CS364B: Frontiers in Mechanism Design
Lecture #14: The Price of Anarchy in Simple Auctions *

Tim Roughgarden†

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1 Introduction to Part IV

With this lecture we commence the fourth part of the course. In the last two parts we took incentive constraints, like DSIC or BIC, as a hard constraint, and subject to this designed the best mechanisms possible. Most of the mechanisms discussed were quite complex — interesting as proofs of concept of what is possible in principle, but not suitable for actual use. This part of the course insists on simple auction formats as a hard constraint, and seeks conditions under which they perform well. That is, we seek auctions with the following two properties.

1. The auction format should be simple.

2. The equilibria should be good (e.g., near-optimal welfare).

Here, when we speak of "simplicity," we're only referring to the description and implementation of the auction itself. As we'll see, intelligently participating in such an auction need not be simple. Contrast this with DSIC mechanisms, which are trivial for bidders. Even in BIC mechanism, the optimal strategy for a bidder (direct revelation) is trivial, as long as the bidder believes in the prior distribution and that other bidders are bidding truthfully. In the "simple" auction we study in the next several lecture, it is not at all clear what the equilibria are.

In the second property above, "equilibria" could mean a few different things. In this lecture, as a warm-up, we'll focus on pure Nash equilibria of full-information games. Bayes-Nash equilibria are better motivated in auction settings, however, and we'll take them up in subsequent lectures.

†Department of Computer Science, Stanford University, 462 Gates Building, 353 Serra Mall, Stanford, CA 94305. Email: tim@cs.stanford.edu.

2 Simultaneous Second-Price Auctions (S2A)

2.1 The Auction Format

We next introduce our first "simple" auction format. Let U denote a set of m non-identical items.

Simultaneous Second-Price Auctions (S2A)

1. Each bidder i submits a bid b_{ij} for each item j.
2. Each item j is sold separately via a second-price auction — the highest bidder for the item wins it, at a price equal to the second-highest bid.

One way to think about a S2A is simply as selling off the items of U on eBay at the same time, with each item in its own single-item auction.

Another way to think about a S2A is as the VCG mechanism in the special case in which every bidder has an additive valuation — i.e., bidder i has private values $v_{i1}, \ldots, v_{im}$ for singletons and in general $v_i(S) = \sum_{j \in S} v_{ij}$. In this case, setting $b_{ij} = v_{ij}$ for every j is a dominant strategy for every bidder i, and if all bidders follows these dominant strategies, then the outcome of the auction has full welfare.

It's more interesting to study the performance of a S2A when bidders' valuations need not be additive — presumably the auction's performance deteriorates as bidders' valuations become increasingly non-additive, and we'd like to understand this precisely. For concreteness, let's return to the scenario of bidders with submodular valuations.

Scenario #6:

- Each bidder i has a private valuation $v_i : 2^U \rightarrow \mathcal{R}^+$ that is *submodular*, meaning that for every pair of sets $S \subseteq T \subseteq U$ and item j,

$$v_i(T \cup \{j\}) - v_i(T) \leq v_i(S \cup \{j\}) - v_i(S). \tag{1}$$

  As always, we also assume that every valuation satisfies $v_i(\emptyset) = 0$ and is monotone (i.e., $S \subseteq T$ implies $v_i(S) \leq v(T)$).

We've mentioned in passing that there are constant-factor polynomial-time approximation algorithms for welfare maximization with submodular valuations, and that there is no known DSIC mechanism with these properties. How good are the equilibria of a S2A in this setting?

For a relatively rich class of valuations like Scenario #6, a S2A is fundamentally different than the mechanisms we've been discussing thus far. Keep in mind that a bidder with a submodular valuation effectively possesses $2^m - 1$ private parameters, one for each non-empty bundle of goods. A S2A only gives a bidder the vocabulary to articulate m numbers. Thus, "direct revelation" doesn't even make sense in an S2A with non-additive valuations. We also don't think of the valuations as being "input" to the mechanism in any sense. For equilibria to make sense, however, one should think of a bidder as being able to answer demand queries with respect to its valuation.

