CS364B: Frontiers in Mechanism Design
Lecture #5: The Gross Substitutes Condition*

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1 Introduction

Thus far, we've studied four scenarios that admit tractable auctions that maximize welfare subject to strong incentive guarantees. The first two were simple enough that English auctions did the trick. The last two scenarios — unit-demand valuations with non-identical items, and downward-sloping valuations with identical items — were not so simple. While we got everything we wanted, we had to work to get it, and the two solutions — the Crawford-Knoer (CK) auction and Ausubel's clinching auction — don't really resemble each other.

This lecture introduces the gross substitutes condition, which generalizes all four of the scenarios we've seen thus far. This condition captures the "frontier of tractability" for a surprisingly wide range of properties, and we'll only have time to touch on a couple of them. In this lecture, we'll motivate the condition as necessary for auctions in the spirit of the CK auction to plausibly work, and we'll see that it more generally represents a natural limit for the guaranteed existence of Walrasian equilibria. Next lecture, we study the computational complexity of welfare maximization (and hence the VCG mechanism), where again gross substitutes valuations represent the most general class of preferences for which strong positive results are possible.

2 A General Valuation Model

The most general welfare-maximization problem we'll consider in this course is the following.

- There is a set U of m non-identical goods.

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- Each bidder $i = 1, 2, 3 \ldots, n$ has a private valuation $v_i(S)$ for each bundle $S \subseteq U$ of goods that it might receive.[1]
  - Assumption #1: $v_i(\emptyset) = 0$.
  - Assumption #2: "free disposal," meaning the monotonicity condition that $v_i(S) \leq v_i(T)$ whenever $S \subseteq T$.

This is a very general model, certainly general enough to encompass the four scenarios we've look at thus far. It will be come clear over the next several lectures that additional conditions are required for interesting positive results.

The next section motivates the gross substitutes condition as one under which a natural auction "works," meaning terminates with a Walrasian equilibrium. Walrasian equilibria, introduced first for unit-demand valuations in Lecture #2, have an analogous definition in the general model, with the notion of a "favorite item" replaced with a "favorite bundle." Precisely, the *demand* $D_i(\mathbf{q})$ of a bidder i at a price vector $\mathbf{q}$ is the set of its favorite bundles: $\operatorname{argmax}\{v_i(S) - \sum_{j \in S} q(j)\}_{S \subseteq U}$, with the empty set allowed. An *allocation* $(S_1, \ldots, S_n)$ is a bundle $S_i \subseteq U$ for each bidder i such that each item j appears in at most one of these bundles. A *Walrasian equilibrium (WE)* is a nonnegative price vector $\mathbf{q}$ on the items and an allocation $(S_1, \ldots, S_n)$ such that:

(WE1) Each bidder i is matched to a favorite bundle

$$S \in \operatorname{argmax} \left\{ v_i(S) - \sum_{j \in S} q(j) \right\}_{S \subseteq U}, \tag{1}$$

with the empty set $S = \emptyset$ is allowed.

(WE2) An item $j \in U$ is unsold only if $q(j) = 0$.

3 The Kelso-Crawford Auction

We next give an extension of the CK auction, by Kelso and Crawford (KC) [5], for non-unit-demand bidders. It is important that such bidders can bid on more than one item at once, and can also bid for new items even if some items are already assigned to them. Like in CK auction, we disallow bid withdrawals — the only way a bidder can become unassigned to an item is by some other bidder outbidding it.

Kelso-Crawford (KC) Auction:

1. Initialize the price of every item j to $q(j) = 0$.
2. For every bidder i, initialize the set S_i of items assigned to i to $\emptyset$.

[1] This is a lot of private parameters. We'll address complexity and representation issues when we need to, in the next lecture.

3. while (TRUE):

   (a) Ask each bidder for their favorite subset of items not assigned to them, given the items they already have and the current prices — an arbitrary set T_i in

   $$\operatorname*{argmax}_{T \subseteq U \setminus S_i} \left\{ v_i(S_i \cup T) - \mathbf{q}^{\epsilon}(S_i \cup T) \right\},$$

   where

   $$\mathbf{q}^{\epsilon}(S_i \cup T) = \sum_{j \in S_i} q(j) + \sum_{j \in T} (q(j) + \epsilon).$$

   (b) If $T_i = \emptyset$ for all bidders i, then halt with the current allocation $(S_1, \ldots, S_n)$ and prices $\mathbf{q}$.

