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Robust Auctions for Revenue via Enhanced Competition

Tim Roughgarden, Inbal Talgam-Cohen
Department of Computer Science, Stanford University, Stanford, CA 94305, tim@cs.stanford.edu, italgam@stanford.edu

Qiqi Yan
Google Inc., Mountain View, CA 94043, contact@qiqiyan.com

Most results in revenue-maximizing mechanism design hinge on "getting the price right" — offering to sell a good to bidders at a price low enough to encourage a sale, but high enough to garner non-trivial revenue. Getting the price right can be hard work, especially when the seller has little or no a priori information about bidders' valuations. Moreover, this approach becomes prohibitively challenging when there are multiple indivisible goods on the market, in which case getting the prices right is a long-standing open problem, even for matching markets with symmetric bidders (each of whom seeks a single good).

In this paper we apply a robust approach to designing auctions for revenue. Instead of relying on prior knowledge regarding bidder valuations, we "let the market do the work" and let prices emerge from competition for scarce goods. We analyze the revenue guarantees of one of the simplest imaginable implementations of this idea: first, enhance competition in the market, whether by increasing demand or by limiting supply; second, run a standard second-price (Vickrey) auction. Enhancing competition is a natural way to bypass lack of knowledge — a seller who does not know how to set prices can instead set quantities (of bidders and/or goods on the market). We prove that simultaneously for many valuation distributions, this achieves expected revenue at least as good as the optimal revenue in the original market or guarantees a constant approximation to it.

Our robust and simple approach thus provides a handle on the elusive optimal revenue in multi-item matching markets, and shows when the use of welfare-maximizing Vickrey auctions is justified even if revenue is a priority. By establishing quantitative trade-offs, our work provides guidelines for a seller in choosing among alternative revenue-extracting strategies: sophisticated pricing based on market research, advertising to draw additional bidders, and limiting supply to create scarcity on the market.

Key words: Bidding and auctions, pricing, matchings

1. Introduction

1.1. The Revenue-Maximization Problem

Consider a set of m indivisible goods for sale, and the problem of matching them to n buyers with private values, each of whom wants no more than a single good. This problem has been studied extensively with respect to the goal of maximizing economic efficiency; e.g., this is the topic of the classic paper on "Multi-Item Auctions" of Demange et al. (1986). In this paper we focus on an alternative important goal — maximizing the seller's revenue.

To demonstrate our setting, consider a for-profit travel website selling overnight accommodation faced with the task of assigning m available rooms to n interested buyers. Each buyer needs a single room for the night, and has different private values for different rooms based on their location, size etc. Uncertainty of the seller regarding buyer values is captured by a probabilistic (Bayesian) model, in which the values for every good $j \in [m]$ are assumed to be independent draws from a distribution F_j (where F_j satisfies a standard regularity condition). The fact that F_j is common to all buyers makes our model *symmetric* with respect to buyers (but not with respect to goods). The seller wishes to maximize its expected revenue by designing a deterministic auction in which no buyer can do better than to participate and reveal his true values (i.e., *dominant strategy* truthful).[1]

When $m = 1$, that is when there is a single good on the market, Myerson (1981) characterizes the revenue-optimal truthful auction under the assumption that the distribution F_1 from which values for the good are drawn is fully known to the seller. The optimal auction in this case turns out to be the well-known second-price auction (Vickrey 1961), with an additional reserve price r tailored to the distribution F_1. The resulting auction is very simple: the bidders report their values to the seller, the bidder with the highest bid above r wins, and the winner pays the second-highest bid above r if there is one or r otherwise. Myerson's characterization of optimal mechanisms also applies to markets with multiple copies (*units*) of the single good, where each bidder seeks at most one copy. More generally, it applies to all *single-parameter* markets, in which every bidder can either win or lose and has a single private value for winning.[2]

Since Myerson's seminal work there have been efforts to extend it in several directions. A direction that has attracted much attention is to generalize the optimal auction characterization *beyond the* $m = 1$ *case, to multi-parameter markets* (e.g., Vincent and Manelli 2007). In particular, there is no known characterization for the matching markets described above, in which there are multiple goods and each buyer seeks at most one good. Another important direction that has become known as "Wilson's doctrine" is to design alternative, *robust* auctions for revenue, in the sense that they do not depend on the seller's full knowledge of the value distributions (Wilson 1987). A third direction is inspired by the *simplicity* of Myerson's auction — a second-price auction with reserve

— and aims to design similarly simple auctions for revenue in more general settings (e.g., Hartline and Roughgarden 2009).

