Stability Analysis of Frame Slotted Aloha Protocol

Jihong Yu, Lin Chen

LRI-CNRS UMR 8623, Univ. Paris-Sud, 91405 Orsay, France. {jihong.yu, chen}@lri.fr

Abstract

Frame Slotted Aloha (FSA) protocol has been widely applied in Radio Frequency Identification (RFID) systems as the *de facto* standard in tag identification. However, very limited work has been done on the stability of FSA despite its fundamental importance both on the theoretical characterisation of FSA performance and its effective operation in practical systems. In order to bridge this gap, we devote this paper to investigating the stability properties of FSA by focusing on two physical layer models of practical importance, the models with single packet reception and multipacket reception capabilities. Technically, we model the FSA system backlog as a Markov chain with its states being backlog size at the beginning of each frame. The objective is to analyze the ergodicity of the Markov chain and demonstrate its properties in different regions, particularly the instability region. By employing drift analysis, we obtain the closed-form conditions for the stability of FSA and show that the stability region is maximised when the frame length equals the backlog size in the single packet reception model and when the ratio of the backlog size to frame length equals in order of magnitude the maximum multipacket reception capacity in the multipacket reception model. Furthermore, to characterise system behavior in the instability region, we mathematically demonstrate the existence of transience of the backlog Markov chain.

Index Terms

Frame slotted Aloha, stability, multipacket reception.

I. Introduction

A. Context and Motivation

Since the introduction of Aloha protocol in 1970 [1], a variety of such protocols have been proposed to improve its performance, such as Slotted Aloha (SA) [16] and Frame Slotted Aloha (FSA) [12]. SA is a well known random access scheme where the time of the channel is divided into identical slots of duration equal to the packet transmission time and the users contend to access the server with a predefined slot-access probability. As a variant of SA, FSA divides time-slots into *frames* and a user is allowed to transmit only a single packet per frame in a randomly chosen time-slot.

Due to the effectiveness to tackle collisions in wireless networks, Aloha-based protocols have been applied extensively to various networked systems ranging from the traditional satellite networks [12], wireless LANs [24] to the emerging Machine-to-Machine (M2M) communications [27]. Specifically, in radio frequency identification (RFID) systems, FSA plays a fundamental role in the identifications of tags [28], [29] and is standardized in EPCGlobal Class-1 Generation-2 (C1G2) RFID standard [4]. In RFID systems, all tags transmit in the first frame in the selected slot respectively, but only tags experiencing no collisions are identified while other nodes, referred to as backlogged nodes (or simply backlogs), retransmit in the subsequent frames until all of them are successfully identified.

It is evident that stability is of primary importance for systems operating on top of Aloha-like protocols. A large body of studies have been devoted to stability analysis in a slotted collision channel [21], [17], [14] where a transmission is successful if and only if just a single user transmits in the selected slot, referred to as single packet reception (SPR).

More recently, application of FSA in RFID systems has received considerable research attention. However, very limited work has been done on the stability of FSA despite its fundamental importance both on the theoretical characterisation of FSA performance and its effective operation in practical systems. Moreover, the emerging multipacket reception (MPR) technologies in wireless networks, such as Code Division Multiple Access (CDMA) and Multiple-Input and Multiple-Output (MIMO), make it possible to receive multiple packets in a time-slot simultaneously, which remarkably boosts system performance. Motivated by the above observation, we argue that a systematic study on the stability properties of FSA incorporating the MPR capability is called for in order to lay the theoretical foundations for the design and optimization of FSA-based communication systems.

B. Summary of Contributions

In this paper, we investigate the stability properties of FSA with single packet reception and multipacket reception capabilities. The main contributions of this paper are articulated as follows:

- Firstly, we model the packet transmission process in a frame as the bins and balls problem [8] and derive the number of successful received packets under both SPR and MPR models.
- Secondly, we formulate a homogeneous Markov chain to characterize the number of the backlogged packets and derive the one-step transition probability.
- Thirdly, by employing drift analysis, we obtain the closed-form conditions for the stability of FSA and derive conditions maximising the stability regions for both SPR and MPR models.

- Finally, to characterise system behavior in the instability region, we mathematically demonstrate the existence of transience of the backlog Markov chain.

Our work demonstrates that the stability region is maximised when the frame length equals the backlog size in the MPR model and when the ratio of the backlog size to frame length equals in order of magnitude the maximum multipacket reception capacity in the MPR model. In addition, it is also shown that FSA-MPR outperforms FSA-SPR remarkably in term of the stability region size.

C. Paper Organisation

The remainder of the paper is organised as follows. Section II gives a brief overview of related work and compares our results with existing results. In Section III, we present the system model, including random access model, traffic model and packet success probability. In Section IV, we summary the main result of this paper. Then the detailed proofs on the stability properties of FSA-SPR and FSA-MPR are given in Section V and Section VI, respectively. Finally, we conclude our paper in Section VII.

II. RELATED WORK

Aloha-based protocols are basic schemes for random medium access and are applied extensively in many communication systems. As a central property, the stability of Aloha protocols has received a lot of research attention, which we briefly review in this section.

Stability of slotted Aloha Tsybakov and Mikhailov [23] initiated the stability analysis of finite-user slotted Aloha in 1979. They found sufficient conditions for stability of the queues in the system using the principle of stochastic dominance and derived the stability region for two users explicitly. For the case of more than two users, the inner bounds to the stability region were shown in [15]. Szpankowski [22] found necessary and sufficient conditions for the stability of queues for a fixed transmission probability vector assuming the arrival rates follow Bernoulli process. However, the derived conditions are not closed-form, meaning that verifying them is difficult. Borst *et al.* extended the results of [22] to more general systems assuming the monotonicity of the service rate [3]. We would like to point out that all the above stability analysis results were derived for the SPR model.

Stability of slotted Aloha with MPR The first attempt at analyzing stability properties of SA with MPR was made by Ghez *et al.* in [6], [7] in an infinite-user single-buffer model. They drew a conclusion that the system could be stabilized under the symmetrical MPR model with a non-zero probability that all packets were transmitted successfully. Sant and Sharma [18] studied a special case of the symmetrical MPR model for finite-user with an infinite buffer case. They derived sufficient conditions on arrival rate for stability of the system under the stationary ergodic arrival process. Although the work aforementioned analyzed the stability of system without MPR or/and with MPR, they are mostly, if not all, focused on SA protocol, while our focus is FSA with both SPR and MPR.

Performance analysis of FSA There exist several studies on the performance of FSA. Schoute [19] investigated dynamic FSA and obtained the expected number of time-slots needed until the backlog becomes zero. The optimal frame setting for dynamic FSA was proved mathematically by Luca and Flaminio [2]. Wieselthier and Anthony [26] introduced an combinational technique to analyse performance of FSA-MPR for the case of finite users. However, these works did not address the stability of FSA, which is of fundamental importance.

In summary, only very limited work has been done on the stability of FSA despite its fundamental importance both on the theoretical characterisation of FSA performance and its effective operation in practical systems. In order to bridge this gap, we devote this paper to investigating the stability properties of FSA under both SPR and MPR models.

III. SYSTEM MODEL

In this section, we introduce our system model which will be used throughout the rest of this paper.

A. Random access model in FSA

We consider a system of infinite identical users operating on one frequency channel. In one slot, a node can complete a packet transmission. FSA organises time-slots with each frame containing L consecutive time-slots. Each user is allowed to randomly and independently choose a time-slot to send his packet at most once per frame.

A packet suffers a collision if more than one packet is transmitted simultaneously in the same time-slot with SPR and if more than M packets are transmitted simultaneously with MPR, where M quantifies the MPR capacity. In our analysis, we do not distinguish packets, i.e., newly generated packets are simply regarded and treated as backlogs.

B. Traffic model

Let N_i denote the total number of new packets arriving during frame i and A_{il} denote the number of new packets arriving during time-slot l in frame i where $l = 1, 2, \cdots, L$. Assume that (A_{il}) are independent and identically distributed random variables with probability distribution:

$$P\{A_{il} = k\} = \Lambda_k (k \geq 0) \tag{1}$$

such that the expected number of arrivals per time-slot $\Lambda = \sum_1^\infty k\Lambda_k$ is finite.

Then as $N_i = \sum_{l=1}^L A_{il}$, the distribution of N_i, denoted as $\{\lambda_n\}_{n \geq 0}$, is

$$\lambda_n = P\{N_i = n\} = P\left\{\sum_{l=1}^L A_{il} = n\right\} = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_L}\prod_l \Lambda_{k_l}, \tag{2}$$

where $\sum_{l=1}^L k_l = n$. The expected number of arrivals during a frame, denoted as λ, is $\lambda = L\Lambda$.