   (c) Otherwise, pick an arbitrary bidder i with $T_i \neq \emptyset$:

      i. $S_i \leftarrow S_i \cup T_i$;

      ii. for all $k \neq i$, $S_k \leftarrow S_k \setminus T_i$;

      iii. for $j \in T_i$, $q(j) \leftarrow q(j) + \epsilon$.[2]

In the special case of unit-demand bidders, the KC auction is identical to the CK auction.

For bidders with general valuations, bidding sincerely in the KC auction can be a disaster.

Example 3.1 Suppose there are two items $U = \{A, B\}$ and two bidders, with $v_1(S) = 3$ if $S = U$ and otherwise, and $v_2(S) = 2$ is $S \neq \emptyset$ and 0 otherwise. The first bidder is "single-minded" in that it only wants one good if it gets the other. The second bidder is unit-demand.

What happens in the CK auction? Suppose in the first iteration we let the first bidder bid. It wants both goods (at price 0). In the second iteration, the second bidder will take one of the two goods, say the first one, bumping up its price to ϵ. In the third iteration, the first bidder will take the good back, increasing the price to 2ϵ. The next two iterations repeat this cycle with the second good, raising its price to 2ϵ. Every 4 iterations the price of both goods go up by 2ϵ.

Eventually, the price of both goods exceeds $\frac{3}{2}$. At this point, the first bidder would prefer to drop out of the auction. Recall that the KC auction does not allow bid withdrawals, however, so this option is not available. The first bidder has to choose between owning both goods at a price slightly higher than $\frac{3}{2}$ (for slightly negative utility) or one good at this price (for utility roughly $-\frac{3}{2}$); the former option is obviously better. So, the cycle continues until the price of both goods reaches 2, at which point the second bidder gives up. The KC auction concludes with the first bidder getting both goods, at a total price of roughly 4, for a utility of roughly -1.

[2] Unless j was previously unassigned (so i is the first bidder on it), in which case the price can stay at 0.

Remark 3.2 The dilemma faced by the first bidder in Example 3.1 occurs frequently in real auctions, such as auctions for wireless spectrum licenses, and is known as the "exposure problem." The first bidder's preferences exhibit "complements" or "synergies" between the goods, and the KC auction offers no way of expressing this fact. In real-world auctions, bidders vulnerable to the exposure problem tend to bid very conservatively to avoid the disastrous outcome in Example 3.1. The design of richer auction formats that allow for the expression of complements — without introducing new opportunities for harmful strategic behavior — has been a central concern of the economics literature on combinatorial auctions over the past 10+ years.

It is obvious in Example 3.1 that sincere bidding is not any kind of best response. But the same issue applies even in our last scenario, multi-unit auctions with downward-sloping valuations (due to demand reduction). Something more serious is going on in Example 3.1: even assuming sincere bidding, the auction does not result in a Walrasian equilibrium (since bidder 1 gets negative utility).

4 The Gross Substitutes Condition

A red flag was raised in Example 3.1 when the price of both items reaches $\frac{3}{2}$: the first bidder wanted to relinquish one of its goods — that is, no preferred bundle of $D_1(\mathbf{q})$ included the item already assigned to it — but this was forbidden by the auction format. Such an event would seem to preclude convergence to a WE. What conditions on a valuation would guarantee that such an event could not occur? Informally, if a bidder i is assigned to the items S_i at some point in the KC auction — meaning i bid on them at some point in the past — then bidder i should still want them, in the sense that there is a preferred bundle at the current prices that includes all of S_i. The gross substitutes condition is simply the precise articulation of this idea.

Definition 4.1 (Gross Substitutes) A valuation v_i defined on item set U satisfies the *gross substitutes (GS)* condition if and only if the following condition holds. For every price vector $\mathbf{p}$, every set $S \in D_i(\mathbf{p})$, and every price vector $\mathbf{q} \geq \mathbf{p}$, there is a set $T \subseteq U$ with

$$(S \setminus A) \cup T \in D_i(\mathbf{q}),$$

where $A = \{j : q(j) > p(j)\}$ is the of items whose prices have increased (in $\mathbf{q}$ relative to $\mathbf{p}$).

