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Abstract— **In this paper we introduce the low-power high-gain observer, developed in [1], to solve problems of output regulation for nonlinear systems. We show how the new tool makes it possible the implementation of high dimensional controllers, that tipically arise when the ideal steady-state control that must be generated to secure zero regulation error is affected by uncertainties.**

I. Introduction

Nonlinear output regulation has been the subject of many researches in the last decades. Although the internal model approach has reached a mature stage (see among others [12], [11], [4], [8] and [3]) some problems are still unsolved as far as structural or parametric uncertainties are considered in the steady-state control laws. In this paper, we consider a class of minimum-phase nonlinear systems, in which the steady-state control action needed to ensure a zero regulation error (the so-called "friend") is assumed to satisfy a possibly nonlinear regression formula. Under such hypothesis, in [5] the authors showed how the theory of high-gain observers stands out as a very effective tool to deal with asymptotic regulation problems. The main drawback in using such tools for regulator design, though, is linked to the fact that high gain structures are hard to implement if the dimension of the internal model is high. As a matter of fact, the standard theory of high-gain observers asks that the high-gain parameter is powered up to the order of the internal model, with the latter that is equal to the highest order of derivative of the "friend" characterizing the regression law mentioned before. This, in turn, makes the theory ineffective in practical applications whenever the regression law involves many time-derivative of the friend. This is the case, in fact, when the friend has uncertainties (see [7]). Such a drawback is particularly evident in [9], in which the authors introduced a new approach to the design of robust internal models, which, taking advantage of the results of [5], allows one to deal with parameter uncertainties in the exosystem when the exogenous input appear as a superposition of some uncertain linear oscillators. The peculiarity of this approach relies on the fact that no explicit adaptation scheme is used for the parameter estimation, rather the robustness features come from a particular *immersion* property of the internal model. A drawback of this method is given by the fact that the order of the internal model is increased of four times with respect to those obtained in the case of known parameters, requiring hence an equally increase in the order of the high-gain based controller. In [7] this idea has been developed further to more general nonlinear exosystems. Although no complete theoretical foundation is provided, the authors proposed a general procedure which can be applied case by case to many instances of nonlinear structures. This approach leads to robust designs of the internal models, requiring no knowledge of the parameters and no adaptation. However, the price to pay resides again in a huge increase of the internal model dimension, by thus raising the problems of implementation mentioned before. In this paper we show how the low power high gain observers, recently introduced in [1], can be applied to the problem of robust output regulation, by substantially overtaking the problems linked to the high power of the high-gain parameter mentioned before.

Notation: We denote a triplet (A_n, B_n, C_n) of dimension n in prime form as

$$A_n = \begin{pmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{pmatrix}, \quad B_n = \begin{pmatrix} 0_{(n-1)\times 1} \\ 1 \end{pmatrix},$$
$$C_n = \begin{pmatrix} 1 & 0_{1\times(n-1)} \end{pmatrix}.$$

We denote with $|\cdot|$ any vector or induced matrix norm, with $\|s(\cdot)\|_\infty := \sup_{t\geq 0} |s(t)|$ the infinity norm of a time varying signal $s(t)$ and with $\|x\|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x-y|$ the distance of a vector $x \in \mathbb{R}^n$ to a set $\mathcal{A} \subset \mathbb{R}^n$

II. The Framework of Output Regulation by Means of High-Gain Tools

In this paper we consider the class of nonlinear systems with unitary relative degree that can be written in the form

$$\begin{aligned} \dot{z} &= f(w, z, e) \\ \dot{e} &= q(w, z, e) + b(w, z, e)u \end{aligned} \tag{1}$$

with state (z, e) taking values in $\mathbb{R}^n \times \mathbb{R}$, input $u \in \mathbb{R}$ and regulation error e. The functions $f(\cdot)$, $q(\cdot)$ and $b(\cdot)$ are assumed to be smooth in their arguments. The exogenous signal w represents reference signals to be tracked and/or possible disturbances to be rejected. We suppose it is generated by an exosystem of the form

$$\dot{w} = s(w) \tag{2}$$

where the state w ranges in a compact set $W \subset \mathbb{R}^{n_w}$ assumed to be forward invariant for the dynamics of (2). This property is usually referred to as Poisson stability (see [4]). Finally, we suppose that the function $b(\cdot)$ is bounded from below

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by a positive constant $\underline{b} > 0$, namely $b(w,z,e) \geq \underline{b}$ for all $(w,z,e) \in W \times \mathbb{R}^n \times \mathbb{R}$. In this framework the problem of semiglobal output regulation reads as follows: given arbitrary compact sets $W \subset \mathbb{R}^s$, $Z \subset \mathbb{R}^n$ and $E \subset \mathbb{R}$ for the initial conditions $(w(0), z(0), e(0))$ of (1), (2), design an error-feedback controller of the form

$$\dot{\xi} = \psi(\xi, e), \qquad u = \gamma(\xi, e)$$

with state $\xi \in \mathbb{R}^d$, for some integer $d > 0$, and a compact set $\Xi \subset \mathbb{R}^d$ such that the trajectories of the closed-loop system originating from $W \times Z \times E \times \Xi$ are bounded and $\lim_{t\to\infty} e(t) = 0$ uniformly in the initial conditions. In this work we show how to solve the aforementioned problem under a certain number of assumptions.