In this paper we contribute to all three goals above by applying a robust approach to revenue maximization. We develop a framework for designing mechanisms that are robust, simple, and guaranteed to work well for a variety of market environments including matching markets. Our mechanisms are based on the natural idea of enhancing bidder competition[3] for the goods, either by adding competing bidders in the manner of Bulow and Klemperer (1996) or by artificially limiting the supply, and then running a variant of the Vickrey auction. Despite avoiding any reference to the value distributions, the expected revenue achieved by these mechanisms exceeds or approximates the expected revenue of the optimal mechanisms tailored to the distributions. Besides leading to good mechanisms, our approach sheds light on trade-offs among possible seller strategies, including how many more buyers are needed, or how many units of a good to produce relative to the market size, in order to replace the need to rigorously learn the preferences of existing buyers.

We now demonstrate our approach via a simple motivating example. Our treatment of robustness in the remainder of this introduction is intuitive; for a formal discussion of how we define robustness and a comparison to other robustness notions see Section 2.

1.2. Motivating Example: Multi-Unit Markets

As a simple motivating example we consider symmetric multi-unit markets (Example 1), and in particular a special case in Example 2 and a generalization in Example 3.

Example 1 (Multi-unit). There are k identical copies, or *units*, of a single good for sale, and $n \geq k$ bidders who each want at most one unit. The bidders' values for a unit are i.i.d. samples from the value distribution.

For a survey on multi-unit auctions see Nisan (2014).[4] For example, the units can be identical rooms at a large hotel. Another example is copies of a digital good such as an e-book, in which case there is no limit on the number of copies that can be made.

Example 2 (Digital goods). A multi-unit market with $k = n$ units, where n is the number of bidders.

We also consider a generalization in which there are *feasibility* constraints, i.e., not all sets of bidders can feasibly be allocated units, even if they include less bidders than the number of available units. The next example demonstrates the kind of feasibility constraints we consider (additional such *matroid* constraints appear in Bikhchandani et al. 2011):

Example 3 (Job Scheduling). A multi-unit market where the units are slots for running jobs on a machine, and a subset of bidders is feasible if each bidder's job can be matched to a slot between its arrival time and deadline.

Technically, a multi-unit market with no constraints can be thought of as finding a matching of bidders to units in a complete bipartite graph, while in the job scheduling example the corresponding bipartite graph has some suitable structure.

Our Approach We present two approaches to robust revenue maximization in the above examples: augmenting demand and limiting supply. Both approaches are inspired by common business practices — augmenting demand corresponds to advertising the auction and drawing more participants, and limiting supply corresponds to practices like "limited editions", limited runs of artwork, or artificial scarcity (for an example of this phenomenon in the diamond market see McEachern 2012, other examples include scarcity of newly-launched technology products etc.). On a theoretical level, both approaches rethink the standard definition of an auction environment, in which the demand and supply are considered exogenous, treating these instead as an endogenous part of the mechanism design problem.

Augmenting Demand in Multi-Unit Markets A well-known result states the following:

Theorem 1 **(Bulow and Klemperer (1996))**. *When selling a single good to bidders whose values are i.i.d. draws from a distribution satisfying regularity, the expected revenue of the revenue-optimal mechanism with n bidders is at most that of the Vickrey auction with $n+1$ bidders.*

In other words, when the demand is augmented by adding a single additional bidder competing for the good, the simple Vickrey auction achieves at least the maximum revenue possible with the original demand. This is despite being oblivious to the value distribution, whereas the optimal Myerson mechanism for n bidders depends on this knowledge to set the reserve price.