In Definition 4.1, S should be thought of as the items that i wants at some iteration at the prices $\mathbf{p}$, $S \setminus A$ as the items that have since been reassigned to other bidders (at higher prices), and T as the new goods that i wants at the current prices $\mathbf{q}$, given that it still possess the items at $S \subseteq A$ at the original prices $\mathbf{p}$. Asserting that $(S \setminus A) \cup T$ is a preferred bundle (i.e., lies in $D_i(\mathbf{q})$) is tantamount to saying that i does not want to relinquish any of the items $S \setminus A$ that it still retains. Conversely, any failure of the GS condition is an opportunity for the KC auction to go awry: if bidder I bids on the items S at prices $\mathbf{p}$, is outbid by other bidder on the items of $S \cap A$, and has no set T available so that $(S \setminus S) \cup T$ is a preferred

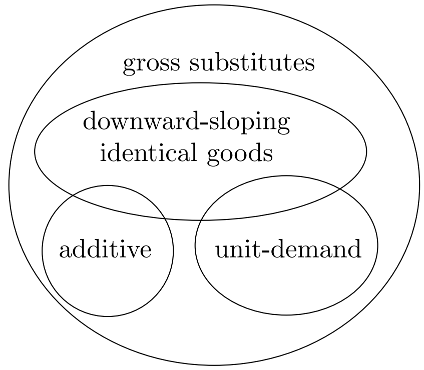

Figure 1: The hierarchy of the scenarios we've studied so far.

bundle at the new prices $\mathbf{q}$, then the KC auction's preclusion of bid withdrawals prevents bidder i from acquiring a preferred bundle, which in turn rules out convergence to a WE.

The valuation of the first bidder in Example 3.1 does *not* meet the GS condition. For example, at the price vector $(1, 1)$ the bidder's favorite bundle is U, while at the price vector $(3, 1)$, the bidder's favorite bundle is $\emptyset$, even though the price of the second item has not increased.

All four scenarios that we've studied thus far are special cases of gross substitutes valuations (Figure 1). With non-identical items and additive valuations, GS holds because a bidder want an item if and only if its valuation for that item is at least its price, independent of its values and the prices for other items. With non-identical items and unit-demand bidders, we effectively proved the GS condition when we proved that the CK auction converges to an approximate WE: if an item is a bidder's favorite and only the prices of other items go up, then the item remains the bidder's favorite. Multi-unit auctions with downward-sloping valuations correspond, in the present language, to a symmetric valuation function (which depends on S only through $|S|$) with decreasing differences ($v_i(|S|+1)-v_i(|S|) \leq v_i(|S|)-v_i(|S|-1)$). Such valuations satisfy the GS condition because, intuitively, losing items only makes the retained items more valuable. For a new example, define a *k-unit valuation* as one with the form

$$v_i(S) = \max_{T \subseteq S \,:\, |T| \leq k} \sum_{j \in T} v_{ij}, \tag{2}$$

where the v_{ij}'s are bidder i's valuations for single items. That is, given a bundle S of items, the bidders throws up all but its k favorites. Such a valuation remains GS if a concave nondecreasing function is applied to the sum in (2); see the exercises.

We next make precise the intuition that the GS condition is the one under which the KC auction converges to a WE.

Theorem 4.2 *If all bidders have gross substitutes valuations and bid sincerely, then the Kelso-Crawford auction terminates at a $m\epsilon$-Walrasian equilibrium.*[3]

[3]In a δ-approximate WE, unsold items have price 0 and every bidder gets a bundle that yields utility within $m\epsilon$ of its preferred bundles.

Proof: Unsold items have price 0 by the same argument used in the CK auction for unit-demand bidders: since bidders only relinquish an item when outbid by another bidder, an item goes unsold only if no bidder even bid on it, in which case its final price is 0.