Assumption 1 *There exists a smooth map $\pi : W \to \mathbb{R}^n$ satisfying*

$$L_s\pi(w) = f(w, \pi(w), 0), \qquad \forall\, w \in W$$

and the set

$$\mathcal{A} = \{(w,z) \in W \times \mathbb{R}^n \mid z = \pi(w)\} \tag{3}$$

is asymptotically and locally exponentially stable for system

$$\dot{w} = s(w) \qquad\qquad \dot{z} = f(w, z, 0) \tag{4}$$

with a domain of attraction $\mathcal{D} \supset W \times Z$.

In this setting a crucial role is played by the function

$$u^\star(w) := -\frac{q(w, \pi(w), 0)}{b(w, \pi(w), 0)} \tag{5}$$

As a matter of fact $u^\star(w)$ is the control action needed to make the set $\mathcal{A} \times \{0\}$ invariant for (1), (2), thus keeping the regulated error to zero in the steady state. By following the framework of [5], we make the further assumption that the function $u^\star(w)$ satisfies a nonlinear regression formula of the kind

$$L_s^d u^\star(w) = \phi(u^\star(w), L_s u^\star(w), \ldots, L_s^{d-1} u^\star(w)) \tag{6}$$

for any $w \in W$, for some positive integer d and for some known locally Lipschitz function $\phi : \mathbb{R}^d \to \mathbb{R}$. Under this hypothesis, in [5] it has been shown that the problem of output regulation is solved by a regulator of the form

$$\dot{\xi} = F(\xi) + Gv\,, \qquad\qquad u = \gamma(\xi) + v \tag{7}$$

with initial conditions in a compact set $\Xi \subset \mathbb{R}^d$ and where the functions $F(\cdot)$, G and $\gamma(\cdot)$ are chosen as

$$\begin{array}{rcl} F(\xi) & := & A_n\xi + B_n\phi_s(\xi) \\ G & := & \mathrm{col}(\lambda_1 g, \lambda_2 g^2, \ldots, \lambda_d g^d) \\ \gamma(\xi) & := & C_n\xi \end{array} \tag{8}$$

being (A_n, B_n, C_n) a triplet in prime form of dimension d, $\{\lambda_1, \ldots, \lambda_d\}$ a set of coefficients of an Hurwitz polynomial, $\phi_s(\cdot)$ a uniformly bounded and locally Lipschitz function which agrees with $\phi(\cdot)$ in $\mathcal{A}$, $g > 1$ a design parameter, and v an additional input defined as

$$v = -\mathrm{sgn}(e)\kappa(|e|)$$

where $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a properly defined class-$\mathcal{K}$ function that can be taken linear when the set $\mathcal{A}$ is locally exponentially stable. In [9], moreover, the authors extended the previous high-gain methodology to the case in which some unknown model mismatch is present in (6), namely they showed that, when $u^\star(w)$ satisfies a relation of the kind

$$L_s^d u^\star(w) = \phi(u^\star(w), L_s u^\star(w), \ldots, L_s^{d-1} u^\star(w)) + \nu(w) \tag{9}$$

for some unknown continuous function $\nu : \mathbb{R}^s \to \mathbb{R}$, then the same regulator (7), (8) achieves practical regulation as summarized in the forthcoming proposition.

Proposition 1 *Let Assumption 1 be fulfilled and assume $u^\star(w)$ satisfies* (9)*. Let Ξ be an arbitrary compact set of $\mathbb{R}^d$, then there exists a $g^\star \geq 1$ and a $\bar{c} > 0$ such that, for any $g > g^\star$ there exists a $\kappa^\star > 0$ such that for any $\kappa > \kappa^\star$ the trajectories of the closed loop system* (1)*,* (2)*,* (7)*,* (8) *with $v = -\kappa e$ and originating from $W \times Z \times E \times \Xi$ are bounded and such that*

$$\limsup_{t\to\infty} |e(t)| \;\leq\; \frac{\bar{c}}{\kappa\, g^d} \max_{w\in W} |\nu(w)| \tag{10}$$

For a proof of Proposition 1 the reader is referred to [9, Theorem 2].

III. Robust Internal Models by Nonlinear Regressions

In this section we briefly recall the main results of [7], in which the function $\phi(\cdot)$ in (6) is assumed to be affected by an uncertain parameter $\theta \in \mathbb{R}^p$, $p > 0$. Given two integers $b > a > 0$ and a smooth function $f : W \to \mathbb{R}$, we denote by $f_{[a,b]} = \mathrm{col}(L_s^a f, \ldots, L_s^b f)$ the vector of the Lie derivatives of $f(\cdot)$. As a starting point we assume that the function $u^\star(w)$ satisfies a regression formula of the kind (6), in which $\phi(\cdot)$ is affine in the unknown parameter vector θ, namely

$$L_s^k u^\star = h(u^\star_{[0,k-1]}) + \psi(u^\star_{[0,k-1]})\theta \tag{11}$$

for some $k > 0$ and for some smooth functions $h : \mathbb{R}^k \to \mathbb{R}$ and $\psi : \mathbb{R}^k \to \mathbb{R}^p$. Taking the i-th Lie derivative of (11), with $i > 0$, and collecting the obtained equations yields

