^*$  and  $\Lambda(T)$  and  $\Sigma(T)$  defined as in the beginning of this section. By Proposition [3.3,](#page-11-2) if we take

(4.10) 
$$
\begin{cases} N(T) \gg \Lambda(T)^{\frac{4}{\min\{4,d\}s-\frac{4}{(4+d)(1-s)-}}\\ \delta := c_0 \Sigma(T)^{-\frac{2}{s}}, \end{cases}
$$

then the solution  $u$  satisfies the estimate

$$
\|\langle \nabla \rangle I_{N(T)}u\|_{S^{0}([t,t+\delta])} \lesssim \|\langle \nabla \rangle I_{N(T)}u(t)\|_{2} \lesssim \Sigma(T),
$$

uniformly in t, provided  $[t, t + \delta] \subset [0, T]$ . Thus, splitting  $[0, T]$  into  $O(\frac{T}{\delta})$  subintervals and applying Lemma [4.2](#page-13-2) on each of these subintervals, we get

<span id="page-15-2"></span>
$$
\sup_{t \in [0,T]} |E(I_{N(T)}u(t))| \lesssim |E(I_{N(T)}u_0)| + \frac{T}{\delta} N(T)^{-\min\{1, \frac{4}{d}\}s + \sum(T)^{2 + \frac{4}{d}}}
$$
  
+ 
$$
\frac{T}{\delta} N(T)^{-\min\{1, \frac{4}{d}\}s + \sum(T)^{2 + \frac{8}{d}}}
$$
  

$$
\lesssim |E(I_{N(T)}u_0)| + N(T)^{-\min\{1, \frac{4}{d}\}s + \sum(T)^{2 + \frac{4}{d} + \frac{2}{s}}}
$$
  
(4.11)

Using interpolation, Sobolev embedding, and Lemma [2.4,](#page-6-3) we estimate

$$
|E(I_{N(T)}u_0)| \lesssim \|\nabla I_N u_0\|_2^2 + \|I_N u_0\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \\
\lesssim N^{2(1-s)} \|u_0\|_{H_x^s}^2 + \|I_N u_0\|_2^{\frac{4}{d}} \|\nabla I_N u_0\|_2^2 \\
\lesssim N^{2(1-s)} \left( \|u_0\|_{H_x^s}^2 + \|u_0\|_{H_x^s}^{2+\frac{4}{d}} \right) \\
\lesssim N^{2(1-s)}.
$$
\n(4.12)

<span id="page-16-1"></span><span id="page-16-0"></span>Moreover, by Lemma [2.4,](#page-6-3) we also have

(4.13) 
$$
\Sigma(T) \lesssim N(T)^{1-s} \Lambda(T).
$$

Substituting  $(4.12)$  and  $(4.13)$  into  $(4.11)$ , we obtain

<span id="page-16-2"></span>
$$
\sup_{t \in [0,T]} |E(I_{N(T)}u(t))| \lesssim N(T)^{2(1-s)} + N(T)^{-\min\{1, \frac{4}{d}\}s + (2 + \frac{4}{d} + \frac{2}{s})(1-s)} + \Lambda(T)^{2 + \frac{4}{d} + \frac{2}{s}}
$$
\n
$$
(4.14) + N(T)^{-\min\{1, \frac{4}{d}\}s + (2 + \frac{8}{d} + \frac{2}{s})(1-s)} + \Lambda(T)^{2 + \frac{8}{d} + \frac{2}{s}}.
$$

Optimizing [\(4.14\)](#page-16-2), we observe that if

(4.15) 
$$
N(T) \sim \Lambda(T)^{\frac{2+\frac{8}{d}+\frac{2}{s}}{\min\{1,\frac{4}{d}\}s - (1-s)(\frac{8}{d}+\frac{2}{s})-}},
$$

then  $N(T)$  satisfies the assumption [\(4.10\)](#page-15-3) and moreover,

<span id="page-16-3"></span>
$$
\sup_{t \in [0,T]} |E(I_{N(T)}u(t))| \lesssim N(T)^{2(1-s)} \le \Lambda(T)^{\frac{2(2+\frac{8}{d}+\frac{2}{s})(1-s)}{\min\{1,\frac{4}{d}\}s-(1-s)(\frac{8}{d}+\frac{2}{s})-}}.
$$