With GS valuations, we claim the KC auction maintains the following invariant:

> for every bidder i, S_i is contained in a set of $D_i(\mathbf{q}^\epsilon)$, where $\mathbf{q}^\epsilon(j)$ equals $q(j)$ for $j \in S_i$ and $(q(j) + \epsilon)$ for $j \notin S_i$.

That is, no bidder ever wants to withdraw its bids for the items it possesses. The base case, where $S_i = \emptyset$ for each i, is trivial.

For the inductive step, consider bidder i. If i is chosen to bid in this iteration, then the inductive hypothesis ensure that there is a set $T_i \subseteq U \setminus S_i$ such that $S_i \cup T_i \in D_i(\mathbf{q}^\epsilon$. Thus, after this iteration i will possess a set in $D_i(\mathbf{q}^\epsilon)$. Otherwise, let A_i be the last set of items that i bid on. By the inductive hypothesis, B_i was a preferred bundle at the prices at the time. In the subsequent iterations, including the current one, the items reassigned from i to other bidders have had their price increased, while the prices of the items in S_i have stayed the same. By the definition of the GS condition, S_i belongs to a set of $D_i(\mathbf{q}^\epsilon)$, completing the inductive step.

By the invariant and the KC auction's stopping rule, at termination $S_i \in D_i(\mathbf{q}^\epsilon)$ for every bidder i. Since the final prices $\mathbf{q}$ differ from $\mathbf{q}^\epsilon$ by at most ϵ on each good, the KC auction terminates with an $m\epsilon$-WE. ■

Taking the limit as $\epsilon \to 0$ gives the following remarkable consequence of the KC auction.

Corollary 4.3 *If valuations $v_1, \ldots, v_n$ satisfy the gross substitutes condition, then there exists a Walrasian equilibrium.*

Proof: (Sketch.) Consider a sequence of $\frac{1}{N}$-WE for $N = 1, 2, 3, \ldots$, which exists by Theorem 4.2. Some allocation $(S_1, \ldots, S_n)$ repeats infinitely often. The corresponding price vectors lies in a compact set, bounded by the valuations, so they have an accumulation point. This point, together with $(S_1, \ldots, S_n)$, is a Walrasian equilibrium. ■

Thus far we've been taking for granted the existence of Walrasian equilibrium. In many cases, however, WE do not exist.

Example 4.4 Recall Example 4.4, with two items $U = \{A, B\}$ and two bidders, with $v_1(S) = 3$ if $S = U$ and otherwise, and $v_2(S) = 2$ is $S \neq \emptyset$ and 0 otherwise. We claim that there is no Walrasian equilibrium in this instance. One way to see this is to first note that the First Welfare Theorem from Lecture #2 — stating that only welfare-maximizing allocations participate in WE — continues to hold for the general valuation model of Section 2. The first bidder gets both goods in such an allocation. For $\mathbf{q}$ to be a corresponding WE price vector, the total price of the two items must be at least 3. But then there is an item with price less than 2, in which case the second bidder does not receive a preferred bundle.

Example 4.4 absolves the KC auction in Example 3.1 — the reason it failed to compute a WE is because none exist.

More generally, the gross substitutes condition is in some sense the frontier for the guaranteed existence of WE. The following result takes some work to prove, and we won't do so here.

Theorem 4.5 ([4]) *If v_i is a valuation that does not satisfy the GS condition, there are unit-demand (and hence GS) valuations $\mathbf{v}_{-i}$ such that $\mathbf{v}$ admits no Walrasian equilibrium.*

The KC auction naturally motivated the gross substitutes condition, and Theorem 4.5 gives evidence that it is a fundamental concept. It is not merely the boundary beyond which the KC auctions fails to work (meaning does not converge to a WE), but the boundary beyond which WE do not generally exist.[4] Over the next couple of lectures we'll see a very different sense in which GS valuations are a frontier of tractability: they are essentially the most general valuations for which we can compute a welfare-maximizing allocation (and hence implement the VCG mechanism) in polynomial time.

5 EPIC Ascending Auctions

To clarify, we've said nothing so far about incentive-compatibility. Recall that even in the special case of scenario #4 — multi-unit auctions with downward-sloping valuations — we gave an explicit example in which the VCG payments were strictly smaller than all WE prices, to prevent incentives for demand reduction. This contrasted with scenario #3 — unit-demand bidders — where VCG payments coincided with the smallest WE price vector. Since a welfare-maximizing EPIC auction must simulate the VCG outcome — both its allocation *and* its payments (recall Lecture #1) — sincere bidding cannot be an EPNE in the KC auction in general.