Figuring out how to bid in a S2A is not easy; see also the discussion in Lecture #8 of CS364A. For example, imagine that you just one item, and you don't care which one. How should you bid? Should you go all-in on a single item? Or should you place low bids on numerous items, hoping to get one for a bargain? A bidder in a S2A must hedge the risks of acquiring too many item — all of which must be paid for, even if only one is desired — versus the risk of acquiring none.

2.2 No Overbidding Conditions

Recall that the *price of anarchy (POA)* is defined as the ratio in objective function value (here, welfare) of the worst equilibrium and an optimal outcome. A drawback of second-price auction formats is the presence of "bluffing" equilibria — low-welfare equilibria caused by overbidding.

Example 2.1 (Overbidding Yields Low Welfare-Equilibria) Consider a single-item Vickrey auction with two bidders with valuations $v_1 = 1$ and $v_2 = \epsilon$ for $\epsilon > 0$ arbitrarily small. Consider the full-information game induced by the Vickrey auction. The bid profile with $b_1 = 0$ and $b_2 = 1$ is a pure Nash equilibrium. The second bidder clearly does not want to deviate. For the first bidder, if the bids high enough to win the item, it has to pay 1 and therefore continues to receive zero utility.

The bad pure Nash equilibrium in Example 2.1 is more of an annoyance than a fatal flaw in the auction format. The second bidder is highly exposed: if the first bidder changes its bid to $1 - \epsilon$, for example, the second bidder winds up with significantly negative utility. For this reason, the relevance of such equilibria is questionable.[1] For this reason, we'll prove welfare bounds only for pure Nash equilibria that satisfy a "no overbidding" assumption.

We use the following notation. We denote a bid vector by $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, where each $\mathbf{b}_i$ is itself an m-vector, indexed by U. We use $S_i(\mathbf{b})$ to denote the items won by bidder i — that is, the items on which it is the highest bidder. We use $p_j(\mathbf{b})$ to denote the price paid by the winner of item j — that is, the second-highest bid for j.

Definition 2.2 (Weak No Overbidding (WNO)) The bid vector $\mathbf{b}$ satisfies *weak no overbidding (WNO)* if

$$\sum_{j \in S_i(\mathbf{b})} b_{ij} \leq v_i(S_i(\mathbf{b})) \tag{2}$$

for every bidder i.

That is, no bidder bids more than its value for the bundle of goods that it wins. Definition 2.2 is badly violated in Example 2.1 — the second bidder wins an item for which it has value ϵ with a bid of 1.

[1] With only one item, something stronger is true: overbidding in a Vickrey auction is dominated by bidding one's true valuation.

The following stronger version of Definition 2.2 is also worth noting. Today's results apply with either definition.

Definition 2.3 (Strong No Overbidding (SNO))

(a) The bid vector $\mathbf{b}_i$ of bidder i satisfies *strong no overbidding (SNO)* if

$$\sum_{i \in S} b_{ij} \leq v_i(S) \tag{3}$$

for every subset $S \subseteq U$.

(b) A bid profile $\mathbf{b}$ is satisfies *strong no overbidding (SNO)* if every constituent bid vector $\mathbf{b}_i$ does.

There are two interpretations of our bounds for pure Nash equilibria that satisfy no overbidding. The first is that each bidder i simply does not consider overbidding — its strategy space consists only of bid vectors that satisfy SNO. In this case, every equilibrium obviously satisfies SNO (and WNO). In the second interpretation, bidders can bid whatever they want; some equilibria will suffers from overbidding, other not.[2] The bounds we give will apply to all equilibria that satisfy no overbidding (the WNO condition is sufficient).

3 The Price of Anarchy in S2A with Submodular Valuations

Could it be that, with no overbidding, every pure Nash equilibrium in a S2A with bidders with submodular valuations is fully efficient? The following example establishes a limit on what we can hope for.