Let

$$
p(s) := \frac{2(2 + \frac{8}{d} + \frac{2}{s})(1 - s)}{\min\{1, \frac{4}{d}\}s - (1 - s)(\frac{8}{d} + \frac{2}{s})}.
$$

Then, a little work shows that the condition  $0 < p(s) < 2$  leads to the restriction

$$
s>s_0(d).
$$

Thus, for  $N(T)$  defined in [\(4.15\)](#page-16-3) and  $s > s_0(d)$ , we have  $0 < p(s) < 2$  and

$$
\sup_{t\in[0,T]}|E(I_{N(T)}u)(t)| \lesssim \Lambda(T)^{p(s)}.
$$

This proves Proposition [4.1.](#page-13-0)

## 5. Proof of Theorem [1.1](#page-2-1)

In this section, we use Proposition [4.1](#page-13-0) together with Lemma [2.11](#page-11-0) to prove Theorem [1.1.](#page-2-1)

We choose a sequence of times  $\{t_n\}_{n\geq 1}$ , such that  $t_n \to T^*$  as  $n \to \infty$  and

$$
||u(t_n)||_{H_x^s} = \Lambda(t_n).
$$

As the solution u blows up at time  $T^*$ , we must have  $\Lambda(t_n) \to \infty$  as  $n \to \infty$ . Set

$$
\psi_n(x) := \rho_n^{\frac{d}{2}}(I_{N(t_n)}u)(t_n, \rho_n x),
$$

where  $N(t_n)$  is given by [\(4.1\)](#page-13-3) with  $T := t_n$  and the parameter  $\rho_n$  is given by

$$
\rho_n := \frac{\|\nabla Q\|_2}{\|\nabla I_{N(t_n)}u(t_n)\|_2}.
$$

By Lemma [2.4](#page-6-3) and Corollary [3.2,](#page-11-3) we get

$$
\rho_n \lesssim \frac{1}{\|u(t_n)\|_{H^s_x}} \lesssim (T^*-t_n)^{\frac{s}{2}}.
$$

<span id="page-17-0"></span>Basic calculations show that  $\{\psi_n\}_{n\geq 1}$  is a bounded sequence in  $H_x^1$ . Indeed,

(5.1) 
$$
\|\psi_n\|_2 = \|I_{N(t_n)}u(t_n)\|_2 \le \|u(t_n)\|_2 = \|u_0\|_2
$$

$$
\|\nabla \psi_n\|_2 = \rho_n \|I_{N(t_n)} \nabla u(t_n)\|_2 = \|\nabla Q\|_2.
$$

By Proposition [4.1](#page-13-0) (with  $T = t_n$ ), we can estimate the energy of  $\psi_n$  as follows:

$$
E(\psi_n) = \rho_n^2 E(I_{N(t_n)} u(t_n)) \lesssim \rho_n^2 \Lambda(t_n)^{p(s)} \lesssim ||u(t_n)||_{H_s^s}^{p(s)-2}.
$$

Thus, as  $p(s) < 2$  for  $s > s_0(d)$ ,

<span id="page-17-4"></span>
$$
E(\psi_n) \to 0
$$
 as  $n \to \infty$ ,

which by the definition of the energy and [\(5.1\)](#page-17-0) implies

(5.2) 
$$
\|\psi_n\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \to \frac{d+2}{d} \|\nabla Q\|_2^2 \text{ as } n \to \infty.
$$