$$u^\star{}_{[k,k+i]} = H_i(u^\star_{[0,k+i-1]}) + \Psi_i(u^\star_{[0,k+i-1]})\theta \tag{12}$$

having denoted with $H_i(\cdot)$ and $\Psi_i(\cdot)$ the functions

$$\begin{aligned} H_i(u^\star_{[0,k+i-1]}) &= \mathrm{col}(h_0(u^\star_{[0,k-1]}), \ldots, h_i(u^\star_{[0,k+i-1]})) \\ \Psi_i(u^\star_{[0,k+i-1]}) &= \mathrm{col}(\psi_0(u^\star_{[0,k-1]}), \ldots, \psi_i(u^\star_{[0,k+i-1]})) \end{aligned}$$

being $h_0(\cdot) = h(\cdot)$, $\psi_0(\cdot) = \psi(\cdot)$ and for $j = 1, \ldots, i$ $h_j(\cdot) = L_s h_{j-1}(\cdot)$, $\psi_j(\cdot) = L_s\psi_{j-1}(\cdot)$. At this point we make the following assumption.

Assumption 2 *There exists $m \geq p - 1$ such that*

$$\mathrm{rank}\left(\Psi_m\left(u^\star_{[0,k+m-1]}(w)\right)\right) = p, \qquad \forall w \in W.$$

If Assumption 2 holds then we have the following result.

Proposition 2 *Let* (11) *holds for some* $k > 0$ *and for some unknown* $\theta \in \mathbb{R}^p$*, let Assumption 2 be satisfied by some* $m \geq p-1$ *and let* $d = k+m+1$*. Then there exists a locally Lipschitz function* $\phi : \mathbb{R}^d \to \mathbb{R}$*, independent on* θ*, such that*

$$L_s^d u^\star(w) = \phi(u^\star(w), \ldots, L_s^{d-1} u^\star(w)) \tag{13}$$

for all $w \in W$.

Proof: Equation (12) for $i = m$ and Assumption 2 yield

$$\theta = \Psi_m^+(u^\star_{[0,k+m-1]})\Big(u^\star_{[k,k+m]} - H_m(u^\star_{[0,k+m-1]})\Big) \tag{14}$$

for all $w \in W$. Moreover, by taking the $(m+1)$-th Lie derivative of (11) we obtain

$$L_s^{k+m+1} u^\star = h_{m+1}(u^\star_{[0,k+m]}) + \psi_{m+1}(u^\star_{[0,k+m]})\theta. \tag{15}$$

Now, let $\bar{\phi} : u^\star_{[0,k+m]}(W) \to \mathbb{R}$ be the function defined as

$$\bar{\phi}\Big(u^\star_{[0,k+m]}\Big) = h_{m+1}(u^\star_{[0,k+m]}) + \psi_{m+1}(u^\star_{[0,k+m-1]}) \cdot \\ \cdot \Psi_m^+(u^\star_{[0,k+m-1]}) \cdot \Big(u^\star_{[k,k+m]} - H_m(u^\star_{[0,k+m-1]})\Big)$$

Using Assumption 2 it turns out that $\bar{\phi}(\cdot)$ is differentiable and, by construction, is such that

$$L_s^{k+m+1} u^\star(w) = \bar{\phi}\left(u^\star_{[0,k+m]}(w)\right) \tag{16}$$

Furthermore, by the Kirszbraun theorem (see for instance [6, Theorem 2.10.43]), there exists a known locally Lipschitz function $\phi : \mathbb{R}^d \to \mathbb{R}$ that agrees with $\bar{\phi}(\cdot)$ on $u^\star_{[0,k+m]}(W)$. This concludes the proof of the proposition. ∎

As a consequence of proposition 2, if Assumption 2 holds, we can construct a regulator of the form of (7), (8) with an extended dimension of $d = k+m+1$ using in the design any bounded Lipschitz extension of the function $\bar{\phi}$ in (16). The main practical drawback of this method, however, relies in the fact that, in general, the order d of the new regression law can be very high, and as a consequence, the implementation of the dynamic regulator (7), (8) may yield implementation issues. Such design indeed includes powers of the high-gain parameter g up to order d, and large values of g could easily lead to the infeasibility of any numerical implementation. In this work, we want to show how the new class of high-gain observers introduced in [1], referred to as *low-power high-gain observer*, can be applied to this framework in order to overcome these numerical problems. In particular, by following the low-power structure proposed in [1], we show how to implement a dynamic regulator of dimension $2d-2$ with the high-gain parameter which is powered only up to two, regardless the value of d.