C. Packet success probability

The process of randomly and independently choosing a time-slot in a frame to transmit packets can be cast into a class of problems that are known as occupance problems, or bins and balls problem [8]. Specifically, consider the setting where a number of balls are randomly and independently placed into a number of bins, the classic occupance problem studies the maximum load of an individual bin.

In our context, time-slots and packets to be transmitted in a frame can be cast into bins and balls, respectively. Given h packets being sent in frame i and the frame length L, the number x of packets sent in one time-slot, referred as to occupancy number, is binomially distributed with parameters h and $\frac{1}{L}$:

$$B_{h,\frac{1}{L}}(x) = \binom{h}{x}(\frac{1}{L})^x(1 - \frac{1}{L})^{h-x}. \tag{3}$$

Applying the distribution of equation (3) to all L slots in the frame, we can get the expected value $b(x)$ of the number of time-slots with occupance number x in a frame as follows:

$$b(x) = L \cdot B_{h,\frac{1}{L}}(x) = L\binom{h}{x}(\frac{1}{L})^x(1 - \frac{1}{L})^{h-x}. \tag{4}$$

We further derive the probability that a packet is transmitted successfully under both SPR and MPR.

Packet success probability of FSA with SPR

with SPR, the number of successfully received packets equals that of time-slots with occupance number $x = 1$. Extending the result of [25], we can derive the probability ξ_{hk} that there exists exactly k successful time-slots in the frame as follows:

$$\xi_{hk} = \begin{cases} \frac{\binom{L}{k}\binom{h}{k}k!G(L-k,h-k)}{L^h}, & 0 < k < \min(h, L) \\ \frac{\binom{L}{h}h!}{L^h}, & k = h \leq L \\ 0, & k > \min(h, L) \\ 0, & k = L < h \end{cases} \tag{5}$$

where

$$G(V, u) = V^u + \sum_{t=1}^u (-1)^t \prod_{j=0}^{t-1}[(u-j)(V-j)](V-t)^{u-t}\frac{1}{t!},$$

with $V \triangleq L - k$ and $u \triangleq h - k$.

Consequently, the expected number of successfully received packets in the frame, denoted as r_h, is

$$r_h = \sum_{k=1}^h k\xi_{hk} = b(1).$$

Packet success probability of FSA with MPR

With MPR, the number of successfully received packets in a frame is the sum of packets in time-slots with occupance number $1 \leq x \leq M$. Here, we provide the formulation of the probability ξ_{hk} that exact k packets are received successfully among h transmitted packets in the frame. Let occupancy numbers x_l stands for the number of packets in the lth time-slot, $l = 1, 2, \cdots, L$. Every L-tuple of integers satisfying

$$\sum_{l=1}^L x_l = h \tag{6}$$

describes a possible configuration of occupancy numbers. Let Z be the set of all possible configurations of occupancy numbers. Denote by $Z_a \in Z$ the configuration with the occupancy numbers $x_1^{Z_a}, \cdots, x_L^{Z_a}$ satisfying (6) and the following conditions:

$$\begin{cases} \sum_{l=1}^L x_l^{Z_a} Y_l = k, \\ P(Z_a) = \frac{h!}{x_1^{Z_a}!x_2^{Z_a}!\cdots x_L^{Z_a}!}L^{-h}, \end{cases}$$

where Y_l denotes the indicator function satisfying

$$Y_l = \begin{cases} 1 & 1 \leq x_l{}^{Z_a} \leq M, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the probability ξ_{hk} can be written as follows:

$$\xi_{hk} = \sum_{Z_a \in Z} P(Z_a) \tag{7}$$

Consequently, we can derive the expected number of successfully received packets in the frame as

$$r_h = \sum_{k=1}^{h} k\xi_{hk} = L \sum_{x=1}^{M} x \binom{h}{x} (\frac{1}{L})^x (1 - \frac{1}{L})^{h-x}. \tag{8}$$

IV. MAIN RESULTS

To streamline the presentation, we summarize the main results in this section and give the detailed proof and analysis in the sections that follow.

Aiming at studying the stability of FSA, we decompose our global objective into the following three questions, all of which are of fundamental importance both on the theoretical characterisation of FSA performance and its effective operation in practical systems:

- **Q1**: Under what condition(s) is FSA stable?
- **Q2**: When is the stability region maximised?
- **Q3**: How does FSA behave in the instability region?

Before answering the questions, we first introduce the formal definition of stability employed by Ghez *et al.* in [6].

Define by a variable X_i the number of backlogged packets in the system at the start of frame i. The discrete-time process $(X_i)_{i \geq 0}$ can be seen as a homogeneous Markov chain.

Definition 1. *An FSA system is stable if $(X_i)_{i \geq 0}$ is ergodic and unstable otherwise.*

By Definition 1, we can transform the study of stability of FSA into investigating the ergodicity of the backlog Markov chain. The rationality of this transformation is two-fold. One interpretation is the property of ergodicity that there exists an unique stationary distribution of a Markov chain if it is ergodic. The other can be interpreted from the nature of ergodicity that each state of the Markov chain can recur in finite time with probability 1.

We then establish the following results characterizing the stability region and demonstrating the behavior of the backlog Markov chain in nonergodicity regions under both SPR and MPR. For notation convenience, we use FSA-SPR and FSA-MPR to denote the FSA system operating with SPR and MPR, respectively.

A. Results for FSA-SPR

Theorem 1. *Under FSA-SPR, consider an irreducible and aperiodic backlog Markov chain $(X_i)_{i \geq 0}$ with nonnegative integers. Denote by h the number of backlogged packets in frame i and $\alpha \triangleq \frac{h}{L}$, for $h \to \infty$, we have* [1]

1) *The system is always stable for all arrival distributions if $\Lambda < \alpha e^{-\alpha}$ and $L = \Theta(h)$. Specially, the stability region is maximised when $\alpha = 1$.*

2) *The system is unstable for all arrival distributions under each of the following three conditions: (a) $L = o(h)$; (b) $L = O(h)$; (c) $L = \Theta(h)$ and $\Lambda > \alpha e^{-\alpha}$.*

Remark 1. *Theorem 1 answers the first two questions and can be interpreted as follows:*

- *When $L = o(h)$, i.e., the backlog size h is far larger than the frame length L, a packet experiences collision with high probability (w.h.p.), thus increasing the backlog size and destabilising the system;*
- *When $L = O(h)$, i.e., the backlog size is far smaller than the frame length, a packet is transmitted successfully w.h.p.. However, the number of successful transmitted packets is still significantly less than that of new arrivals in the frame. The system is thus unstable.*
- *When $L = \Theta(h)$, i.e., the backlog size has the same order of magnitude as the frame length, the system is stable when the backlog can be reduced gradually, i.e., when the expected arrival rate is less than the successful rate.*

It is well known that an irreducible aperiodic Markov chain falls into one of three mutually exclusive classes: positive recurrent, null recurrent, and transient. So, our next step after deriving the stability conditions is to show whether the backlog

---

[1]For two variables X, Y, we use the following asymptotic notations:
- $X = o(Y)$ if $\frac{X}{Y} = 0$, as $Y \to \infty$;
- $X = O(Y)$ if $\frac{X}{Y} = \infty$, as $Y \to \infty$;
- $X = \Theta(Y)$ if $\theta_1 \leq \frac{X}{Y} \leq \theta_2$, as $Y \to \infty$, where $\theta_2 \geq \theta_1 > 0$.

Markov chain in the instability region is transient or recurrent, which answers the third question. To facilitate the demonstration, we focus on the Poisson arrival processes. However, our analysis can be extended to other arrival processes.