Applying Lemma [2.11](#page-11-0) to the sequence  $\{\psi_n\}_{n\geq 1}$  (with  $M := \|\nabla Q\|_2$  and  $m :=$  $\left(\frac{d+2}{d}\|\nabla Q\|_2^2\right)^{\frac{d}{2d+4}}$ , we derive the existence of a sequence  $\{x_n\}_{n\geq 1} \subset \mathbb{R}^d$  and of a function  $V \in H^1(\mathbb{R}^d)$  such that  $||V||_2 \ge ||Q||_2$  and, up to a subsequence,

$$
\psi_n(\cdot + x_n) \rightharpoonup V
$$
 weakly in  $H_x^1$  as  $n \to \infty$ ,

<span id="page-17-1"></span>that is,

d

(5.3) 
$$
\rho_n^{\frac{d}{2}}(I_{N(t_n)}u)(t_n, \rho_n \cdot + x_n) \rightharpoonup V \quad \text{weakly in } H_x^1 \quad \text{as } n \to \infty.
$$

To prove Theorem [1.1,](#page-2-1) we have to eliminate the smoothing operator  $I_{N(t_n)}$  from [\(5.3\)](#page-17-1). We do so at the expense of trading the weak convergence in  $H_x^1$  for convergence in the sense of distributions. Indeed, for any  $\sigma < s$  we have

$$
\|\rho_n^{\frac{\alpha}{2}}(u(t_n) - I_{N(t_n)}u(t_n))(\rho_n \cdot + x_n)\|_{\dot{H}_x^{\sigma}}\n= \rho_n^{\sigma} \|P_{\geq N(t_n)}u(t_n)\|_{\dot{H}_x^{\sigma}}\n\lesssim \rho_n^{\sigma} N(t_n)^{\sigma - s} \|P_{\geq N(t_n)}u(t_n)\|_{\dot{H}_x^s}\n\lesssim \Lambda(t_n)^{-\sigma} \Lambda(t_n)^{\frac{(\sigma - s)p(s)}{2(1 - s)}} \|P_{\geq N(t_n)}u(t_n)\|_{H_x^s}\n\lesssim \Lambda(t_n)^{1 - \sigma + \frac{(\sigma - s)p(s)}{2(1 - s)}}.
$$

<span id="page-17-2"></span>Plugging the explicit expression for  $p(s)$  in the above computation, we find that for

<span id="page-17-3"></span>
$$
\sigma < \tilde{s} := \frac{2d + 8s + s^2d(2 - \min\{1, \frac{4}{d}\})}{4d + 16s - s^2(d\min\{1, \frac{4}{d}\} + 8)},
$$

the exponent of  $\Lambda(t_n)$  in [\(5.4\)](#page-17-2) is negative. Hence,

(5.5) 
$$
\|\rho_n^{\frac{d}{2}}(u(t_n) - I_{N(t_n)}u(t_n))(\rho_n \cdot + x_n)\|_{H_x^{\tilde{s}-}} \to 0 \text{ as } n \to \infty.
$$

Combining [\(5.3\)](#page-17-1) and [\(5.5\)](#page-17-3) finishes the proof of Theorem [1.1.](#page-2-1)

## <span id="page-18-1"></span>6. Proof of Theorem [1.2](#page-2-2)

By Theorem [1.1,](#page-2-1) there exists a blowup profile  $V \in H_x^1$ , with  $||V||_2 \ge ||Q||_2$ , and there exist sequences  $\{t_n, \rho_n, x_n\}_{n\geq 1} \subset \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}^d$  such that  $t_n \to T^*$ ,

(6.1) 
$$
\frac{\rho_n}{(T^* - t_n)^{\frac{s}{2}}} \lesssim 1 \text{ for all } n \ge 1,
$$

<span id="page-18-0"></span>and

(6.2) 
$$
\rho_n^{\frac{d}{2}} u(t_n, \rho_n \cdot + x_n) \rightharpoonup V \quad \text{weakly as } n \to \infty.
$$

From  $(6.2)$  it follows that for any  $R > 0$  we have

$$
\liminf_{n \to \infty} \rho_n^d \int_{|x| \le R} |u(t_n, \rho_n x + x_n)|^2 \ge \int_{|x| \le R} |V|^2 dx,
$$

which, by a change of variables, yields

$$
\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \le R\rho_n} |u(t_n, x)|^2 dx \ge \int_{|x| \le R} |V|^2 dx.
$$