IV. Internal Model Design via Low-Power High-Gain Observers

In this section we design a regulator of the form (7) solving the semiglobal practical output regulation problem for system (1), (2), under the same assumptions of Proposition 1. In particular, inspired by the result in [1] we propose a new dynamic regulator that achieves an asymptotic bound of the same form of (10) without implementing terms $(g, \ldots, g^d)$. To this end, let (A, B, C) be a triplet in prime form of dimension 2, $N := B^\top B \in \mathbb{R}^{2\times 2}$, $\Gamma := \begin{pmatrix} C & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{d\times(2d-2)}$, $\Psi := \mathrm{blkdiag}\,(C, \ldots, C, I_2) \in \mathbb{R}^{d\times(2d-2)}$ and $D_2(g) := \mathrm{diag}(g, g^2)$, with g the high-gain parameter. We define $\phi_s(\cdot)$ as any bounded locally Lipschitz function which agrees with $\phi(\cdot)$ on the set $\mathcal{A}$. The proposed *low-power* regulator of dimension $2d-2$ is defined by (7) with

$$F(\xi) := \begin{pmatrix} F_1(\xi) \\ \vdots \\ F_{d-1}(\xi) \end{pmatrix}, \; G := \begin{pmatrix} D_2(g)K_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \gamma(\xi) := \Gamma\xi\,, \tag{17}$$

where

$$\begin{aligned} F_1(\xi) &:= A\xi_1 + N\xi_2\,, \\ F_i(\xi) &:= A\xi_i + N\xi_{i+1} + D_2(g)K_i\eta_i\,, \\ &\qquad\qquad i = 2, \ldots, d-2 \\ F_{d-1}(\xi) &:= A\xi_{d-1} + B\phi_s(\Psi\xi) + D_2(g)K_{d-1}\eta_{d-1}\,, \end{aligned}$$

in which $\xi = (\xi_1, \ldots, \xi_{d-1}) \in \mathbb{R}^{2d-2}$, $\xi_i \in \mathbb{R}^2$, $K_i = \mathrm{col}(k_{i1}, k_{i2})$ for $i = 1, \ldots, d-1$, and

$$\eta_i := (B^\top \xi_{i-1} - C\xi_i), \qquad i = 2, \ldots, d-1\,.$$

The following proposition presents the main result of this paper.

Proposition 3 *Let Assumption 1 be fulfilled and assume* $u^\star(w)$ *satisfies* (9)*. Let* Ξ *be an arbitrary compact set of* $\mathbb{R}^{2d-2}$*. Then, there exists a choice of the coefficients* (k_{i1}, k_{i2})*,* $i = 1, \ldots, d-1$*, a* $g^\star \geq 1$ *and a* $\bar{\mu} > 0$ *such that for any* $g > g^\star$ *there exists a* $\kappa^\star > 0$ *such that for any* $\kappa > \kappa^\star$ *the trajectories of the closed loop system* (1), (2), (7), (17) *with* $v = -\kappa e$ *and originating from* $W \times Z \times E \times \Xi$*, are bounded and such that*

$$\limsup_{t\to\infty} |e(t)| \leq \frac{\bar{\mu}}{\kappa\, g^d} \max_{w\in W} |\nu(w)|. \tag{18}$$

Proposition 3 can be proved similarly to Theorem 2 in [9]. In particular the closed loop system (1), (2), (7), (17) is a system with unitary relative degree between the input v and the output e and zero dynamics described by

$$\begin{aligned} \dot{w} &= s(w)\,, \qquad \dot{z} = f(w, z, 0) \\ \dot{\xi}_1 &= A\xi_1 + N\xi_2 + D_2(g)K_1(c(w,z) - C\xi_1) \\ \dot{\xi}_i &= A\xi_i + N\xi_{i+1} + D_2(g)K_i\eta_i \\ &\qquad\qquad i = 2, \ldots, d-2 \\ \dot{\xi}_{d-1} &= A\xi_{d-1} + B\phi_s(\xi') + D_2(g)K_{d-1}\eta_{d-1} \end{aligned} \tag{19}$$

Under Assumption 1, the zero dynamics (19) of the closed loop system can be made ISS (input-to-state stable) relatively to a compact attractor with respect to the input $\nu(w)$, as summarized in the forthcoming technical lemma (proved in the Appendix). To this end, let define the function $\tau^e : W \to \mathbb{R}^{2d-2}$ as

$$\begin{aligned} \tau^e(w) &:= \mathrm{col}(\tau_1^e(w), \ldots, \tau_{d-1}^e(w)), \\ \tau_i^e(w) &:= \mathrm{col}(L_s^{i-1}u^\star(w), L_s^i u^\star(w)) \end{aligned}$$

Lemma 1 *There exists a constant $g^\star \geq 1$ such that, for any choice of $g > g^\star$, the zero dynamics* (19) *are ISS relatively to the set*

$$\mathcal{E} = \left\{(w, \xi, z) \in W \times \mathbb{R}^{2d-2} \times \mathbb{R}^n \mid \xi = \tau^e(w),\ z = \pi(w)\right\} \tag{20}$$

with respect to the input $\nu(w)$ with an asymptotic bound of the form

$$\limsup_{t\to\infty} \|(w, z, \xi)\|_{\mathcal{E}} \leq \frac{\mu}{g^d} \max_{w\in W} |\nu(w)| \tag{21}$$

for some positive constant $\mu > 0$.

Finally, in view of Lemma 1, the small-gain arguments of [12, Theorem 2 and 3] can be used to show that the claim of Proposition 3 holds for some constant $\bar{\mu} > 0$.

We stress that with the proposed design we obtained the same asymptotic properties (namely a gain $1/g^d$) of the standard high-gain design (7), (8), without requiring powers of the high-gain parameter g of order larger than 2, thus avoiding numerical problems.

