Aperiodic Tilings: Breaking Translational Symmetry

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Abstract

Classical results on aperiodic tilings are rather complicated and not widely understood. Below an alternative approach to these results is offered in hope to provide additional intuition not apparent in classical works.

1 Palettes and Tilings

Physical computing media are asymmetric. Their symmetry is broken by irregularities, physical boundaries, external connections, etc. Such peculiarities, however, are highly variable and extraneous to the fundamental nature of the media. Thus, they are pruned from theoretical models such as cellular automata, and reliance on them is frowned upon in programming practice.

Yet, computation, like many other highly organized activities, is incompatible with perfect symmetry. Some standard mechanisms must assure breaking the symmetry inherent in idealized computing models. A famous example of such mechanisms is aperiodic tiling: hierarchical self-similar constructions, first used for computational purposes in a classical – although rather complicated – work [Berger 66]. They were further developed in [Robinson 71, Myers 74, Gurevich Koriakov 72]; [Allauzen Durand 96] give a helpful exposition.

Definition 1 *Let G be the grid of unit length edges between integer points on an infinite plane. A* **tiling** *T is its mapping into a finite set of* **colors**. *Its* **crosses** *and* **tiles** *are ordered color combinations of four edges sharing a corner or forming a square respectively. A* **Palette** *P of T is a set including all its tiles (*+**palette** *for crosses).*
We say P with a mapping f of its colors into a smaller color alphabet **enforces** *a set S of tilings if replacing colors according to f turns each P-tiling into a tiling in S.*

Turning each edge orthogonally around its center turns G into its dual graph and palettes into +palettes and vice versa. Thus, one can use either type as convenient.

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2 2-Adic Coordinates

The set of all tilings with a given palette P has translational symmetry, *i.e.* any shift produces another P-tiling. We want a palette that forces a complete spontaneous breaking of this symmetry, *i.e.* prevents individual tilings from being *periodic*. So, each location in a given tiling will be uniquely characterized by a sort of *coordinates*. Their infinitely many values cannot be reflected in the finite variety of tile's colors. Rather they will be represented *distributively*, *i.e.* in the colors of the surrounding tiles, and computable from them to any given number of digits.

Let us first so distribute, say, the horizontal Cartesian integer coordinates $x = (2i{+}1)2^k$ of vertical edges by reflecting one bit (i mod 2) in their color. We view this bit as the direction of a ***bracket***. In this tiling C_1, the brackets of the same ***rank*** k are equidistant (Figure 1).

It is convenient to visualize the bits of even rank, picturing them }/{ or red, separately from odd, depicted]/[. The bits of either shape at each side of the origin form the progression of balanced parenthetical expressions, called ***domains***. Each domain has four ***grandchildren*** of the second lower rank: two within its outermost brackets and one to each side. The two ***children*** have the other shape and are centered at each border of the ***parent***, thus connecting it with its grandchildren.

Handling the vertical coordinate similarly yields a neat 2-d tiling C called ***central***. Figure 2 depicts intersections of orthogonal domains of equal ranks as squares shown by using ***boldness*** bit to interrupt each line outside the orthogonal domains of the same rank.

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[  ]  [  ]  [  ]  [  ]  [  ]  [  ]  [  ]
 {     }     {     }     {     }     {
    [                 ]           [
          {                             }
                      [
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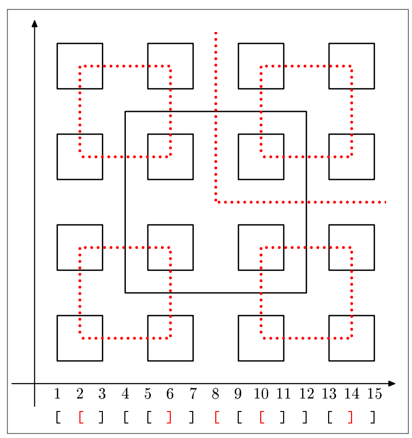
Figure 1: Brackets of C_1 split by rank.