As by hypothesis  $\frac{(T^* - t_n)^{\frac{s}{2}}}{\alpha(t_n)} \to 0$  as  $n \to \infty$ , [\(6.1\)](#page-18-1) implies that  $\frac{\rho_n}{\alpha(t_n)} \to 0$  as  $n \to \infty$ . Therefore,

$$
\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^d}\int_{|x-y|\leq \alpha(t_n)}|u(t_n,x)|^2dx\geq \int_{|x|\leq R}|V|^2dx.
$$

Letting  $R \to \infty$ , we obtain

$$
\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^d}\int_{|x-y|\le\alpha(t_n)}|u(t_n,x)|^2dx\ge\|V\|_2^2.
$$

As  $||V||_2 \geq ||Q||_2$ , this implies

$$
\limsup_{t\to T^*}\sup_{y\in\mathbb{R}^d}\int_{|x-y|\leq \alpha(t)}|u(t,x)|^2dx\geq \|Q\|_2^2.
$$

As for any fixed time t, the map  $y \to \int_{|x-y| \leq \alpha(t)} |u(t,x)|^2 dx$  is continuous and goes to zero as  $y \to \infty$ , there exists  $x(t) \in \mathbb{R}^d$  such that

$$
\sup_{y \in \mathbb{R}^d} \int_{|x-y| \le \alpha(t)} |u(t,x)|^2 dx = \int_{|x-x(t)| \le \alpha(t)} |u(t,x)|^2 dx.
$$

This finally implies

$$
\limsup_{t \to T^*} \int_{|x - x(t)| \le \alpha(t)} |u(t, x)|^2 dx \ge ||Q||_2^2,
$$

which proves Theorem [1.2.](#page-2-2)

## 7. Proof of Theorem [1.3](#page-3-0)

In this section, we upgrade Theorem [1.1](#page-2-1) to Theorem [1.3](#page-3-0) under the additional assumption  $||u_0||_2 = ||Q||_2$ .

With the notation used in the proof of Theorem [1.1,](#page-2-1) we have

$$
\|\psi_n\|_2 \le \|u_0\|_2 = \|Q\|_2 \le \|V\|_2.
$$

On the other hand, using the semi-continuity of weak convergence,

$$
||V||_2 \le \liminf_{n \to \infty} ||\psi_n||_2 \le ||Q||_2.
$$

Therefore,

$$
||V||_2 = ||Q||_2 = \lim_{n \to \infty} ||\psi_n||_2.
$$

Thus, as  $\psi_n(\cdot + x_n) \rightharpoonup V$  weakly in  $L^2_x$  (up to a subsequence which we still denote by  $\psi_n(\cdot + x_n)$ , we conclude that

$$
\psi_n(\cdot + x_n) \to V
$$
 strongly in  $L_x^2$ .

Moreover, by the Gagliardo-Nirenberg inequality and the boundedness of  $\{\psi_n\}_{n\geq 1}$ in  $H_x^1$ , we have

$$
\psi_n(\cdot + x_n) \to V \text{ in } L_x^{2 + \frac{4}{d}}.
$$

Combining this with [\(5.2\)](#page-17-4) and the sharp Gagliardo-Nirenberg inequality, we obtain

$$
\|\nabla Q\|_2 \le \|\nabla V\|_2.
$$

By the semi-continuity of weak convergence, we also have

$$
\|\nabla V\|_2 \le \liminf_{n\to\infty} \|\nabla \psi_n\|_2 = \|\nabla Q\|_2,
$$

and so

$$
\|\nabla V\|_2 = \|\nabla Q\|_2 = \lim_{n \to \infty} \|\nabla \psi_n\|_2.
$$

Thus, as  $\psi_n(\cdot + x_n) \to V$  in  $H_x^1$ , we conclude that

$$
\psi_n(\cdot + x_n) \to V
$$
 strongly in  $H_x^1$ .