V. Example

In this section we propose an example in which the low-power high-gain observer and the design approach by nonlinear regression are applied together to address a robust output regulation problem. The control goal is to asymptotically reject a disturbance which can be indistinguishably generated by an uncertain linear, Duffing or Van der Pool oscillator without the need of changing anything in the controller. To this end, a regression formula of the kind (6) is derived by following the procedure presented in Section III, with an overall order of $d = 7$. Note that such dimension could easily lead to implementation issues when standard high-gain tools are used, also for low values of the high-gain parameter g. The proposed controller, based on the low-power high-gain observers, instead experiments no particular problem, also with high values of g. In this example we consider as the controlled plant the following forced nonlinear oscillator

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_2^3 \\ \dot{x}_2 &= 2x_2 - 2x_1 + u - w \end{aligned} \tag{22}$$

where u is the control input and w is the exogenous disturbance, which is assumed to be generated from a linear, a Duffing or a Van der Pool oscillator with unknown parameters, namely we can assume w to satisfy

$$\ddot{w} = \alpha w + \beta\dot{w} + \eta w^3 + \gamma w^2 \dot{w} \tag{23}$$

which for different configurations of the parameters includes, among all the others, also the dynamics of interest. In our example, the control goal is to regulate $e = x_2$ to zero by error feedback control, asymptotically rejecting the exogenous disturbance w. Hence to be consistent to the notation used in Section IV we call $e = x_2$ and $z = x_1$ and we rewrite system (22) as

$$\dot{z} = -2z + e^3, \qquad \dot{e} = 2e - 2z + u - w$$

which clearly fits into the considered framework. Note that the steady state control law able to maintain the error to zero is exactly $u^\star(w) = w$, which satisfies (6) with $\phi(u^\star(w), \dot{u}^\star(w))$ given by (23). Therefore, following the notation of Section III, we define the following regression

$$L_s^2 u^\star = \psi(u^\star_{[0,1]})\theta \tag{24}$$

being $\psi(u^\star_{[0,1]}) = \mathrm{col}(u^\star, \dot{u}^\star, u^{\star 3}, u^{\star 2}\dot{u}^\star)$ and $\theta = \mathrm{col}(\alpha, \beta, \eta, \gamma)$. Taking the 4-th Lie derivative of (24) and collecting the equations yields

$$u^\star_{[2,6]} = \Psi_4(u^\star_{[0,5]})\theta \tag{25}$$

where $\Psi_4(\cdot) \in \mathbb{R}^{5\times 4}$ is the regression matrix. Assumption 2 can be then tested either numerically (see [7]) or analytically. Taking the 5-th Lie derivative of (24) yields

$$u^{\star(7)} = \psi_5(u^\star_{[0,6]})\Big(\Psi_4^+(u^\star_{[0,5]})u^\star_{[2,6]}\Big) \tag{26}$$

which has the form (6), with $d = 7$ and no parameter appearing. A controller of the form (7), (17) with an overall dimension of $2(d-1) = 12$ has been used with a properly saturated version of (26) entering in the last state equation. The controller design is completed by the choice $v = -ke$, where $k > 0$ is chosen large enough. Figure 1 shows the simulation results of the overall closed-loop systems subject to a disturbance $w(t)$ which in the first 10 seconds is a sinusoid with $\alpha = -9$, at time $t = 10s$ switches to the output of a Duffing oscillator obtained with $\alpha = 2$ and $\eta = -1$, and at time $t = 20s$ switches to the output of a Van der Pool oscillator obtained with $\alpha = -4$, $\beta = 1$ and $\gamma = -1$. In order to dominate the dynamics of the 7^{th} derivative of the considered exosystem, we used a gain $g = 200$. While with this design the implementation has not problems, with standard high gain tools we would have had a term of $g^d = 200^7$, which is a 17-digit number which could easily lead to numerical problems at the implementation phase.

VI. Conclusions

In this paper we introduced the low-power high-gain observers in the problem of output regulation. With respect to standard high-gain tools, the controller exhibits a larger dimension ($2d - 2$ instead of d), allowing however the implementation of much higher dimensional internal models. In fact, in the new design, the power of the high-gain parameter g limits to 2 (instead of d) thus avoiding numerical problems arising when standard high-gain tools are used for large dimensional internal models. The advantages of this tool have been shown in the example in Section V. The structure of the modified low-power high-gain observer recently introduced in [2] can be also used in order to avoid peaking phenomenon.