Theorem 2. *With the same notations as in Theorem 1 under Poisson arrivals, $(X_i)_{i \geq 0}$ is always transient in the instability region, i.e., under each of the following three conditions: (1) $L = o(h^{1-\epsilon})$, $\forall\ 0 < \epsilon < 1$; (2) $L = \Theta(h)$ and $\Lambda > \alpha e^{-\alpha}$; (3) $L = O(h)$.*

Remark 2. *If a state of a Markov chain is transient, then the probability of returning to itself for the first time in an finite time is less than 1. Hence, Theorem 2 implies that once out of the stability region, the system is not guaranteed to return to stable state in a finite time.*

B. Results for FSA-MPR

Theorem 3. *Consider an FSA-MPR system where a receiver can decode at most M simultaneously transmitted packets. Using the same notations as in Theorem 1, we have*

   1) *The system is always stable for all arrival distributions if $\Lambda < \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}$ and $L = \Theta(h)$. Specially, let α^* denote the value of α that maximises the stability region, it holds that $\alpha^* = \Theta(M)$.*
   2) *The system is unstable for all arrival distributions under each of the following conditions: (1) $L = o(h)$; (2) $L = O(h)$; (3) $L = \Theta(h)$ and $\Lambda > \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}$.*

Remark 3. *Comparing the results of Theorem 3 to Theorem 1, we can quantify the performance gap between FSA-SPR and FSA-MPR in terms of stability. For example, when $\alpha = 1$, the stability region is maximised in FSA-SPR with $\Lambda^{SPR} < e^{-1}$, while the stability region in FSA-MPR is $\Lambda^{MPR} < e^{-1} \sum_{x=1}^{M} \frac{1}{(x-1)!}$. Note that for $M > 2$, it holds that*

$$1 + 1 + \frac{1}{2} < \sum_{x=1}^{M} \frac{1}{(x-1)!} < 1 + 1 + \sum_{x=1}^{M} \frac{1}{x(x+1)} < 2 + \left(\sum_{x=1}^{M} \frac{1}{x} - \frac{1}{x+1} \right) = 3 - \frac{1}{M}.$$

The maximum stability region of FSA-MPR is thus between 2.5 and 3 times that of FSA-SPR.

Theorem 4. *With the same notations as in Theorem 3 under Poisson arrivals, $(X_i)_{i \geq 0}$ is transient under each of the following three conditions: (1) $L = o(h^{1-\epsilon})$, $\forall\ 0 < \epsilon < 1$; (2) $L = \Theta(h)$ and $\Lambda > \alpha$; (3) $L = O(h)$.*

Remark 4. *Theorem 4 demonstrates that despite the stability gain of FSA-MPR over FSA-MPR, the properties in the unstable region is almost the same.*

V. STABILITY ANALYSIS OF FRAME SLOTTED ALOHA WITH SINGLE PACKET RECEPTION

In this section, we will analyse the stability of FSA-SPA and prove Theorem 1 and 2.

A. Characterising backlog Markov chain

As mentioned in Sec. IV, we characterize the number $(X_i)_{i \geq 0}$ of the backlogged packets in the system at the beginning of frame i as a homogeneous Markov chain. We now calculate the one-step transition probability as a function of $\xi_{h,k}$ and $\{\lambda_n\}_{n \geq 0}$. Denoting by h the number of the backlogged packets in frame i, we obtain the one-step transition probability $\mathbf{P} = \{P_{hk}\}$ as follows:

- For $h = 0$:

$$\begin{cases} P_{00} = \lambda_0, \\ P_{0k} = \lambda_k, \quad k \geq 1, \end{cases}$$

- For $h \geq 1$:

$$\begin{cases} P_{h,h-k} = \sum_{n=0}^{\min(h,L)-k} \lambda_n \xi_{h,n+k}, & 1 \leq k \leq \min(h,L), \\ P_{h,k} = \lambda_0 \xi_{h,0} + \sum_{n=1}^{\min(h,L)} \lambda_n \xi_{h,n}, & k = h, \\ P_{h,h+k} = \sum_{n=0}^{\min(h,L)} \lambda_{n+k} \xi_{h,n}, & k \geq 1. \end{cases} \tag{9}$$

The rationale for the calculation of transition probability is explained as follows:

- When $h = 0$, i.e., there are no backlogged packets in the frame, the backlog size remains 0 if no new packets arrive and increases by k if k new packets arrive in the frame.
- When $h > 0$, we have three possibilities, corresponding to the cases where the backlog size decreases, remains unchanged and increases, respectively:

- The state $1 \le k \le \min(h, L)$ corresponds to the case where the backlog size decreases when $n \le \min(h, L) - k$ new packets arrive and the number of successfully transmitted packets is between n and $\min(h, L)$.
- The backlog size remains unchanged if either of two following events happens: (a) no new packets are generated and all the retransmitted backlogged packets fail; (b) $n \le \min(h, L)$ new packets arrive and the number of the successfully received packets is also n.
- The backlog size increases if the number of successfully transmitted packets is less than that of new arrivals.

In order to establish the ergodicity of the backlog Markov chain $(X_i)_{i \ge 0}$, it is necessary to ensure $(X_i)_{i \ge 0}$ is irreducible and aperiodic. To this end, we conclude this subsection by providing the sufficient conditions on $\{\lambda_n\}$ for the irreducibility and the aperiodicity of $(X_i)_{i \ge 0}$ as follows:

$$0 < \lambda_n < 1, \ \forall \ n \ge 0. \tag{10}$$

We would like to point out that most of traffic models can satisfy (10). Throughout the paper, it is assumed that (10) holds and hence $(X_i)_{i \ge 0}$ is irreducible and aperiodic.

B. Stability analysis

Recalling Definition 1, to study the stability of FSA, we need to analyse the ergodicity of the backlog Markov chain $(X_i)_{i \ge 0}$. We first introduce two auxiliary lemmas which will be useful in the ergodicity demonstration.

Lemma 1 ([13]). *Given an irreducible and aperiodic Markov chain $(X_i)_{i \ge 0}$ such that*
1) *the state space is nonnegative integers with the transition probability matrix $\{P_{hk}\}$,*
2) *the expected drift at the state h, $|D_h| < \infty$, $\forall \ h$,*
3) *$\limsup_{h \to \infty} D_h < 0$,*
it holds that $(X_i)_{i \ge 0}$ is ergodic.

Lemma 2 ([9]). *Under the assumptions of Lemma 1, if for some integer $Q \ge 0$ and some constant $B \ge 0$, $c \in [0, 1]$ such that*
1) *$D_h > 0$, for all $h \ge Q$,*
2) *$\phi^h - \sum_k P_{hk} \phi^h \ge -B(1 - \phi)$, for all $h \ge Q$, $\phi \in [c, 1]$,*
then $(X_i)_{i \ge 0}$ is not ergodic.

Armed with Lemma 1 and Lemma 2, we further prove Theorem 1.

Proof of Theorem 1: In the proof, we first formulate the expected drift and then study the ergodicity of Markov chain based on drift analysis.

Let D_h be the expected drift of the backlog Markov chain $(X_i)_{i \ge 0}$ at state h ($h \ge 0$), formally defined as follows:

$$D_h = E[X_{i+1} - X_i | X_i = h]. \tag{11}$$

And denote by C_i the number of successful transmissions in frame i, we have

$$X_{i+1} - X_i = N_i - C_i.$$

It then follows that

$$D_h = E[N_i - C_i | X_i = h] = \lambda - E[C_i | X_i]. \tag{12}$$

Since all of the backlogged packets in frame i are retransmitted, we have

$$P\{C_i = k | X_i = h\} = \xi_{h,k}, \ 0 \le k \le \min(h, L),$$

where $\xi_{00} = 0$. Recalling the definition of r_h as the expected number of successfully received packets in the frame, we have

$$E[C_i = k | X_i = h] = r_h. \tag{13}$$

Following (12) and (13), we obtain the value of the expected drifts as follows:

$$D_h = \lambda - r_h. \tag{14}$$

After formulating the expected drift, we then proceed by two steps.

Step 1: $L = \Theta(h)$ **and** $\Lambda < \alpha e^{-\alpha}$**.**

We first show that $|D_h|$ is finite. From (14), we have

$$|D_h| = |\lambda - r_h| \le |\lambda| + |r_h| \le \lambda + \sum_{k=1}^{h} k \xi_{h,k} \le \lambda + L, \tag{15}$$

which demonstrates that $|D_h|$ is finite.

We then derive the limit of D_h:

$$\lim_{h \to \infty} D_h = \lambda - \lim_{h \to \infty} r_h = L \left\{ \Lambda - \lim_{h \to \infty} \binom{h}{1} \frac{1}{L} \left[1 - \frac{1}{L} \right]^{h-1} \right\} = L(\Lambda - \alpha e^{-\alpha}), \tag{16}$$

where $\alpha \triangleq \frac{h}{L}$. We thus have

$$\lim_{h \to \infty} D_h < 0,$$

when $L = \Theta(h)$ and $\Lambda < \alpha e^{-\alpha}$.

It then follows from Lemma 1 that $(X_i)_{i \geq 0}$ is ergodic. Specially, when $\alpha = 1$, the system stability region is maximized, i.e., $\Lambda < e^{-1}$.

Step 2: $L = o(h)$ **or** $L = O(h)$ **or** $L = \Theta(h)$ **and** $\Lambda > \alpha e^{-\alpha}$.