In particular, this implies

$$
E(V)=0.
$$

Collecting the properties of  $V$  we find

$$
V \in H_x^1
$$
,  $||V||_2 = ||Q||_2$ ,  $||\nabla V||_2 = ||\nabla Q||_2$ , and  $E(V) = 0$ .

The variational characterization of the ground state, [39], implies that

$$
V(x) = e^{i\theta} Q(x + x_0)
$$

<span id="page-19-5"></span>for some  $(e^{i\theta}, x_0) \in (S^1 \times \mathbb{R}^d)$ . Thus,

(7.1) 
$$
\rho_n^{\frac{d}{2}}(I_{N(t_n)}u)(t_n, \rho_n x + x_n) \to e^{i\theta}Q(x + x_0) \text{ strongly in } H_x^1 \text{ as } n \to \infty.
$$

Theorem [1.3](#page-3-0) follows from [\(5.5\)](#page-17-3) and [\(7.1\)](#page-19-5).

#### **REFERENCES**

- <span id="page-19-2"></span>[1] P. Begout, A. Vargas, Mass concentration phenomena for the  $L^2$ -critical nonlinear Schrödinger equation, preprint.
- <span id="page-19-1"></span>[2] J. Bourgain, Refinements of Strichartz inequality and applications to 2D-NLS with critical nonlinearity, I.M.R.N. 5 (1998), 253-283.
- [3] J. Bourgain, Scattering in the energy space and below for 3D NLS, J. d'Analyse Math. 75 (1998), 267-297.
- [4] J. Bourgain, New global well-posedness results for non-linear Schrödinger equations, AMS Publications (1999).
- [5] T. Cazenave, F.B. Weissler, Critical nonlinear Schrödinger Equation, Non. Anal. TMA 14 (1990), 807–836.
- <span id="page-19-0"></span>[6] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10, American Mathematical Society, 2003.
- <span id="page-19-4"></span>[7] M. Christ, M. Weinstein, Dispersion of small amplitude solutions of the generalized Kortewegde Vries equation, J. Funct. Anal. 100 (1991), 87-109.
- <span id="page-19-3"></span>[8] R. R. Coifman, Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. AMS 212 (1975), 315–331.