5.1 An Impossibility Result

Can we get the VCG outcome with GS valuations using an ascending auction? Perhaps surprisingly, the answer is no.

Theorem 5.1 ([4]) *There is no ascending auction for which sincere bidding yields the VCG outcome for every profile of gross substitutes valuations.*

We emphasize that Theorem 5.1 is not about information, not computation — it says that restricting an auction to ascending price trajectory precludes it from learning all of the valuation information necessary to compute correctly the VCG payments. In this sense, general gross substitutes valuations are harder than the previous four scenarios that we studied. The exercises outline the three-bidder, four-item example used to prove Theorem 5.1.

[4]If some unit-demand valuations are disallowed, then it is possible to guarantee the existence of WE a bit beyond gross substitutes valuations [3, 8].

Theorem 5.1 depends on exactly what is permitted as an "ascending auction." In Theorem 5.1, an ascending auction is one that maintains a price vector on items at all times, and these item prices can only increase over time. Interaction with bidders takes place solely via demand queries. The final allocation and payments is a function of the entire history of the auction. Thus, if two valuation profiles yield to the same histories — the same answers to all demand queries — then the same allocation and payments are computed for these two profiles. All of the ascending auctions we've seen in the course satisfy this definition.

5.2 Allowing Multiple Ascending Price Trajectories

There have been two proposals for enlarging the auction design space to elude the impossibility result in Theorem 5.1. In the first, Ausubel [1] shows how to compute the VCG outcome using $n + 1$ ascending price trajectories. This is in the spirit of allowing $n + 1$ welfare-maxmization computations to compute the VCG outcome and payments in a direct-revelation mechanism. The auction in [1] runs one ascending auction with all bidders present, and one with each bidder excluded in turn. Each of these $n + 1$ auctions can be thought of as an extension of the clinching auction for identical goods (Lecture #4), where the new complication is that the demand for a good can go up or down over the course of the auction (when demand goes up, items are "unclinched" and a refund at the current price is given as compensation). Despite these complications, the demand query information from the different trajectories can be stitched together at the end of the auction to reconstruct the VCG allocation and payments. This auction is an impressive proof of concept — auctions that compute only a Walrasian equilibrium (like the KC auction) can be extended to compute VCG prices while using only item prices — but is fairly complex and does not resemble most auctions that are used in practice.

5.3 Ascending Auctions with Package Bidding

The second proposal is to allow "package bidding," meaning prices on bundles of item in addition to on individual items [2, 7]. The most permissive model is to allow a price $p_i(S)$ for each bidder-bundle pair. This is richer than the auctions we've been looking at so far in two ways: prices can be non-linear (meaning the price for a bundle need not be the sum of its item prices) and non-anonymous (meaning different prices for different bidders).

For example, the following ascending auction is in the spirit of the Crawford-Knoer auction for unit-demand bidders, with bidder-bundle pairs (i, S) playing the previous role of items j.

- While there are new (non-empty) bids:
  - The seller computes the allocation $(S_1, \ldots, S_n)$ maximizing its revenue $\sum_{i=1}^{n} p_i(S)$ and the current prices.
  - For each bidder i with $S_i = \emptyset$, pick $S_i \in \text{argmax}_{S \subseteq U}\{v_i(S) - (p_i(S) + \epsilon)\}$ and increment $p_i(S)$ by ϵ (making it more attractive to the seller next iteration).

The first property of this ascending auction with package bidding is that, assuming sincere bidding, it terminates at an analog of a Walrasian equilibrium, an (approximate) *competitive equilibrium*. This means that the allocation maximize's the seller revenue at the current prices (which holds here by definition) and that each bidder receives a utility-maximizing bundle (up to ϵ). The reasoning for the latter property parallels that of why the CK auction converges to a Walrasian equilibrium with unit-demand bidders — a bidder receives a bundle that it bid on (as its favorite) at some point, and since then the prices of other bundles have only gone up. There is a First Welfare Theorem for competitive equilibria, meaning that the corresponding allocation must be welfare-maximizing.[5] The fact that the ascending auction above terminates with a welfare-maximizing allocation (up to $n\epsilon$) does not contradict the many intractability results for welfare-maximization with general valuations. Indeed, the auction is exponential, "squared": it can require an exponential number of iterations to make non-trivial progress (there are exponentially many prices to maintain), and in each round the seller's revenue-maximization problem in generally NP-hard.