We remark that the *regularity* constraint on the distribution is standard; for a bidder whose value is drawn from the distribution, regularity means that the *revenue curve* describing the trade-off between selling to the bidder often at a low price and selling less often at a higher price is concave. This is satisfied by all common distributions (uniform, normal, power-law, etc.), and without it no result along the lines of Theorem 1 is possible — see Section 3 for details.

The Bulow and Klemperer theorem generalizes to the multi-unit setting in Example 1: When there are k units of the good, the expected revenue of the revenue-optimal mechanism with n bidders is at most that of the Vickrey auction with $n+k$ bidders (Bulow and Klemperer 1996). It also applies to constrained settings such as the job scheduling example (Example 3): When the best schedule is able to match $\rho \leq k$ bidders to the k slots without violating an arrival time/deadline, the expected revenue of the revenue-optimal mechanism with n bidders is at most that of the Vickrey auction with $n+\rho$ bidders, where the ρ additional bidders can be scheduled simultaneously (Dughmi et al. 2012).

In general, a *Bulow-Klemperer-type* theorem states that instead of running the optimal mechanism on the original auction environment, we can get as much revenue in expectation by running a variant of the Vickrey auction on an environment suitably augmented with additional bidders. This can be seen as a theoretical justification to treat bidder participation in auctions as a first-order concern when aiming for revenue, perhaps even at the expense of sophisticated pricing.

Limiting Supply in Multi-Unit Markets The flip side of increasing demand is limiting supply.

Mechanism 1 Supply-Limiting Mechanism

1. Set a supply limit $\ell = n/2$ equal to half the number of bidders.
2. Run the Vickrey auction subject to supply limit ℓ.

Mechanism 1 is a supply-limiting mechanism for digital goods (Example 2). In the second step of the algorithm, the *Vickrey auction subject to supply limit* ℓ is a simple variation on the standard second-price auction: it assigns copies to the ℓ buyers with highest bids (even though there are enough copies for all buyers), and charges them each the $(\ell+1)$th highest bid. The resulting mechanism is simple and natural, and does not rely on knowledge of value distributions.

Intuitively, enhancing competition by limiting the supply has a similar effect on revenue as enhancing competition by adding bidders in Bulow-Klemperer-type theorems. The difference between the two approaches is that the former requires augmenting the resources — in this case bidders — available to the auction, while the latter requires the ability to withhold supply (many sellers, e.g. companies like `Apple Inc.`, have the ability to do both). This difference translates into different revenue guarantees: while in Bulow-Klemperer-type theorems the expected revenue of the augmented Vickrey auction usually exceeds that of the optimal mechanism, the expected revenue of the supply-limiting mechanism approximates that of the optimal mechanism. In particular, *the supply-limiting mechanism achieves at least half of the optimal revenue in expectation, despite remaining oblivious to the value distribution on which the optimal mechanism depends.*

The performance guarantee for Mechanism 1 can be seen as justification to the following "rule of thumb" for sellers: assuming no production costs and buyers whose values are distributed similarly, produce a number of units equal to a constant fraction of the market size. I.e., when sellers lack the necessary information to set prices, they can set quantities instead, and this works well simultaneously for many value distributions.

The Connection between Augmenting Demand and Limiting Supply Our technical approach to establishing approximation guarantees of supply-limiting mechanisms utilizes the intuition above, by which limiting supply has a similar effect as increasing demand. This intuition is formulated by basing the proofs of the approximation factors on a reduction among markets, which enables the application of an appropriate Bulow-Klemperer-type theorem.

Reduction 2 Digital Goods

0. **Start with original market with n buyers and n units**

Denote the optimal expected revenue by OPT.

1. **Restrict to market with $n/2$ buyers and supply limit $n/2$**

The optimal expected revenue is $\mathrm{OPT}/2$, by subadditivity of revenue in the buyers (Lemma 6 below) and since the supply limit has no effect here.