In this step, we prove the instability of $(X_i)_{i \geq 0}$ by applying Lemma 2. Taking into consideration the impact of different relation between L and h on the limit of D_h, we have:

- when $L = o(h)$, it holds that

$$\Lambda - \lim_{h \to \infty} \binom{h}{1} (\frac{1}{o(h)})(1 - \frac{1}{o(h)})^{h-1} = \Lambda - \lim_{h \to \infty} \frac{h}{o(h)} e^{-\frac{h}{o(h)}} = \Lambda > 0;$$

- when $L = O(h)$, it holds that

$$\Lambda - \lim_{h \to \infty} \binom{h}{1} (\frac{1}{O(h)})(1 - \frac{1}{O(h)})^{h-1} = \Lambda - \lim_{h \to \infty} \frac{h}{O(h)} e^{-\frac{h}{O(h)}} = \Lambda > 0;$$

- when $L = \Theta(h)$ and $\Lambda > \alpha e^{-\alpha}$, it holds that

$$\Lambda - \lim_{h \to \infty} \binom{h}{1} \frac{1}{\Theta(h)} \left[1 - \frac{1}{\Theta(h)}\right]^{h-1} = \Lambda - \alpha e^{-\alpha} > 0.$$

Substituting these results into (16), we have $\lim_{h \to \infty} D_h > 0$, which proves the first condition in Lemma 2.

Next, we will show that the second condition is also satisfied. Referring to [20], The key step here is to show that the downward part of the expected drift at state h ($h \geq 0$), formally defined as

$$d_{h^-} \triangleq \sum_{k=\max(h-L,0)}^{h} (k - h) P_{h,k}, \tag{17}$$

is lower-bounded. According to the transition probability (9), we have

$$d_{h^-} = - \sum_{k=1}^{\min(h,L)} k \sum_{n=0}^{\min(h,L)-k} \lambda_n \xi_{h,n+k}.$$

After some algebraic operations, we further get

$$d_{h^-} = - \sum_{n=0}^{\min(h,L)} \lambda_n \sum_{k=0}^{\min(h,L)-n} k \xi_{h,n+k} \geq - \sum_{n=0}^{\min(h,L)} \lambda_n \sum_{k=0}^{\min(h,L)-n} (k+n) \xi_{h,n+k} \geq - \sum_{n=0}^{\min(h,L)} \lambda_n \sum_{k=0}^{\min(h,L)} k \xi_{h,k}$$

$$\geq - \sum_{n=0}^{\min(h,L)} \lambda_n r_h \geq - \min(h, L). \tag{18}$$

It then holds that d_{h^-} is lower-bounded, which completes the proof of Step 2 and also the proof of Theorem 1. ∎

C. System behavior in instability region

It follows from Theorem 1 that the system is not stable in the following three conditions: $L = o(h)$; $L = O(h)$; and $L = \Theta(h)$ but $\Lambda > \alpha e^{-\alpha}$. In this section, we further investigate the system behavior in the instability region, i.e., when $(X_i)_{i \geq 0}$ is nonergodic. The key results are given in Theorem 2.

Before proving Theorem 2, we first introduce the following lemma [10] on the conditions for the transience of a Markov chain.

Lemma 3 ([10]). *Let $(X_i)_{i \geq 0}$ be an irreducible and aperiodic Markov chain with the nonnegative integers as its state space and one-step transition probability matrix $\mathbf{P} = \{P_{hk}\}$. $(X_i)_{i \geq 0}$ is transient if and only if there exists a sequence $\{y_i\}_{i \geq 0}$ such that*

1) y_i ($i \geq 0$) is bounded,
2) for some $h \geq N$, $y_h < y_0, y_1, \cdots, y_{N-1}$,
3) for some integer $N > 0$, $\sum_{k=0}^{\infty} y_k P_{hk} \leq y_h$, $\forall h \geq N$.

Armed with Lemma 3, we now prove Theorem 2.

Proof of Theorem 2: The key to prove Theorem 2 is to show the existence of a sequence satisfying the properties in Lemma 3, so we first construct the following sequence (19) and then prove that it satisfies the required conditions.

$$y_i = \frac{1}{(i+1)^\theta}, \quad \theta \in (0, 1). \tag{19}$$

It can be easily checked that $\{y_i\}$ satisfies the first two properties in Lemma 3.

Next, We distinguish three cases in the rest of proof.

Case 1: $L = o(h^{1-\epsilon})$.

In this case, we first derive an equivalent condition on ξ_{hk} shown in the following lemma for the third property in Lemma 3. The proof is detailed in Appendix A.

Lemma 4. *If $\lim_{k\to\infty} k^2 \sup_{h\geq k} \xi_{hk} = 0$, then $(X_i)_{i\geq 0}$ is always transient for $L = o(h^{1-\epsilon})$.*

Lemma 4 provides another approach to prove the transience of $(X_i)_{i\geq 0}$ in the instability region, i.e., whether ξ_{hk} is of the property that $\lim_{k\to\infty} k^2 \sup_{h\geq k} \xi_{hk} = 0$ when $L = o(h^{1-\epsilon})$. To this end, we introduce the following lemma by which an approximate result on ξ_{hk} can be derived.

Lemma 5 ([5]). *Let h packets each is sent in a slot picked at random among L time-slots in the frame i. And Let $h, L \to \infty$ such that $\rho_j = L\frac{e^{-h/L}}{j!}(\frac{h}{L})^j$ remains bounded, then the probability $P(m_j)$ of finding exactly m_j time-slots with j packets can be approximated by the following Poisson distribution with the parameter ρ_j,*

$$P(m_j) = e^{-\rho_j}\frac{\rho_j^{m_j}}{m_j!}. \tag{20}$$

Now, we want to show Lemma 5 is applicable When $L = o(h^{1-\epsilon})$ for h large enough. This is true because

$$0 \leq \rho_j \leq \frac{h^j}{j!L^{j-1}e^{h^\epsilon}} \leq \frac{h^j}{j!L^{j-1}} \cdot \frac{(\lceil\frac{1}{\epsilon}\rceil j)!}{(h^\epsilon)^{\lceil\frac{1}{\epsilon}\rceil j}} \leq \frac{(\lceil\frac{1}{\epsilon}\rceil j)!}{j!L^{j-1}}, \tag{21}$$

meaning that ρ_j is bounded if j is finite.

For FSA-SPR, the probability ξ_{hk} that k packets are received successfully is equivalent to the probability $P(m_j = k)$ when $j = 1$, we thus have

$$\xi_{hk} = e^{-\rho_1}\frac{\rho_1^k}{k!}, \tag{22}$$

where $\rho_1 \leq \lceil\frac{1}{\epsilon}\rceil!$.

Using the inequality $\frac{n^k}{k!} \leq (\frac{ne}{k})^k$ yields

$$0 \leq \xi_{hk} \leq e^{-\rho_1}(\frac{\rho_1 e}{k})^k.$$

Thus,

$$0 \leq k^2 \xi_{hk} \leq e^{-\rho_1}k^2(\frac{\rho_1 e}{k})^k.$$

Furthermore, since $e\rho_1 \leq e\lceil\frac{1}{\epsilon}\rceil!$ is constant, we have

$$\lim_{k\to\infty} e^{-\rho_1}k^2(\frac{\rho_1 e}{k})^k = 0,$$

which means ξ_{hk} in FSA-SPR satisfies the property that $\lim_{k\to\infty} k^2 \sup_{h\geq k} \xi_{hk} = 0$ when $L = o(h^{1-\epsilon})$. The first case on the transience of $(x_i)_{i\geq 0}$ is proved.

Case 2: $L = \Theta(h)$ **and** $\Lambda > \alpha e^{-\alpha}$.

In this case, we directly prove that y_i satisfies the third property in Lemma 3 by algebraic operations. The key steps we need here are to obtain upper bounds of ξ_{hk} and the arrival rate in a new way. To this end, we first recomputed ξ_{hk} as follows when $L = \Theta(h)$:

$$\begin{cases} \xi_{hk} &= \frac{\binom{L}{k}k!(L-k)^{h-k}}{L^h} - \frac{\binom{L}{k+1}(k+1)!(L-k-1)^{h-k-1}}{L^h} \\ &\leq \frac{\binom{L}{k}k!}{L^k}\left((1-\frac{k}{L})^{h-k} - (1-\frac{k+1}{L})^{h-k}\right) \\ &\leq \frac{\binom{L}{k}k!}{L^k} \leq (1-\frac{k/2}{L})^{k/2}, & k < \min(h, L), \\ \xi_{hh} &= \frac{\binom{L}{h}h!}{L^h} \leq (1-\frac{h/2}{L})^{h/2} \leq (1-\alpha/2)^{h/2}, & k = h \leq L. \end{cases} \tag{23}$$

The rational behind the above inequalities is as follows: the event of exactly k successful packets is equivalent to the event that at least k successful packets deducts the part of at least $k+1$ successful packets.