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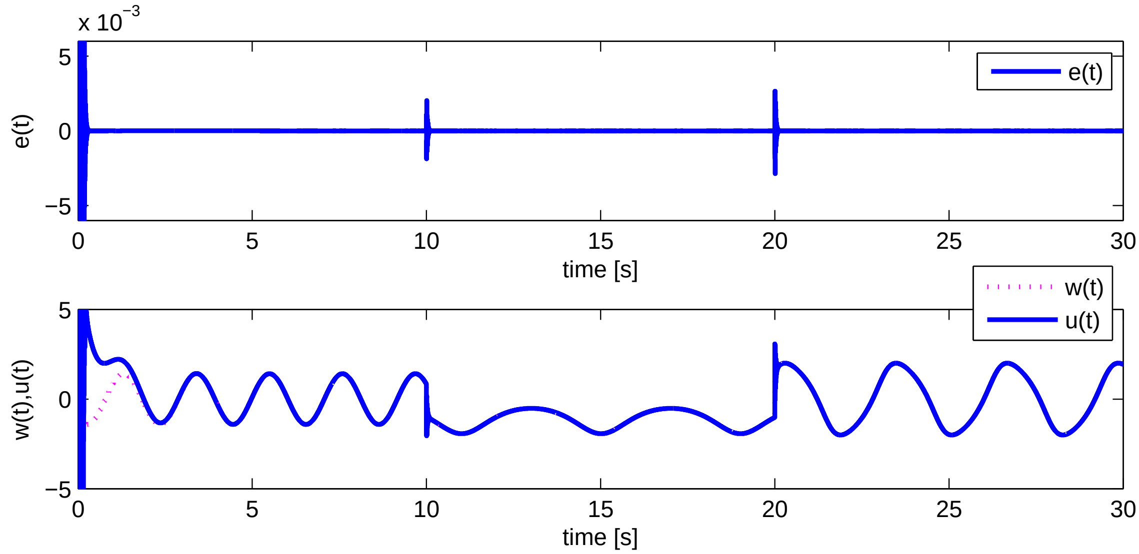

Fig. 1. Simulation results.

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Appendix
Proof of Lemma 1

Proof: Consider the subsystem ξ of (19) and the change of variables

$$\xi \mapsto \varepsilon := D_g(\xi - \tau^e(w)) \tag{27}$$

with $D_g := \text{blkdiag}(D_g^1, \ldots, D_g^{d-1})$, where $D_g^i := g^{2-i} D_2(g)^{-1}$ and let define the function

$$c(w,z) := -\frac{q(w,z,0)}{b(w,z,0)}$$

We start analysing the dynamics of $\varepsilon = \text{col}(\varepsilon_1, \ldots, \varepsilon_{d-1})$ component-wise, where for each $i = 1, \ldots, d-1$, $\varepsilon_i = \text{col}(\varepsilon_{i1}, \varepsilon_{i2}) \in \mathbb{R}^2$. For $i = 1$ we have

$$\begin{aligned}\dot\varepsilon_{11} &= g\varepsilon_{12} - gk_{11}\varepsilon_{11} + gk_{11}(c(w,z) - \tau^e_{11}(w))\\ \dot\varepsilon_{11} &= g\varepsilon_{22} - gk_{12}\varepsilon_{11} + gk_{12}(c(z,w) - \tau^e_{11}(w))\end{aligned}$$

for $i = 2, \ldots, d-2$ we have

$$\begin{aligned}\dot\varepsilon_{i1} &= g\varepsilon_{i2} + gk_{i1}\varepsilon_{(i-1)2} - gk_{i1}\varepsilon_{i1}\\ \dot\varepsilon_{i2} &= g\varepsilon_{(i+1)2} + gk_{i2}\varepsilon_{(i-1)2} - gk_{i2}\varepsilon_{i1}\end{aligned}$$

and finally, for $i = d-1$ we obtain

$$\begin{aligned}\dot\varepsilon_{(d-1)1} &= g\varepsilon_{(d-1)2} + gk_{(d-1)1}(\varepsilon_{(d-2)2} - \varepsilon_{(d-1)1})\\ \dot\varepsilon_{(d-1)2} &= g^{1-d}[\Delta_\phi(\tilde\xi, \tau^e(w)) - \nu(w)]\\ &\qquad + gk_{(d-1)1}(\varepsilon_{(d-2)2} - \varepsilon_{(d-1)1})\end{aligned}$$

with

$$\Delta_\phi(\varepsilon, \tau^e(w)) := \phi_s(\Psi(D_g^{-1}\varepsilon + \tau^e(w))) - \phi(\Psi\tau^e(w)) .$$

By defining $\delta(z,w) = c(w,z) - u^\star(w)$, the system ε can be compactly rewritten as

$$\dot\varepsilon = gM\varepsilon + g^{1-d}B_{2d-2}[\Delta_\phi(\varepsilon, \tau^e(w)) - \nu(w)] + gH\delta(w,z)$$

being $B_{2d-2} := \text{col}(0, \ldots, 0, 1) \in \mathbb{R}^{2d-2}$, $H := \text{col}(k_{11}, k_{12}, 0, \ldots, 0) \in \mathbb{R}^{2d-2}$ and M defined as

$$M := \begin{pmatrix} E_1 & N & 0 & & \cdots & \cdots & 0\\ Q_2 & E_2 & N & 0 & & & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots\\ \vdots & \ddots & Q_i & E_i & N & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & & \ddots & Q_{d-2} & E_{d-2} & N\\ 0 & \cdots & \cdots & \cdots & 0 & Q_{d-1} & E_{d-1} \end{pmatrix}$$

where for $i = 1, \ldots, d-1$, $E_i \in \mathbb{R}^{2\times 2}$, $Q_i \in \mathbb{R}^{2\times 2}$ are defined as

$$E_i = \begin{pmatrix} -k_{i1} & 1\\ -k_{i2} & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & k_{i1}\\ 0 & k_{i2} \end{pmatrix}$$