Figure 2: (Right) Boldness bit lines in C. Courtesy of A. Shen and B. Durand:



C_1 has a special, *i.e.* unmatched, bracket in the origin, directed arbitrarily and unranked. No palette can enforce a set of tilings with unique special points (designated by a Borel function commuting with shifts) since the set of all tilings is compact[1] while the set of locations of their special point and the group $\mathbb{Z}$ of their shifts is not. We will extend $\mathbb{Z}$ to a compact group and also ***define ranks*** in other tilings, *e.g.*, shifted C_1, using this property:

Remark 1 *A shift by $(2i{+}1)2^k$ in C_1 reverses all brackets of rank $k{-}1$, none of lower ranks, and every second bracket of any rank $> k$.*

[1]and so has finite shift-invariant measures, *e.g.*, defined by condensation points of frequencies of finite configurations in some quasiperiodic tiling

So, the shifts by $(2i{+}1)2^k$ change our bits only at a $2/2^k$ fraction of locations. This fraction can be used as a metric on the group of shifts which can then be completed for it. The result is a remarkable compact group g of 2-adic integers or ***2-adics*** acting on a similarly completed set of 2-adic coordinates. A 2-adic a is a formal infinite sum $\sum_{i\geq 0} 2^i a_i = \ldots + 4a_2 + 2a_1 + a_0$, where $a_i \in \{0,1\}$, viewed as an infinite to the left sequence of bits. The usual algorithms for addition and multiplication make g a ring with $\mathbb{Z}$ as a sub-ring (*e.g.,* $-1 = \ldots{+}8{+}4{+}2{+}1$).[2]

The natural action of $\mathbb{Z}$ (by shifts) can be extended to the action of the whole g on C_1 and its images. Indeed, the brackets of rank k are unaffected by terms a_i with $i > k{+}1$. Thus, a 2-adic shift a of C_1 can be defined as the pointwise limit of the sequence of shifts by integers approximating a. With inverse shifts, this sequence diverges for the unranked bracket in the origin of C_1 and of its integer shifts. This bracket is determined not by its location, but by an arbitrary ***default*** included as an additional (external, unmoved by shifts) point in the tilings. The ***reflection*** reverses the default, all brackets and the signs of their locations. With added reflection, the action of g is free and transitive: each of these tilings can be obtained from any other, *e.g.,* from C_1. In 2 dimensions we can also add the diagonal reflection exchanging the vertical and horizontal coordinates.[3]

3 Enforcing the Coordinate System with a Palette

Theorem 1 *The set of 2-adic shifts and reflections of tiling C can be enforced by a palette.*

To prove it, we use multi-component colors in T which f projects onto their first component – bracket bit. The second component includes two ***enforcement*** bits that extend C to the enhanced tiling CE, with a +palette ce of 7 crosses modulo 8 reflections.[4] One of its bits is the already described boldness bit (Figure 2). The other is a ***pointer*** in the direction of the nearest orthogonal line of the same rank. On the (unranked) axes these bits are set by a default central cross.

Definition 2 *A* **box** *is an* **open** *ce-tiled rectangle,* i.e. *one with the border edges removed. Its k-***block** *is a square with monochromatic sides that is*[5] *a tile (for $k{=}0$) or a combination of four $(k-1)$-blocks sharing a corner. The four segments connecting block's center to its sides are called* (k–1)**-medians**. *The rest of the open block is called a* **frame**.
*We call a box k-***tiled** *if removing an outer layer which is thinner than a k-block turns it into a box composed of (open at the box border) k-blocks.*[6]

Remark 2 *Each ce-cross appears at meetings with 0-medians in open 3-blocks in C.*
Borders between open blocks in a box are monochromatic
since all ce-crosses have 1 or 4 inward pointers.

Lemma 1 *Each non-empty k-frame pattern in a box is enclosed in its open k-block.*
All open k-blocks are congruent and have equal frames.

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[2] Odd 2-adics have inverses. This allows extending g to a famous locally compact field with fractions $a/2^i$.

[3] We allow fewer tilings than [Allauzen Durand 96] which permits different shifts at each side of the origin.

[4] [Robinson 71] uses only 6 tiles (with reflections) but colors their corners, in addition to sides.