Next, we introduce a lemma to bound the probability distribution of the arrival rate. When the number of new arrivals per slot A_{il} is Poisson distributed with the mean Λ, the number of new arrivals per frame N_i (A_{il} and N_i is formally defined in Sec. III.) is also a Poisson random variable with the mean $\lambda = L\Lambda > he^{-\alpha}$.

Lemma 6 ([11]). *Given a Poisson distributed variable X with the mean μ, the following inequality holds:*

$$Pr[X \leq x] \leq \frac{e^{-\mu}(e\mu)^x}{x^x}, \; \forall \; x < \mu.$$

Applying Lemma 6, we have

$$P\{N_i \leq he^{-\alpha}\} \leq \frac{e^{-\lambda}(e\lambda)^{\frac{h}{e^\alpha}}}{(\frac{h}{e^\alpha})^{\frac{h}{e^\alpha}}} \leq e^{-\frac{h}{e^\alpha}(\frac{L\Lambda}{he^{-\alpha}}-1)}\left(\frac{L\Lambda}{he^{-\alpha}}\right)^{\frac{h}{e^\alpha}} \leq \frac{1}{a^{\frac{h}{e^\alpha}}}, \tag{24}$$

where $a \triangleq \frac{he^{-\alpha}}{L\Lambda} \cdot e^{\frac{L\Lambda}{he^{-\alpha}}-1} > 1$, following the fact that $e^x > 1 + x$, for $\forall \; x > 0$.

Armed with (23) and (24), we prove that the third property in Lemma 3 is also satisfied when $L = \Theta(h)$ by considering different value of L.

(1) $h \leq L$.

By employing the sequence $\{y_i\}_{i \geq 0}$, we have

$$\sum_{k=0}^{\infty} y_k P_{hk} = \sum_{k=0}^{\lfloor \frac{h}{e^\alpha} \rfloor} y_k P_{hk} + \sum_{k=\lfloor \frac{h}{e^\alpha} \rfloor+1}^{h} y_k P_{hk} + \sum_{k=h+1}^{\infty} y_k P_{hk} \tag{25}$$

$$\leq \sum_{k=0}^{\lfloor \frac{h}{e^\alpha} \rfloor} \sum_{n=0}^{k} \lambda_n + y_{h+1} + \sum_{k=\lfloor \frac{h}{e^\alpha} \rfloor+1}^{h} y_k \sum_{n=0}^{k} \lambda_n (1 - \frac{n+h-k}{2L})^{(n+h-k)/2}$$

$$\leq \frac{\lfloor \frac{h}{e^\alpha} \rfloor + 1}{a^{\lfloor \frac{h}{e^\alpha} \rfloor}} + y_{h+1} + \sum_{k=\lfloor \frac{h}{e^\alpha} \rfloor+1}^{h} y_k \sum_{n=0}^{k} \lambda_n (1 - \frac{n}{2L})^{\frac{n}{2}}$$

$$\leq \frac{\lfloor \frac{h}{e^\alpha} \rfloor + 1}{a^{\lfloor \frac{h}{e^\alpha} \rfloor}} + y_{h+1} + \sum_{k=\lfloor \frac{h}{e^\alpha} \rfloor+1}^{h} y_k \Big(\sum_{n=0}^{\lfloor \frac{h}{e^\alpha} \rfloor} \lambda_n (1 - \frac{n}{2L})^{\frac{n}{2}} + \sum_{n=\lfloor \frac{h}{e^\alpha} \rfloor+1}^{k} \lambda_n (1 - \frac{n}{2L})^{\frac{n}{2}} \Big)$$

$$\leq \frac{\lfloor \frac{h}{e^\alpha} \rfloor + 1}{a^{\lfloor \frac{h}{e^\alpha} \rfloor}} + \sum_{k=\lfloor \frac{h}{e^\alpha} \rfloor+1}^{h} y_k \Big(\sum_{n=0}^{\lfloor \frac{h}{e^\alpha} \rfloor} \lambda_n + (1 - \frac{\frac{h}{e^\alpha}}{2L})^{\frac{h}{2e^\alpha}} \Big) + y_{h+1}$$

$$\leq \frac{\lfloor \frac{h}{e^\alpha} \rfloor + 1}{a^{\lfloor \frac{h}{e^\alpha} \rfloor}} + \frac{h - \lfloor \frac{h}{e^\alpha} \rfloor}{(\frac{h}{e^\alpha} + 2)^\theta} \Big(\frac{1}{a^{\frac{h}{e^\alpha}}} + (1 - \frac{\alpha}{2e^\alpha})^{\frac{h}{2e^\alpha}} \Big) + y_{h+1} \leq \frac{1}{(h+1)^\theta}, \quad \text{as } h \to \infty, \tag{26}$$

where the last inequality holds because

- $\frac{\lfloor \frac{h}{e^\alpha} \rfloor + 1}{a^{\lfloor \frac{h}{e^\alpha} \rfloor}} + \frac{h - \lfloor \frac{h}{e^\alpha} \rfloor}{(\frac{h}{e^\alpha} + 2)^\theta} \Big(\frac{1}{a^{\frac{h}{e^\alpha}}} + (1 - \frac{\alpha}{2e^\alpha})^{\frac{h}{2e^\alpha}} \Big) \sim \Theta\big(\frac{h}{a_0^{he^{-\alpha}}} \big)$ where $a_0 = \max\Big(a, \frac{1}{\sqrt{1-\frac{\alpha}{2e^\alpha}}} \Big) > 1$,
- $y_h - y_{h+1} = \frac{1}{(h+1)^\theta}(1 - (1 - \frac{1}{h+2})^\theta) \geq \frac{\theta}{(h+1)^\theta(h+2)}$ where we use the fact that $(1 - \frac{1}{h+2})^\theta \leq 1 - \frac{\theta}{h+2}$ following Taylor's theorem.

(2) $he^{-\alpha} < L < h$.

With the same reasoning as (26) and noticing the fact that at most $L - 1$ packets are successfully received when $L < h$, we have

$$\sum_{k=0}^{\infty} y_k P_{hk} = \sum_{k=0}^{h-L-1} y_k P_{hk} + \sum_{k=h-L}^{h} y_k P_{hk} + \sum_{k=h+1}^{\infty} y_k P_{hk} \leq y_{h+1} + \sum_{k=h-L}^{h} y_k \sum_{n=0}^{L+k-h} \lambda_n (1 - \frac{n+h-k}{2L})^{(n+h-k)/2}$$

$$\leq y_{h+1} + \sum_{k=h-L}^{h} y_k \sum_{n=0}^{L} \lambda_n (1 - \frac{n}{2L})^{\frac{n}{2}} \leq y_{h+1} + \sum_{k=h-L}^{h} y_k \Big(\sum_{n=0}^{\lfloor \frac{h}{e^\alpha} \rfloor} \lambda_n + \sum_{n=\lfloor \frac{h}{e^\alpha} \rfloor+1}^{L} (1 - \frac{n}{2L})^{\frac{n}{2}} \Big)$$

$$\leq \frac{L}{(h-L+1)^\theta} \Big(\frac{1}{a^{\frac{h}{e^\alpha}}} + (1 - \frac{\alpha}{2e^\alpha})^{\frac{h}{2e^\alpha}} \Big) + y_{h+1} \leq \frac{1}{(h+1)^\theta}, \quad \text{as } h \to \infty. \tag{27}$$

(3) $L \leq he^{-\alpha}$.

With the same reasoning as (26) and (27), we have

$$\sum_{k=0}^{\infty} y_k P_{hk} = \sum_{k=0}^{h-L-1} y_k P_{hk} + \sum_{k=h-L}^{h} y_k P_{hk} + \sum_{k=h+1}^{\infty} y_k P_{hk} \leq y_{h+1} + \sum_{k=h-L}^{h} y_k \sum_{n=0}^{L+k-h} \lambda_n (1 - \frac{n+h-k}{2L})^{(n+h-k)/2}$$

$$\leq y_{h+1} + \sum_{k=h-L}^{h} y_k \sum_{n=0}^{L} \lambda_n \leq \frac{L}{(h-L+1)^\theta} \cdot \frac{1}{a^{\frac{h}{e^\alpha}}} + y_{h+1} \leq \frac{1}{(h+1)^\theta}, \quad \text{as } h \to \infty. \tag{28}$$

Consequently, the third property in Lemma 3 holds, which completes the proof of the transience of $(x_i)_{i \geq 0}$ for Case 2. Next, we proceed with the proof for the third case.

Case 3: $L = O(h)$.