The first property ensures that, whatever the valuations, the auction above converges to the VCG (i.e., welfare-maximizing) allocation. But we've seen many occasions (e.g., scenarios #4 and #5) where this property is much easier to obtain than also computing the VCG payments. So what payments does the auction above terminate with? Its second property is that, if the valuations satisfy the gross substitutes condition, then the ascending auction above terminates with the VCG allocation and payments (up to ϵ terms). The gross substitutes condition is very close to a necessary condition for this property as well (see [6]). In other words, we have identified yet another property — convergence to of natural package bidding auctions to the VCG outcome — for which gross substitutes valuations are essentially the frontier of attainability. We can think of general and gross substitutes valuations as playing analogous roles with bundle prices that gross substitutes and unit-demand valuations play with item prices (for the properties of guaranteed existence and coincidence with the VCG outcome, respectively). See Figure 2.

5.4 Ascending Auctions in Practice

We discussed the practice of ascending auctions as some length last quarter, but we recap here a few relevant points. One approach, used frequently in the 1990s, is just run a Kelso-Crawford-style auction, which is essentially a bunch of simultaneous ascending auctions. There can be big trouble when there synergies between items, due to the "exposure problem," and even with substitutes valuations are vulnerable to demand reduction (in both theory and practice). In their defense, they are simple to understand, and despite their flaws often work reasonably well.

There has been an increasing profusion of proposals over the past 10-15 years about exactly how to incorporate package bidding in combinatorial auctions. An increasing number

[5]Let $\mathbf{p}$ and $(S_1, \ldots, S_n)$ form a competitive equilibrium and let $(S_1^*, \ldots, S_n^*)$ be welfare-maximizing. By the second condition of competitive equilibria, $\sum_{i=1}^n [v_i(S_i) - p_i(S_i)] \geq \sum_{i=1}^n [v_i(S_i^*) - p_i(S_i^*)]$. By the first condition, $\sum_{i=1}^n p_i(S_i) \geq \sum_{i=1}^n p_i(S_i^*)$. Putting these together implies that $\sum_{i=1}^n v_i(S_i) \geq \sum_{i=1}^n v_i(S_i^*)$ and hence $(s_1, \ldots, S_n)$ is also welfare-maximizing.

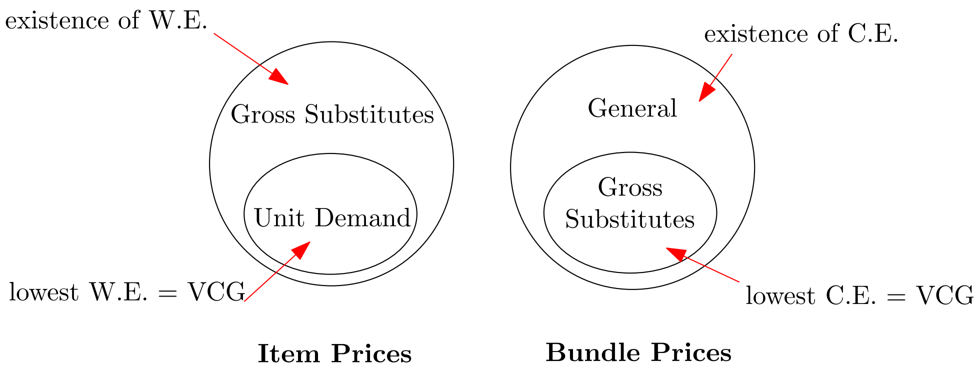

Figure 2: The analogy between item prices and bundle prices, along with the role of Gross Substitutes.

of real-world combinatorial auctions use some form of package bidding. That said, it is a trickly problem and there is currently no consensus on the best way to do it.

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