Example 3.1 (POA Can Be $\frac{1}{2}$) Consider two bidders and a set $U = \{x, y\}$ of two items. Both bidders are unit-demand (and hence submodular). The first really wants x but will settle for y if need be: $v_1(\{x\}) = v_1(U) = 2$ and $v_1(\{y\}) = 1$. The second bidder is the opposite: $v_1(\{y\}) = v_1(U) = 2$ and $v_1(\{x\}) = 1$. The optimal allocation clearly has welfare 4 (giving x to the first bidder and y to the second). There are also pure Nash equilibria that yield this outcome. The price of anarchy, however, is determined by the worst equilibrium (with no overbidding). Consider the bid profile with $b_{1x} = 0$, $b_{1y} = 1$, $b_{2x} = 1$, and $b_{2y} = 0$. This profile satisfies the SNO condition (Definition 2.3). Each bidder gets its less desired item for free in this profile, so each has utility 1. If a bidder deviates to win the item it really wants, it has to pay the other bidders' bid (1) and thus its utility does not increase. This bad pure Nash equilibrium has welfare 2, so that POA in this example is at most $\frac{1}{2}$.

The main result of this lecture is matching positive result.

[2] It is harder for a bid profile to be an equilibrium in this second interpretation, because there is a larger set of potentially profitable unilateral deviations.

Theorem 3.2 ([1]) *If every bidder i has a submodular valuation v_i, and $\mathbf{b}$ is a pure Nash equilibrium of a S2A that satisfies weak no overbidding, then the welfare $\sum_{i=1}^n v_i(S_i(\mathbf{b}))$ of the allocation under $\mathbf{b}$ is at least 50% of the maximum possible.*

For comparison, recall that there are constant-factor polynomial-time approximation algorithms for welfare-maximization with bidders submodular valuations, but no such DSIC mechanisms are known. It is remarkable that the equilibria of such a simple auction format are competitive with state-of-the-art approximation algorithms. To be fair, comparing equilibria with polynomial-time approximation algorithms can be an "apples vs. oranges" discussion, since equilibria are not always easy to compute.

Theorem 3.2 is stated only for pure Nash equilibria. Such equilibria need not exist in general games, although they are guaranteed to exist (while also respecting the WNO or SNO condition) in S2As with submodular valuations. More interesting are bounds for Bayes-Nash equilibria and other more equilibrium concepts; we'll prove these in subsequent lectures.

3.1 Warm-Up: Unit-Demand Bidders

We begin by proving Theorem 3.2 for the special case of unit-demand valuations. Recall that it's hard to figure out what to bid, and what equilibria look like, even in this special case. Also, recall from Example 3.1 that, even with these valuations, the worst-case POA in S2As is no better than $\frac{1}{2}$. Proving a matching positive result already requires most of the ideas needed for the general case, but with less notation.

In CS364A, we introduced a template for proving bounds on the price of anarchy.

1. Given an arbitrary pure Nash equilibrium (PNE) $\mathbf{b}$ — in this case, one that also satisfies a no-overbidding condition — the PNE hypothesis is invoked once per player i, with a hypothetical deviation suggested by the optimal outcome, to derive a lower bound on i's equilibrium utility.

2. The n inequalities on individuals' equilibrium utilities are summed over the players. This gives a relationship between the equilibrium and optimal welfares, modulo an additional function of the equilibrium and an optimal outcome.

3. The entangled term is related back to the only two quantities we care about, the equilibrium and optimal welfares.

4. Solve for the POA.

This same template is useful for auctions games such as S2As. The primary new challenge in auction games is the level of indirection between players' actions and the ensuing outcome. The welfare objective function is defined on allocations, and players affect the allocation indirectly through bids. In particular, there are many different bid profiles that induce a welfare-maximizing allocation — which one should we single out for the purpose of deriving lower bounds on bidders' equilibrium utilities?