When $L = O(h)$, it is easy to see that the expected number of new arrivals per frame $\lambda = L\Lambda \gg h$. Since N_i is Poisson distributed as mentioned in Case 2 above, by employing (24), we also have

$$P\{N_i \leq h\} \leq \frac{1}{a_1^h}, \tag{29}$$

where $a_1 \triangleq \frac{h}{L\Lambda} \cdot e^{\frac{L\Lambda}{h}-1} \gg 1$, following the fact that $e^x > 1 + x$, for $\forall\, x > 0$.

We now prove the transience of $(x_i)_{i \geq 0}$ when $L = O(h)$. To that end, by combining (19) and (29), for h large enough, we have

$$\sum_{k=0}^{\infty} y_k P_{hk} = \sum_{k=0}^{h} y_k P_{hk} + \sum_{k=h+1}^{\infty} y_k P_{hk} \leq \sum_{k=0}^{h} \frac{1}{(k+1)^\theta} \sum_{n=0}^{k} \lambda_n + \frac{1}{(h+2)^\theta}$$

$$\leq \sum_{k=0}^{h} \sum_{n=0}^{k} \lambda_n + \frac{1}{(h+2)^\theta} \leq \frac{h+1}{a_1^h} + \frac{1}{(h+2)^\theta} \leq \frac{1}{(h+1)^\theta}, \quad \text{as } h \to \infty. \tag{30}$$

where the last inequality is true following the same reasoning as (26).

Consequently, it follows Lemma 3 that the backlog Markov chain $(X_i)_{i \geq 0}$ is also transient when $L = O(h)$, which completes the proof of Theorem 2. ∎

VI. STABILITY ANALYSIS OF FRAME SLOTTED ALOHA SYSTEM WITH MULTIPACKET RECEPTION

In this section, we study stability properties of FSA-MPR where the receiver can correctly receive multiple packets transmitted simultaneously. This MPR capability can be achieved through the smart antenna techniques such as MIMO, or coding techniques such as CDMA. Specifically, we exploit MPR-M model that only up to M simultaneous packets can be decoded successfully at the receiver, which differs FSA-MPR from FSA-SPR. The value of M is fixed and is known beforehand.

Note that FSA-SPR investigated previously is a special case of MPR-M with $M = 1$, meaning the only difference between FSA-MPR and FSA-SPR is the packet success probability. Hence, with the packet success probability of FSA-MPR, we can develop the analysis on the stability properties of FSA-MPR using similar ideas in the FSA-SPR case. Firstly, conditions for the stability of FSA-MPR are established and we further analyse the system behavior in instability region subsequently.

A. Stability analysis

We employ Lemma 1 and Lemma 2 as mathematical tools to study the stability properties of FSA-MPR, more specifically, in the proof of Theorem 3.

Proof of Theorem 3: When h is large, with Poisson approximation, (8) can be approximated as follows:

$$r_h = L \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!} = L\alpha \sum_{x=0}^{M-1} e^{-\alpha} \frac{\alpha^x}{x!} \leq L\alpha \tag{31}$$

where $\alpha \triangleq \frac{h}{L}$. And we define $\Phi(\alpha) \triangleq \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}$ in the rest of this paper.

We then develop our proof in three steps.

Step 1: $L = \Theta(h)$ **and** $\Lambda < \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}$.

According to (15) and (31), the expected drift at state h of $(x_i)_{i \geq 0}$ in FSA-MPR is finite as shown in the following inequality:

$$|D_h| \leq \lambda + |r_h| \leq \lambda + L\alpha,$$

which demonstrates the second conditions in Lemma 1 for the ergodicity of $(x_i)_{i \geq 0}$.

Furthermore, when $L = \Theta(h)$, we have

$$\lim_{h \to \infty} D_h = L(\Lambda - \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}) < 0,$$

if $\Lambda < \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}$.

It then follows from Lemma 1 that $(x_i)_{i \geq 0}$ is ergodic if $\Lambda < \Phi(\alpha)$ when $L = \Theta(h)$.

Step 2: stability region maximiser α^*.

In this step, we show that $\alpha^* = \Theta(M)$. Since the proof consists mainly of algebraic operations of function optimization, we state the following lemma proving Step 2 and detail its proof in Appendix B.

Lemma 7. *Let* α^* *denote the value of* α *that maximises the stability region, it holds that* $\alpha^* = \Theta(M)$

Step 3: $L = o(h)$ **or** $L = O(h)$ **or** $L = \Theta(h)$ **and** $\Lambda > \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}$.

In this step, we prove the instability of $(x_i)_{i \geq 0}$ by applying Lemma 2.

According to the analysis on the impact of different relation between L and h in the the second step proof of Theorem 1, we know that $\lim_{h \to \infty} D_h > 0$, if the conditions in the second part of Theorem 3 are satisfied.

In addition, similar to (18), we can prove that the lower-bounded downward part of expected drift in FSA-MPR as follows:

$$d_{h^-} \geq -M \min(h, L),$$

which proves the instability of FSA-MPR following Lemma 2, which completes the proof of Theorem 3. ∎

B. System behavior in instability region

It follows from Theorem 3 that the system is not stable in the following three conditions: $L = o(h)$; $L = O(h)$; $L = \Theta(h)$ but $\Lambda > \sum_{x=1}^{M} e^{-\alpha} \frac{\alpha^x}{(x-1)!}$. In this subsection, we further investigate the system behavior in the instability region, i.e., when $(X_i)_{i \geq 0}$ is nonergodic. The key results are given in Theorem 4, whose proof is detailed as follows.

Proof of Theorem 4: We prove the theorem in three cases as the proof of Theorem 2.

Case 1: $L = o(h^{1-\epsilon})$.

Recall Lemma 4, we can demonstrate the transience of $(x_i)_{i \geq 0}$ by proving $\lim_{k \to \infty} k^2 \xi_{hk} = 0$ in FSA-MPR. To this end, we derive ξ_{hk}.

In (21), we have shown that ρ_j is bounded when $L = o(h^{1-\epsilon})$ for $j = 1, 2, \cdots, M$, so the probability of finding exactly m_j time-slots with j packets in FSA-MPR can also be approximated by the Poisson distribution with the parameter ρ_j, following from Lemma 5 .

Consequently, we can derive the probability that all h packets fail to be received, i.e., there is not any slot with $1 \leq j \leq M$ packets, as follows:

$$\xi_{h0} = e^{-(\rho_1 + \rho_2 + \cdots + \rho_M)}. \tag{32}$$

Further, we can get the following inequalities:

$$\xi_{hk} \leq 1 - e^{-(\rho_1 + \rho_2 + \cdots + \rho_M)} \leq 1 - e^{-M\rho_M}, \; 1 \leq k < L,$$

where we use the fact that the probability of exact $k \geq 1$ successfully received packets among h packets is less than that of at least one packet received successfully in the first inequality. And the second inequality above follows from the fact that when $L = o(h^{1-\epsilon})$, it holds that

$$\rho_M > \rho_{M-1} > \cdots > \rho_2 > \rho_1.$$

As a result, we have

$$0 \leq \lim_{k \to \infty} k^2 \sup_{h \geq k} \xi_{hk} = \lim_{k \to \infty} k^2 (1 - e^{-M\rho_M}) \leq \lim_{h \to \infty} h^2 (1 - e^{-M\rho_M}) \leq \lim_{h \to \infty} \frac{e^{M\rho_M} - 1}{(1/h^2)}$$

$$\leq \lim_{h \to \infty} \frac{(M+3)(M+2)(M+1)ML^3}{2e^{h/L}} \leq 0,$$

where we use L'Hospital's rule and the fact that $\rho_M \to 0$ at an exponential rate as $h \to \infty$.

Thus, according to Lemma 4, the backlog Markov chain $(x_i)_{i \geq 0}$ is transient when $L = o(h^{1-\epsilon})$.

Case 2: $L = \Theta(h)$ **and** $\Lambda > \alpha$.

In this case, we have $\lambda > L\Lambda > h$. Using similar reasoning as (30), we have :

$$\sum_{k=0}^{\infty} y_k P_{hk} \leq \frac{h+1}{a_2^h} + \frac{1}{(h+2)^\theta} \leq \frac{1}{(h+1)^\theta}, \text{ as } h \to \infty,$$

where $a_2 = \frac{\alpha}{\Lambda} e^{\frac{\Lambda}{\alpha} - 1} > 1$.

Thus when $L = \Theta(h)$ and $\Lambda > \alpha$, $(x_i)_{i \geq 0}$ is transient.

Case 3: $L = O(h)$.