By using Lemma 1 in [1] it is possible to show that the eigenvalues of matrix M can be assigned arbitrarily with a suitable choice of the coefficients k_{ij}. In particular let fix k_{ij}, so that M is Hurwitz and let $P = P^\top > 0$ be the solution of the Lyapunov equation $PM + M^T P = -I$. Consider the Lyapunov candidate

$$V(\varepsilon) = \sqrt{\varepsilon^\top P \varepsilon}$$

and note that $\sqrt{\underline{\lambda}}|\varepsilon| \leq V(\varepsilon) \leq \sqrt{\bar{\lambda}}|\varepsilon|$, being $\underline{\lambda}$ and $\bar{\lambda}$ respectively the smallest and largest eigenvalues of P. Taking the Dini derivative of V along the solutions yields

$$D^+V = g\frac{\varepsilon^\top(PM + M^\top P)\varepsilon}{2\sqrt{\varepsilon^\top P\varepsilon}} + \frac{\varepsilon^\top}{\sqrt{\varepsilon^\top P\varepsilon}}\Big(gPH\delta(w,z) + g^{1-d}B_{2d-2}(\Delta_\phi(\varepsilon, \tau^e(w)) - \nu(w))\Big)$$

Since $\tau^e(W)$ is compact and $\phi(\cdot)$ and $\phi_s(\cdot)$ are locally Lipschitz, then there exists $\ell > 0$ such that $|\Delta_\phi(\varepsilon, \tau^e(w))| \leq \ell|D_g^{-1}\varepsilon| \leq \ell g^{d-1}|\varepsilon|$. Moreover, since

$$\frac{|\varepsilon|}{\sqrt{\varepsilon^\top P\varepsilon}} \leq \frac{1}{\sqrt{\underline{\lambda}}} \qquad \forall\, \varepsilon \in \mathbb{R}^{2d-2}\,.$$

we obtain

$$D^+V \leq -\Big(\frac{g}{2\bar{\lambda}} - a_1\Big)V + a_2 g|\delta(w,z)| + \frac{a_3}{g^{d-1}}|\nu(w)|$$

having defined $a_1 = \ell/\sqrt{\underline{\lambda}}$, $a_2 = |H|\bar{\lambda}/\sqrt{\underline{\lambda}}$ and $a_3 = 1/\sqrt{\underline{\lambda}}$. Define $g^\star = \max\{2\bar{\lambda}a_1, 1\}$ and pick $g > g^\star$. Then there exists $c_1 > 0$ such that

$$D^+V \leq -gc_1 V + a_2 g|\delta(w,z)| + a_3 g^{1-d}|\nu(w)|$$

and therefore

$$V(t) \leq \exp(-gc_1\, t)V(0) + \frac{1 - \exp(-gc_1\, t)}{c_1}\Big(a_2\,\|\delta(w,z)\|_\infty + \frac{a_3}{g^d}\,\|\nu(w)\|_\infty\Big) \tag{28}$$

and

$$|\varepsilon(t)| \leq a_4\exp(-gc_1\, t)|\varepsilon(0)| + a_5\,\|\delta(w,z)\|_\infty + \frac{a_6}{g^d}\,\|\nu(w)\|_\infty$$

being $a_4 = \sqrt{\bar{\lambda}/\underline{\lambda}}$, $a_5 = a_2/(c_1\sqrt{\underline{\lambda}})$ and $a_6 = a_3/(c_1\sqrt{\underline{\lambda}})$. Define $V_\xi(w,\xi) = V(D_g(\xi - \tau^e(w)))$ and let $\mathcal{B}$ be the set

$$\mathcal{B} := \big\{(w,\xi) \in W \times \mathbb{R}^{2d-2} \;\mid\; \xi = \tau^e(w)\big\}$$

From (27) we have that for any $g > 1$ and any $w \in W$

$$\|(w,\xi)\|_\mathcal{B} \leq |\xi - \tau^e(w)| \leq |D_g^{-1}\varepsilon| \leq \frac{g^{2d-2}}{\sqrt{\underline{\lambda}}}V_\xi(w,\xi)$$

Moreover let $w_p \in W$ be such that $\|(w,\xi)\|_\mathcal{B} = |(w,\xi) - (w_p, \tau^e(w_p))|$. Since $\tau^e(\cdot)$ is smooth and W compact, there exists L_τ so that the following bound holds

$$\begin{aligned}V_\xi(w,\xi) &\leq \sqrt{\bar{\lambda}}|\varepsilon| = \sqrt{\bar{\lambda}}\big|D_g(\xi - \tau^e(w))\big| \\ &\leq \sqrt{\bar{\lambda}}g^{2-2d}|\xi - \tau^e(w_p) + \tau^e(w_p) - \tau^e(w)| \\ &\leq \sqrt{\bar{\lambda}}g^{2-2d}\left(\|(w,\xi)\|_\mathcal{B} + |\tau^e(w_p) - \tau^e(w)|\right) \\ &\leq \sqrt{\bar{\lambda}}g^{2-2d}(1 + L_\tau)\,\|(w,\xi)\|_\mathcal{B}\end{aligned}$$