With unit-demand valuations, there is an intuitive way to proceed. Let $\mathbf{b}$ be a PNE that satisfies the WNO condition. Fix an optimal allocation; since bidders are unit-demand, we can assume that each bidder i received at most one item in it, denoted $j^*(i)$. We then define the hypothetical deviation $\mathbf{b}_i^*$ for bidder i by

$$b_{ij}^* = \begin{cases} v_{ij} & \text{if } j = j^*(i) \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, in $\mathbf{b}_i^*$, bidder i goes all-in for the item $j^*(i)$ that it receives in the optimal allocation. This deviation satisfies the SNO condition (Definition 2.3(a)).

Because $\mathbf{b}$ is a PNE, the fact that bidder i bids $\mathbf{b}_i$ instead of going all-in for $j^*(i)$ via $\mathbf{b}_I^*$ implies that $\mathbf{b}_i$ nets it higher utility:

$$v_i(S_i(\mathbf{b})) - \underbrace{\sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b})}_{\geq 0} \geq v_i(S_i(\mathbf{b}_i^*, \mathbf{b}_{-i})) - \sum_{j \in S_i(\mathbf{b}_i^*, \mathbf{b}_{-i})} p_j(\mathbf{b}_i^*, \mathbf{b}_{-i}). \tag{4}$$

This corresponds to the first step of our template for deriving POA bounds.

We next simplify the inequality (4). On the left-hand side, since price are nonnegative, we can throw out the second term. On the right-hand side, by the definition of $\mathbf{b}_i^*$, only the good $j^*(i)$ matters. If $v_{ij} > \max_{k \neq i} b_{kj}$, then bidder i wins the item $j^*(i)$ in the bid profile $(\mathbf{b}_i^*, \mathbf{b}_{-i})$ at a price of $\max_{k \neq i} b_{kj}$; if not, then bidder i loses the item and pays nothing. In any event, the contribution of item $j^*(i)$ to the right-hand side of (4) is at least $v_{ij^*(i)} - \max_{k \neq i} b_{kj^*(i)} \geq v_{ij^*(i)} - \max_{k=1}^n b_{kj^*(i)}$. Other items contribute nothing (we can ignore the edge case where i wins additional items with a bid of 0). Combining these observations, the following inequality is only weaker than (4):

$$v_i(S_i(\mathbf{b})) \geq v_{ij^*(i)} - \max_{k=1}^{n} b_{kj^*(i)}. \tag{5}$$

Next, as per the second step in our analysis template, we sum (5) over the bidders i:

$$\underbrace{\sum_{i=1}^{n} v_i(S_i(\mathbf{b}))}_{\text{welfare of } \mathbf{b}} \geq \underbrace{\sum_{i=1}^{n} v_{ij^*(i)}}_{\text{OPT welfare}} - \sum_{i=1}^{n} \max_{k=1}^{n} b_{kj^*(i)}. \tag{6}$$

The third step is to related the "entangled term" $\sum_{i=1}^n \max_{k=1}^n b_{kj^*(i)}$ to the only two quantities we care about, the optimal and equilibrium welfares. In this case, we can directly bound this term from above by the equilibrium welfare. Precisely:

$$\begin{aligned} \sum_{i=1}^{n} \max_{k=1}^{n} b_{kj^*(i)} &\leq \sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} \max_{k=1}^{n} b_{kj} && (7) \\ &= \sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} b_{ij} && (8) \\ &\leq \sum_{i=1}^{n} v_i(S_i(\mathbf{b})), && (9) \end{aligned}$$

where (7) follows from the facts that every item is allocated at most once in the optimal solution and exactly once in $\mathbf{b}$, equation (8) follows from the fact that i is the highest bidder on every item $j \in S_i(\mathbf{b})$ that it wins in $\mathbf{b}$, and inequality (9) follows from the assumption that $\mathbf{b}$ satisfies the WNO condition (2).

For the fourth and final step of the analysis template, we combine (6) and (9) and rearrange to derive that the welfare of $\mathbf{b}$ is at least that of the maximum possible. This complete the proof of the POA bound for bidders with unit-demand valuations.