[5] The rest of the requirement is redundant but useful in the proof.

[6] *ce* prevents crossing of blocks' borders, making decompositions of boxes into blocks unique.

Proof. $k > 2$ cases follow from $k-1$ by viewing 1-blocks as tiles. Let 1-blocks a and b be adjacent in a 2-block c; (L, l) and (R, r) be pairs of medians of a and b with l, r directed to the side s of c, $L-R$ crossing a median m of c at a cross x. x forces L, R to be both pale or both bold. This forces opposite brackets on l, r which, too, must be both pale or both bold depending on the bracket of $L-R$. l, r cannot be both bold since this would require the pointer of s to agree with their opposite brackets.

Thus, all external medians of 2-frames are pale, internal medians bold, their brackets face the frame's center forcing the inward pointer on the 1-medians, like m. ∎

Lemma 2 *Any 1-tiled box is k-tiled. (Follows from $k=2$ case by seeing k–2-blocks as tiles.)*

Proof. The 8 colors of edges in open 1-blocks determine their location in a 2-frame, forcing their regular alternation according to the pattern of 2-frames which, in turn, extend to 2-blocks (by Remark 2 and Lemma 1). ∎

Corollary 1 *Any $2^k \times 2^k$ box, extendible to a 3 times wider co-centric 1-tiled box, extends to a $(k+4)$-block.*

Proof. The box is k-tiled covering the inner box with four blocks sharing a corner. Viewed as 1-blocks meeting at an appropriate cross in a 5-block, they can be extended to it. ∎

Induction Basis. For the simplest enforcement of tiling decomposition into 1-blocks we can use a 2-periodic ***parity*** bit to mark ***odd*** lines carrying 0-medians.
All pointers on odd lines point to odd crossing lines, thus forcing a period 2 on them.
One needs only to assure an odd line exists. This can be easily done with a ***parity pointer*** on even lines, pointing to a crossing odd line.

Proof of Theorem 1. Let T be a *ce*-tiling decomposed, for each k, into k-blocks with equal frames. Then a shift of CE matches T on all lines of rank $< k$.
The shifted CE converge pointwise to T, except possibly on their (unranked) axes.
By Remark 1, the shift increments grow in rank, and so sum to one 2-adic shift.
Finally, reflections match the brackets on axes. ∎

3.1 Parsimonious Enforcement of the Grid of 1-blocks.

First, we reduce the needed parity colors. A parity pointer on a single edge suffices, so it needs to accompany only one color if we show that *ce*-tilings cannot skip colors. Indeed, all *ce*-crosses are either ***bends***, *i.e.* have 4 inward pointers, or ***passes***, *i.e.* have 1. Thus, a third of crosses are bends, within the accuracy of $O(n)$ for $n \times n$ boxes. Moreover, crosses of all orientations are equally frequent, since their orientation alternates on each line. ∎

Tedious case investigation of [Levitsky 04] shows *ce* bits themselves forcing 1-blocks, rendering parity bits redundant. A k-***bar*** is a maximal bold or pale segment, k being its length. $k > 1$ and no bold 3-bar exists since it is easy to see that its middle link would be connected by a tile to a 1-bar. Levitsky first proofs that each *ce*-tiling has bold 2-bars.

Here is a simpler argument for this. The average bar length is 3 since a third of crosses are bends. Then, absent bold 2-bars, this average would allow positive density only of bold 4-bars and pale 2-bars. Such tiling has period 6 and maps onto a 6×6 torus with three bold 4×4 squares. But Z_6 cannot have three disjoint pairs of points of equal parity! $\blacksquare$

The rest of [Levitsky 04] analysis assures bold 2-bars two tiles away at each side of any bold 2-bar. This involves a case-by-case demonstration that no violation can be centered in a 10×10 box. The analysis is laborious but may be verifiable by a computer check.

4 Acknowledgments

These remarks were developed in my attempts to understand the classical constructions of aperiodic tilings while working on [Durand, Levin, Shen 01]. My main source of information was [Allauzen Durand 96] and its explanations by B. Durand and A. Shen to whom I owe all my knowledge on this topic.

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