In the analysis of FSA-SPR system, we have proven that when $L = O(h)$, the Markov chain $(x_i)_{i \geq 0}$ is always transient, independent of ξ_{hk}. Noticing that ξ_{hk} is the only difference between FSA-SPR and FSA-MPR, it holds that $(x_i)_{i \geq 0}$ is also transient under FSA-MPR. ∎

VII. CONCLUSION

In this paper, we have studied the stability of FSA-SPR and FSA-MPR by modeling the system backlog as a Markov chain. By employing drift analysis, we have obtained the closed-form conditions for the stability of FSA and shown that the stability region is maximised when the frame length equals the backlog size in the single packet reception model and when the ratio of the backlog size to frame length equals in order of magnitude the maximum multipacket reception capacity in the multipacket reception model. Furthermore, to characterise system behavior in the instable region, we have mathematically demonstrated the existence of transience of the Markov chain. Our results provide theoretical guidelines on the design of stable FSA-based protocols in practical applications such as RFID and M2M systems.

REFERENCES

[1] N. Abramson. The aloha system: Another alternative for computer communications. In *Proceedings of Fall 1970 AFIPS fall joint computer conference*, pages 281–285, 1970.

[2] L. Barletta, F. Borgonovo, and M. Cesana. A formal proof of the optimal frame setting for dynamic-frame aloha with known population size. arXiv preprint arxiv:1202.3914.

[3] S. Borst, M. Jonckheere, and L. Leskela. Stability of parallel queueing systems with coupled service rates. *Discrete Event Dynamic Systems*, 18(4):447–472, 2008.

[4] EPCglobal Inc. Radio-frequency identity protocols class-1 generation-2 UHF RFID protocol for communications at 860 mhz - 960 mhz version 1.0.9. *EPCglobal Inc.*, 17, 2005.

[5] W. Feller. *An introduction to probability theory and its applications*, volume 2. John Wiley & Sons, 2008.

[6] S. Ghez, S. Verdu, and S. C. Schwartz. Stability properties of slotted aloha with multipacket reception capability. *IEEE Transactions on Automatic Control*, 33(7):640–649, 1988.

[7] S. Ghez, S. Verdu, and S. C. Schwartz. Optimal decentralized control in the random access multipacket channel. *IEEE Transactions on Automatic Control*, 34(11):1153–1163, 1989.

[8] N. Johnson and S. Kotz. *Urn models and their application: an approach to modern discrete probability theory*. Wiley, 1977.

[9] M. Kaplan. A sufficient condition for nonergodicity of a markov chain. *IEEE Transection on Information Theory*, 25(4):470–471, 1979.

[10] J. F. Mertens, E. S. Cahn, and S. Zamir. Necessary and sufficient conditions for recurrence and transience of markov chains, in terms of inequalities. *Journal of Applied Probability*, 15(4):848–851, 1978.

[11] M. Mitzenmacher and E. Upfal. *Probability and computing: Randomized algorithms and probabilistic analysis*. Cambridge University Press, 2005.

[12] H. Okada, Y. Igarashi, and Y. Nakanishi. Analysis and application of framed aloha channel in satellite packet switching networks-fadra method. *Electronics and Communications in Japan*, 60:72–80, 1977.

[13] A. G. Pakes. Some conditions for ergodicity and recurrence of markov chains. *Operations Research*, 17(6):1058–1061, 1969.

[14] I. E. Pountourakis and E. D. Sykas. Analysis, stability and optimization of aloha-type protocols for multichannel networks. *Computer Communications*, 15(10):619–629, 1992.

[15] R. R. Rao and A. Ephremides. On the stability of interacting queues in a multiple-access system. *IEEE Transactions on Information Theory*, 34(5):918–930, 1988.

[16] L. G. Robert. Aloha packet system with and without slots and capture. *ACM SIGCOMM Computer Communication Review*, 5(2):28–42, 1975.

[17] W. A. Rosenkrantz and D. Towsley. On the instability of the slotted aloha multiaccess algorithm. *IEEE transactions on automatic control*, 28(10):994–996, 1983.

[18] J. Sant and V. Sharma. Performance analysis of a slotted-aloha protocol on a capture channel with fading. *Queueing Systems*, 34(1-4):1–35, 2000.

[19] F. C. Schoute. Dynamic frame length aloha. *IEEE Transactions on Communications*, 31(4):565–568, 1983.

[20] L. I. Sennott, P. A. Humblet, and R. L. Tweedie. Mean drifts and the non-ergodicity of markov chains. *Operations Research*, 31(4):783–789, 1983.

[21] W. Szpankowski. Packet switching in multiple radio channels: Analysis and stability of a random access system. *Computer Networks*, 7(1):17–26, 1983.

[22] W. Szpankowski. Stability conditions for some distributed systems: Buffered random access systems. *Advances in Applied Probability*, 26(2):498–515, 1994.

[23] B. S. Tsybakov and V. A. Mikhailov. Ergodicity of a slotted aloha system. *Problemy Peredachi Informatsii*, 15(4):73–87, 1979.

[24] S. Vasudevan, D. Towsley, D. Goeckel, and R. Khalili. Neighbor discovery in wireless networks and the coupon collector's problem. In *Proceedings of the 15th annual international conference on Mobile computing and networking*, pages 181–192, 2009.

[25] H. Vogt. Efficient object identification with passive rfid tags. In *International Conference on Pervasive Computing*, pages 98–113, 2002.

[26] J. E. Wieselthier, A. Ephremides, and A. Larry. An exact analysis and performance evaluation of framed aloha with capture. *IEEE Transactions on Communications*, 37(2):125–137, 1989.

[27] H. Wu, C. Zhu, R. J. La, and X. Liu. Fasa: Accelerated s-aloha using access history for event-driven m2m communications. *IEEE/ACM Transactions on Networking*, 21(6):1904–1917, 2013.

[28] L. Zhu and T. P. Yum. Optimal frame aloha-based anti-collision algorithm for rfid systems. *IEEE Transactions on communications*, 58(12):3583–3592, 2010.

[29] L. Zhu and T. P. Yum. A critical survey and analysis of rfid anti-collision mechanisms. *IEEE Communications Magazine*, 49(5):214–221, 2011.

APPENDIX A
PROOF OF LEMMA 4

Proof: Recall the definition of transition probability and we can get the following equivalent:

$$\sum_k y_k P_{hk} \leq y_h \iff \sum_{k=1}^{\min(h,L)} (y_{h-k} - y_h)P_{h,h-k} + \sum_{k=1}^{\infty} (y_{h+k} - y_h)P_{h,h+k} \leq 0.$$

Define $f'(h)$ and $g'(h)$ as follows:

$$
\begin{cases}
f'(h) = (h+1)^\theta & \sum_{k=1}^{\min(h,L)} \left(\frac{1}{h+1-k} - \frac{1}{h+1}\right) \\
& \cdot \sum_{n=0}^{\min(h,L)-k} \lambda_n \xi_{h,n+k}, \\
g'(h) = (h+1)^\theta & \sum_{k=1}^{\infty} \left(\frac{1}{h+1+k} - \frac{1}{h+1}\right) \\
& \cdot \sum_{n=0}^{\min(h,L)} \lambda_{n+k} \xi_{h,n}.
\end{cases}
\tag{33}
$$

We have

$$\sum_k y_k P_{hk} \leq y_h \iff f'(h) + g'(h) \leq 0.$$

Furthermore, the expected drift of X_i at state h can be also computed from the one-step transition probabilities (9),

$$D_h = -\sum_{k=1}^{\min(h,L)} k P_{h,h-k} + \sum_{k=1}^{\infty} k P_{h,h+k} = f(h) + g(h),$$

where $f(h)$ and $g(h)$ are defined as follows:

$$
\begin{cases}
f(h) = \sum_{k=1}^{\min(h,L)} k \sum_{n=0}^{\min(h,L)-k} \lambda_n \xi_{h,n+k}, \\
g(h) = \sum_{k=1}^{\infty} k \sum_{n=0}^{\min(h,L)} \lambda_{n+k} \xi_{h,n}.
\end{cases}
\tag{34}
$$

It is noted that in the proof of Theorem 1, we have show that $\lim_{h\to\infty} D_h \geq 0$ in the nonergodicity region. Consequently, it then holds that $(X_i)_{i\geq0}$ is transient if the following equation holds:

$$\lim_{h\to\infty} [f'(h) + g'(h) + \theta D_h] = 0.
\tag{35}$$

Noticing that $D_h = f(h) + g(h)$, we prove (35) by showing that: (1) $\lim_{h\to\infty}[f'(h) + \theta f(h)] = 0$; (2) $\lim_{h\to\infty}[g'(h) + \theta g(h)] = 0$.