3.2 Submodular and XOS Bidders

We now proceed to the general case of Theorem 3.2, with bidders with arbitrary submodular valuations. The main steps of the proof are the same as for the unit-demand special case. The key challenge is how to implement the very first step. A unit-demand bidder i only wins one item $j^*(i)$ in an optimal allocation, so it's easy to construct a hypothetical deviation in which i goes all out for $j^*(i)$. With general submodular valuations, a bidder i might receive a bundle S_i of items — what hypothetical deviation constitutes going all out for S_i? How should the valuation $v_i(S_i)$ be distributed amongst i's bids for the items of S_i?

The following lemma provides an elegant solution to this challenge. It shows that every submodular function can be represented as a maximum of additive valuations. The representation can be exponentially large, but we will only use this representation for the sake of analysis.

Lemma 3.3 *If v_i is a submodular valuation on the item set U, then there exist additive valuations $\mathbf{a}_i^1, \ldots, \mathbf{a}_i^r$ on U such that, for every $S \subseteq U$,*

$$v_i(S) = \max_{\ell=1}^{r} \left\{ \sum_{j \in S} a_{ij}^{\ell} \right\}. \tag{10}$$

For example, if v_i is a unit-demand valuation with singleton values $v_{i1}, \ldots, v_{im}$, then v_i is the maximum of m additive valuations $\mathbf{a}_i^1, \ldots, \mathbf{a}_i^m$, where

$$a_{ij}^{\ell} = \begin{cases} v_{ij} & \text{if } \ell = j \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Lemma 3.3: We define $m!$ additive valuations, one for each ordering π of the items of U. For such a π, we set

$$a_{ij}^{\pi} = v_i(S_j^{\pi} \cup \{j\}) - v_i(S_j^{\pi}),$$

where S_j^{π} is the set of items that precede j in π. In words, we add the items of U to the empty set one-by-one according to the order π, and set a_{ij}^{π} to the increase in i's valuation when the item j is added.

Fix a set $S \subseteq U$. To prove (10), we show that one of the additive valuations agrees with v_i on S and that the rest only underestimate it. We show the second statement first. Fix an arbitrary ordering π of U, and write $S = \{1, 2, \ldots, \ell\}$, with j denoting the jth item of S in the ordering π. Then,

$$\begin{aligned}
\mathbf{a}_i^\pi(S) &= \sum_{j \in S} a_{ij}^\pi \\
&= \sum_{j \in S} \left(v_i(S_j^\pi \cup \{j\}) - v_i(S_j^\pi) \right) \\
&\leq \sum_{j \in S} \left(v_i(\{1, 2, \ldots, j\}) - v_i(\{1, 2, \ldots, j-1\}) \right) \\
&= v_i(S) - v_i(\emptyset) = v_i(S),
\end{aligned}$$

where the inequality follows from submodularity (1). (Note that S_j^π certainly includes all of $\{1, 2, \ldots, j-1\}$.) When π is an ordering of U in which all of the items of S come first (in arbitrary relative order), the inequality is trivially an equality. Since S was arbitrary, the proof is complete. ■

The converse of Lemma 3.3 does not hold. That is, the class of valuations representable as maxima of additive valuations is strictly larger than the class of submodular valuations (see Exercises). The former valuations are called *XOS* (for "exclusive or of or singletons") or *fractionally subadditive* valuations (see Exercises). As we'll see, Theorem 3.2 holds more generally for bidders with fractionally subadditive valuations.

Proof of Theorem 3.2: Let $\mathbf{b}$ be a pure Nash equilibrium that satisfies weak no-overbidding (Definition 2.2). Fix a welfare-maximizing allocation, which awards the items S_i^* to each bidder i. By Lemma 3.3, for each bidder i we choose an additive valuation $\mathbf{a}_i^*$ that satisfies

$$v_i(S_i^*) = \sum_{j \in S_i^*} a_{ij}^* \tag{11}$$

and, for every $S \subseteq U$,

$$v_i(S) \geq \sum_{j \in S} a_{ij}^*. \tag{12}$$

For bidder i, we use the hypothetical deviation

$$b_{ij}^* = \begin{cases} a_{ij}^* & \text{if } j \in S_i^* \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, in $\mathbf{b}_i^*$, bidder i goes all-in for the bundle S_i^* that is receives in the optimal allocation. This deviation satisfies the SNO condition (Definition 2.3(a)).