- <span id="page-20-12"></span>[9] R. R. Coifman, Y. Meyer, Ondelettes and operateurs III, Operateurs multilineaires, Actualites Mathematiques, Hermann, Paris (1991).
- <span id="page-20-7"></span>[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation, Math. Res. Lett.  $9$  (2002), 659–682.
- [11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Polynomial upper bounds for the orbital instability of the 1D cubic NLS below the energy norm, Discrete Contin. Dyn. Syst. 9 (2003), 31–54.
- [12] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Polynomial upper bounds for the instability of the nonlinear Schrödinger equation below the energy norm, Commun. Pure Appl. Anal. 2 (2003), 33–50.
- <span id="page-20-8"></span><span id="page-20-0"></span>[13] J. Colliander, S. Raynor, C. Sulem, J. D. Wright, Ground state mass concentration in the  $L^2$ -critical nonlinear Schrödinger equation below  $H^1$ , Math. Res. Lett. 12 (2005), 357-375.
- [14] D. Fang, S. Zhong, Cauchy problem for the  $L^2$ -critical nonlinear Schrödinger equation below  $H<sup>1</sup>$ , Nonlinear Anal. 62 (2005), 117-130.
- [15] P. Gerard, Description du defaut de compacite de l'injection de Sobolev, ESAIM Control Optim. Calc. Var. 3 (1998), 213–233.
- [16] J. Ginibre, G. Velo, *Scattering theory in the energy space for a class of nonlinear Schrödinger* equations, J. Math. Pure. Appl. 64 (1985), 363–401.
- <span id="page-20-10"></span>[17] J. Ginibre, G. Velo, Smoothing properties and retarded estimates for some dispersive evolution equations, Comm. Math. Phys. 144 (1992), 163–188.
- <span id="page-20-5"></span>[18] R. T. Glassey, On the blowing up of solution to the Cauchy problem for nonlinear Schrödinger operators, J. Math. Phys. 8 (1977), 1794–1797.
- [19] N. Hayashi, Y. Tsutsumi, Remarks on the scattering problem for nonlinear Schrödinger equations, Diff. Eq. Math. Phys. (1986), 162–168.
- <span id="page-20-15"></span><span id="page-20-1"></span>[20] T. Hmidi, S. Keraani, Blowup theory for the critical nonlinear Schrödinger equations revisited, Int. Math. Res. Not. 46 (2005), 2815–2828.
- [21] T. Hmidi, S. Keraani, Remarks on the blowup for the  $L^2$ -critical nonlinear Schrödinger equations, preprint.
- [22] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincare Phys. Theor. 46 (1987), 113–129.
- [23] T. Kato, On nonlinear Schrödinger equations, II. H<sup>s</sup>-solutions and unconditional wellposedness, J. d'Analyse Math. **67** (1995), 281–306.
- <span id="page-20-11"></span><span id="page-20-6"></span>[24] M. Keel, T. Tao, Endpoint Strichartz Estimates, Amer. Math. J. 120 (1998), 955–980.
- $[25]$  S. Keraani, On the blow up phenomenon of the critical nonlinear Schrödinger equation, J. Funct. Anal. 235 (2006), 171–192.
- [26] M. K. Kwong, Uniqueness of positive solutions of  $\Delta u u + u^p = 0$  in  $\mathbb{R}^n$ , Arch. Rat. Mech. Anal. 105 (1989), 243-266.
- [27] K. Mcleod, J. Serrin, Nonlinear Schrödinger equation. Uniqueness of positive solutions of  $\Delta u + f(u) = 0$  in  $\mathbb{R}^n$ , Arch. Rat. Mech. Anal. 99 (1987), 115-145.
- <span id="page-20-3"></span>[28] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power, Duke Math. J. 69 (1993), 427–454.
- [29] F. Merle, P. Raphael, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Ann. of Math.  $161$  (2005), 157–222.
- [30] F. Merle, P. Raphael, Profiles and quantization of the blow up mass for critical nonlinear  $Schrödinger$  equation, Comm. Math. Phys.  $253$  (2005), 675–704.
- <span id="page-20-4"></span>[31] F. Merle, Y. Tsutsumi,  $L^2$ -concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical power nonlinearity, J. Diff. Eq. 84 (1990), 205-214.
- <span id="page-20-13"></span>[32] G. Staffilani, On the generalized Korteweg-de Vries-type equations, Differential Integral Equations 10 (1997), 777–796.
- <span id="page-20-9"></span>[33] R. S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44 (1977), 705–714.
- $[34]$  T. Tao, On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation, Dyn. PDE 1 (2004), 1-48.
- <span id="page-20-14"></span>[35] M. E. Taylor, Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials, Mathematical Surveys and Monographs, 81, American Mathematical Society, Providence, RI, 2000.
- <span id="page-20-2"></span>[36] N. Tzirakis, Mass concentration phenomenon for the quintic nonlinear Schrödinger equation in one dimension, SIAM J. Math. Anal. **37** (2006), 1923-1946.
- [37] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, Ph.D. Thesis, UCLA.
- <span id="page-21-1"></span>[38] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, to appear Duke Math. J.
- [39] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (1982/83), 567–576.
- $\left[40\right]$  M. I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal. 16 (1985), 472–491.
- [41] M. I. Weinstein, Lyapunov stability of ground states of nonlinear Schrödinger equations, CPAM 39 (1986), 51–68.
- <span id="page-21-0"></span>[42] V.E. Zakharov, E.A. Kuznetsov, Quasi-classical theory for three-dimensional wave collapse, Sov. Phys. JETP 64 (1986), 773–380.

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