We first prove $\lim_{h\to\infty}[f'(h) + \theta f(h)] = 0.$

From (33) and (34), we get

$$f'(h) + \theta f(h) = (h+1) \sum_{k=1}^{\min(h,L)} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \cdot \sum_{n=0}^{\min(h,L)-k} \lambda_n \xi_{h,n+k}, \qquad (36)$$

which is nonnegative since

$$(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} > 0, \ \forall 1 \le k \le h.$$

Define $\tau_k = \sup_{h \ge k} \xi_{hk}$, we have

$$0 \le f'(h) + \theta f(h) \le (h+1) \sum_{k=1}^{\min(h,L)} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \sum_{n=0}^{\min(h,L)-k} \lambda_n \tau_{n+k}$$

$$\le (h+1) \sum_{n=0}^{\infty} \lambda_n \sum_{k=1}^{h} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \tau_{n+k},$$

which can also be written as

$$f'(h) + \theta f(h) \le z_1(h) + z_2(h) \qquad (37)$$

where $z_1(h)$ and $z_2(h)$ are defined as follows:

$$\begin{cases} z_1(h) = (h+1) \sum_{n=0}^{\infty} \sum_{k=1}^{\lfloor \frac{h+1}{2} \rfloor} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \cdot \lambda_n \tau_{n+k}, \\ z_2(h) = (h+1) \sum_{n=0}^{\infty} \sum_{k=\lfloor \frac{h+3}{2} \rfloor}^{h} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \cdot \lambda_n \tau_{n+k}. \end{cases}$$

Next we show that $z_1(h)$ and $z_2(h)$ go to zero independently. Given $0 < x \le h$, define $m_h(x)$ as follows:

$$m_h(x) = \frac{h+1}{x^2} \left[(\frac{h+1}{h+1-x})^\theta - 1 \right] - \frac{\theta}{x},$$

which is positive and nondecreasing in x for $h \ge 1$. We also have

$$m_h \left(\lfloor \frac{h+1}{2} \rfloor \right) \le \frac{1}{h+1} [4(2^\theta - 1) - 2\theta] = \frac{A}{h+1},$$

where A is a positive constant only depending on θ. Thus,

$$z_1(h) = \sum_{n=0}^{\infty} \lambda_n \sum_{k=1}^{\lfloor \frac{h+1}{2} \rfloor} k^2 m_h(k) \tau_{n+k} \le \frac{A}{h+1} \sum_{n=0}^{\infty} \lambda_n \sum_{k=1}^{\lfloor \frac{h+1}{2} \rfloor} k^2 \tau_{n+k} \le \frac{A}{h+1} \sum_{n=0}^{\infty} \lambda_n \sum_{k=1}^{n+\lfloor \frac{h+1}{2} \rfloor} k^2 \tau_{n+k}.$$

If the condition $\lim_{k \to \infty} k^2 \sup_{h \ge k} \xi_{hk} = 0$ holds, i.e., $lim_{k \to \infty} k^2 \tau_k = 0$, then it holds that $lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k^2 \tau_k = 0$. Consequently, for $\forall \epsilon > 0$, we can choose h large enough such that $\sum_{k=1}^{n} k^2 \tau_k < n\epsilon$ for $n \ge \lfloor \frac{h+1}{2} \rfloor$. It holds that

$$z_1(h) \le \epsilon \frac{A}{h+1} \sum_{n=0}^{\infty} \lambda_n (n + \frac{h+1}{2}) = \epsilon A \left(\frac{\lambda}{h+1} + \frac{1}{2} \right) \to 0.$$

On the other hand, if h is large enough such that for $\forall k \ge \lfloor (h+3)/2 \rfloor$, it holds that $\tau_k < \epsilon/k^2$, we have

$$z_2(h) \le \epsilon \sum_{n=0}^{\infty} \lambda_n \sum_{k=\lfloor \frac{h+3}{2} \rfloor}^{h} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \frac{h+1}{(n+k)^2} \le \epsilon \sum_{n=0}^{\infty} \lambda_n \sum_{k=\lfloor \frac{h+3}{2} \rfloor}^{h} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \frac{h+1}{(\lfloor \frac{h+3}{2} \rfloor)^2}$$

$$\le \frac{4\epsilon}{h+1} \sum_{k=\lfloor \frac{h+3}{2} \rfloor}^{h} \left[(\frac{h+1}{h+1-k})^\theta - 1 - \frac{\theta k}{h+1} \right] \le \frac{2\theta(1+\theta)\epsilon}{h+1} \sum_{k=\lfloor \frac{h+3}{2} \rfloor}^{h} (\frac{k}{h+1})^2 \le \theta(1+\theta)\epsilon,$$

where we use the following inequalities

$$0 \le \frac{1}{(1+x)^\theta} - 1 + \theta x \le \theta(1+\theta) \frac{x^2}{2}, \quad (x \ge 0, 0 < \theta < 1). \qquad (38)$$

Therefore, it holds that $\lim_{h \to \infty} [f'(h) + \theta f(h)] = 0$.

We then prove $\lim_{h \to \infty} [g'(h) + \theta g(h)] = 0$.

From (33) and (34), we get

$$g'(h) + \theta g(h) = (h+1) \sum_{k=1}^{\infty} \left[(\frac{h+1}{h+1+k})^\theta - 1 + \frac{\theta k}{h+1} \right] \cdot \sum_{n=0}^{\min(h,L)} \lambda_{n+k} \xi_{h,n}.$$

Since $\left[(\frac{h+1}{h+1+k})^\theta - 1 + \frac{\theta k}{h+1} \right] \ge 0$, after some algebraic operations, we have

$$g'(h) + \theta g(h) \le (h+1) \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{n} \left[(\frac{h+1}{h+1+k})^\theta - 1 + \frac{\theta k}{h+1} \right] \cdot \xi_{h,n-k}$$

Using the inequalities (38), we have

$$0 \leq g'(h) + \theta g(h) \leq \theta \frac{(\theta+1)}{2}(h+1) \sum_{n=1}^{N} \lambda_n \sum_{k=1}^{n} \frac{k^2}{(h+1)^2} \xi_{h,n-k} + \theta(h+1) \sum_{n=N+1}^{\infty} \lambda_n \sum_{k=1}^{n} \frac{k}{h+1} \xi_{h,n-k}$$

$$\leq \frac{1}{h+1} \sum_{n=1}^{N} n^2 \lambda_n + \sum_{n=N+1}^{\infty} n \lambda_n.$$

For $\forall\, \epsilon > 0$, we choose N such that $\sum_{n=N+1}^{\infty} n\lambda_n < \epsilon/2$; then choose h large enough so that $\frac{1}{h+1} \sum_{n=1}^{N} n^2 \lambda_n < \epsilon/2$; we have $\lim_{h \to \infty}[g'(h) + \theta g(h)] = 0$, which completes the proof of the second part and also Lemma 4. ∎

APPENDIX B
PROOF OF LEMMA 7

Proof: We write $\Phi(\alpha)$ as

$$\Phi(\alpha) = e^{-\alpha} \sum_{i=1}^{M} \frac{\alpha^i}{(i-1)!},$$

whose derivative can be calculated as:

$$\Phi'(\alpha) = e^{-\alpha} \left[\sum_{i=0}^{M-1} \frac{\alpha^i}{i!} - \frac{\alpha^M}{(M-1)!} \right].$$

We distinguish two cases.

- *Case 1:* $\alpha \geq M$. Note that $N! \leq N^{N-1} \; \forall N \in \mathbb{N}$, we have

$$\Phi'(\alpha) < \frac{e^{-\alpha}}{(M-1)!} \cdot \left(\sum_{i=1}^{M} M^{M-i} \alpha^{i-1} - \alpha^M \right) < \frac{e^{-\alpha}}{(M-1)!} \cdot \left(M\alpha^{M-1} - \alpha^M \right) < 0,$$

  meaning that $\Phi(\alpha)$ is monotonously decreasing when $\alpha \geq M$.

- *Case 2:* $\alpha \leq \frac{M-1}{e}$. Using the inequality $N! \geq (\frac{N}{e})^N$, we have

$$\Phi'(\alpha) \geq \frac{e^{-\alpha}}{(M-1)!} \cdot \left[(\frac{M-1}{e})^{M-1} + \alpha(\frac{M-1}{e})^{M-1} - \alpha^M \right] \geq \frac{e^{-\alpha}}{(M-1)!} \cdot \left(\alpha^{M-1} + \alpha^M - \alpha^M \right) > 0,$$

  meaning that $\Phi(\alpha)$ is monotonously increasing when $\alpha \leq \frac{M-1}{e}$.

Combining the analysis in both cases, we have proved that α^* maximising $\Phi(\alpha)$ falls into the interval $[\frac{M-1}{e}, M]$, i.e., $\alpha^* = \Theta(M)$. ∎