We now follow the proof for the unit-demand special case. Because $\mathbf{b}$ is a pure Nash equilibrium,

$$v_i(S_i(\mathbf{b})) - \underbrace{\sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b})}_{\geq 0} \geq v_i(S_i(\mathbf{b}_i^*, \mathbf{b}_{-i})) - \sum_{j \in S_i(\mathbf{b}_i^*, \mathbf{b}_{-i})} p_j(\mathbf{b}_i^*, \mathbf{b}_{-i}). \tag{13}$$

We can throw out the second term on the left-hand side to get an only weaker inequality. On the right-hand side, our additive lower bound $\mathbf{a}_i^*$ on v_i essentially lets us get away with analyzing each item of S_i^* separately. More formally, for each $j \in S_i^*$, either $a_{ij}^* \geq \max_{k \neq i} b_{kj}$ (so bidder i wins j at the price $\max_{k \neq i} b_{kj}$) or not. *If* bidder i's valuation v_i was precisely the additive valuation $\mathbf{a}_i^*$, each item $j \in S_i^*$ would contribute at least

$$a_{ij}^* - \max_{k \neq i} b_{kj} \geq a_{ij}^* - \max_{k=1}^{n} b_{kj}$$

to its utility. Since i's valuation is at least as large as $\mathbf{a}_i^*$ (12), the ensuing sum over items

$$\sum_{j \in S_i^*} \left(a_{ij}^* - \max_{k=1}^{n} b_{kj} \right)$$

remains a legitimate lower bound on i's utility on the right-hand side of (13). Summarizing, and discarding any contributions to the right-hand size of (13) by items outside of S_i^* (which can only be won at price 0), we have

$$v_i(S_i(\mathbf{b})) \geq \sum_{j \in S_i^*} \left(a_{ij}^* - \max_{k=1}^{n} b_{kj} \right). \tag{14}$$

Summing (5) over the bidders i, we have

$$\underbrace{\sum_{i=1}^{n} v_i(S_i(\mathbf{b}))}_{\text{welfare of } \mathbf{b}} \geq \underbrace{\sum_{i=1}^{n} v_i(S_i^*)}_{\text{OPT welfare}} - \sum_{i=1}^{n} \sum_{j \in S_i^*} \max_{k=1}^{n} b_{kj}. \tag{15}$$

We bound the "error term" as in the unit-demand case:

$$\begin{aligned}
\sum_{i=1}^{n} \max_{k=1}^{n} \sum_{j \in S_i^*} b_{kj} &\leq \sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} \max_{k=1}^{n} b_{kj} && (16)\\
&= \sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} b_{ij} && (17)\\
&\leq \sum_{i=1}^{n} v_i(S_i(\mathbf{b})), && (18)
\end{aligned}$$

where (16) follows from the facts that every item is allocated at most once in the optimal solution and exactly once in $\mathbf{b}$, equation (17) follows from the fact that i is the highest bidder on every item $j \in S_i(\mathbf{b})$ that it wins in $\mathbf{b}$, and inequality (18) follows from the assumption that $\mathbf{b}$ satisfies the WNO condition (2). Rearranging terms completes the proof. ■

Theorem 3.2 and its proof extend easily to a parameterized version of the WNO condition. As long as every bidder bids at most γ times its value for the set it receives at equilibrium, the equilibrium welfare is at least $1/(\gamma + 1)$ times that of an optimal allocation (see Exercises).

References

[1] G. Christodoulou, A. Kovács, and M. Schapira. Bayesian combinatorial auctions. In *35th International Colloqium on Automata, Languages and Programming (ICALP)*, pages 820–832, 2008.