Therefore, having defined $\underline{\delta} = g^{2d-2}/\sqrt{\underline{\lambda}}$ and $\bar{\delta} = \sqrt{\bar{\lambda}}g^{2-2d}(1 + L_\tau)$ we obtain $\underline{\delta}||(w,\xi)||_\mathcal{B} \leq V_\xi(w,\xi) \leq \bar{\delta}||(w,\xi)||_\mathcal{B}$ and thus we conclude that the (w,ξ) subsystem is ISS relative to the set $\mathcal{B}$ with respect to the inputs $\nu(w)$ and $\delta(w,z)$. Consider now the set $\mathcal{A}$ defined in (3). From Assumption 1 and due to converse Lyapunov results (see for instance Theorem 4 in [12]), there exist a locally Lipschitz function $V_z : W \times \mathbb{R}^n \to \mathbb{R}$, a constant $c_z > 0$ and two class-K functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$ such that

$$\underline{\alpha}(\|(w,z)\|_\mathcal{A}) \leq V_z(w,z) \leq \bar{\alpha}(\|(w,z)\|_\mathcal{A})$$
$$\dot{V}_z(z) \leq -c_z V_z(w,z)$$

for any $(w,z) \in W \times Z$. With $w_p \in W$ such that $\|(w,z)\|_\mathcal{A} = |(w,z) - (w_p, \pi(w_p))|$ and bearing in mind that $c(\cdot)$ and $\pi(\cdot)$ are smooth and $\pi(W)$ is compact, there exist two constants $L_c, L_\pi > 0$ such that

$$\begin{aligned}|\delta(w,z)| = |c(w,z) - c(w,\pi(w))| &\leq L_c(|z - \pi(w)|) \\ &= L_c(|z - \pi(w_p) + \pi(w_p) - \pi(w)|) \\ &\leq L_c(\|(w,z)\|_\mathcal{A} + L_\pi|w - w_p|) \\ &\leq L_c(1 + L_\pi)\underline{\alpha}^{-1}(V_z(w,z))\end{aligned}$$

Due to the exponential stability of $\mathcal{A}$ in Assumption 1 and since w and z range in compact sets, there exists a constant $c_2 > 0$ such that $|\delta(w,z)| \leq c_2 V_z(w,z)$. Define $\chi = \mathrm{col}(w,z,\xi)$ and let $V_\chi(\chi) = V_\xi(w,\xi) + g(a_2c_2/c_z + 1)V_z(w,z)$. Then we obtain

$$\begin{aligned}D^+V_\chi \leq &- gc_1V_\xi(w,\xi) + a_2g|\delta(w,z)| + a_3g^{1-d}|\nu(w)| \\ &- gc_z(\frac{a_2c_2}{c_z} + 1)V_z(w,z) \\ \leq &- gc_1V_\xi(w,\xi) - gc_zV_z(w,z) + a_3g^{1-d}|\nu(w)| \\ \leq &- c_\chi V_\chi(\chi) + a_3g^{1-d}|\nu(w)|\end{aligned}$$

being $c_\chi = \min\{gc_1, c_z^2/(a_2c_2 + c_z)\}$. Consider now the set $\mathcal{E}$ defined in (20). Since $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$ are linearly bounded, then there exist two constants $\underline{\beta}, \bar{\beta} > 0$ such that

$$\begin{aligned}V_\chi(\chi) &= V_\xi(w,\xi) + g(\frac{a_2c_2}{c_z} + 1)V_z(w,z) \\ &\leq \bar{\delta}\,\|(w,\xi)\|_\mathcal{B} + g(\frac{a_2c_2}{c_z} + 1)\bar{\alpha}(\|(w,z)\|_\mathcal{A}) \\ &\leq \bar{\delta}\,\|\chi\|_\mathcal{E} + g(\frac{a_2c_2}{c_z} + 1)\bar{\alpha}(\|\chi\|_\mathcal{E}) \leq \bar{\beta}||\chi||_\mathcal{E} \\ \|\chi\|_\mathcal{E} &\leq \|(w,\xi)\|_\mathcal{B} + g(\frac{a_2c_2}{c_z} + 1)\,\|(w,z)\|_\mathcal{A} \\ &\leq \underline{\delta}^{-1}V_\xi(\xi) + g(\frac{a_2c_2}{c_z} + 1)\underline{\alpha}^{-1}(V_z(z)) \leq \underline{\beta}^{-1}V_\chi(\chi)\end{aligned}$$

Hence $\underline{\beta}\,\|\chi\|_\mathcal{E} \leq V_\chi(\chi) \leq \bar{\beta}\,\|\chi\|_\mathcal{E}$ and we obtain that the zero dynamics (19) of the closed loop system are ISS relatively to set $\mathcal{E}$ with respect to the input $\nu(w)$. The bound claimed in the statement follows by noting that $\lim_{t\to\infty} V_z(t) = 0$ and from (28) we have

$$\limsup_{t\to\infty} V_\chi(t) = \frac{a_3}{c_1g^d}\max_{w\in W}|\nu(w)|$$

Therefore from the bounds obtained for $\|\chi\|_\mathcal{E}$ we obtain (21) with $\mu = a_3/(c_1\underline{\delta})$. ∎