2. **Augment to get market with n buyers and supply limit $n/2$**

The expected revenue of the Vickrey auction is $\mathrm{OPT}/2$, by the Bulow-Klemperer-type theorem for multi-unit markets applied to the restricted market.

In particular, Reduction 2 shows that Mechanism 1 guarantees half the optimal revenue in expectation as follows: Starting with the *original market*, define a new market by dropping half of the bidders and setting a supply limit of $\ell = n/2$. Consider the resulting *restricted market* with half of the original bidders and corresponding supply limit. One can show that if we were to restrict the optimal mechanism to run on this market instead of the original one, its expected revenue would have been at least half of its original expected revenue. Now conceptually add back the $n/2$ removed bidders but without changing the supply to get the *augmented market*, and run the Vickrey auction. It follows from the Bulow-Klemperer theorem for multi-unit markets that the expected revenue is at least as high as the optimal expected revenue for the restricted market. Therefore the supply-limiting mechanism guarantees at least half of the optimal expected revenue in the original market.

1.3. Our Contribution

As demonstrated in Section 1.2, *our main contribution is in formulating and proving robust revenue guarantees of competition enhancement in auctions, through increased demand or limited supply, for a variety of markets, including types of markets where the optimal mechanism remains unknown (and is presumably very complex).* We show that under minimal regularity assumptions, the simple and robust mechanisms above and their revenue guarantees — Vickrey with additional bidders and Vickrey with a supply limit — generalize to significantly more complex settings. In other words, we identify markets in which such mechanisms are guaranteed to achieve optimal or approximately-optimal expected revenue. We remark that by using these mechanisms for revenue, the seller also guarantees that the welfare is approximately optimal, and in fact one can achieve other trade-offs between revenue and welfare by setting suitable supply limits.

Our technical contribution is in proving novel Bulow-Klemperer-type theorems for different markets, and designing supply-limiting mechanisms whose approximation guarantees follow from the Bulow-Klemperer-type theorems. While Bulow-Klemperer-type theorems have been studied before, they have never been attempted beyond single-parameter buyers, i.e., for multiple different goods. To our knowledge, supply-limiting mechanisms have also not been studied before, nor has the connection between increasing demand and limiting supply been explicitly formulated as in our reductions.

Proving Bulow-Klemperer-type theorems for matching markets is the most technically challenging component of this work. The analysis for multi-parameter settings is challenging due to dependency issues — the competition for item j that drives its price depends on the buyers' values for the other items. We overcome dependency challenges via a technique from the analysis of randomized algorithms called the principle of deferred decision, combined with the combinatorial properties of optimal matchings.

Results: Augmented Demand We prove the first generalization of Bulow and Klemperer's theorem (Theorem 1) to multi-parameter markets.

Theorem 2 **(Bulow-Klemperer-Type Theorem for Matching Markets (Informal)).** *For every matching market with n bidders and m goods, assuming symmetry and regularity, the expected revenue of the Vickrey auction with m additional bidders is at least the optimal expected revenue in the original market.*

The formal statement appears in Theorem 6. We emphasize that the symmetry assumption in this theorem is across bidders, not goods. That is, values of different bidders for the same good are i.i.d. samples from the same distribution, but different goods can have different value distributions. This kind of symmetry makes sense in practical applications, where the seller knows it is selling very different kinds of goods, but sees the bidders — whose identities and characteristics are unknown — as homogeneous (Chung and Ely 2007).

In addition to Theorem 2, we prove Bulow-Klemperer-type theorems that achieve better guarantees for matching markets with more supply than demand ($n \le m$), and that apply to *asymmetric* markets where bidders' values for a good may belong to different distributions (see Section 7).

Results: Limited Supply We design supply-limiting mechanisms for both single-parameter and multi-item markets. The former include digital good markets (recall Mechanism 1), as well as more general multi-unit markets, possibly with constraints or asymmetric bidders (see Section 4).

For multi-item matching markets, we first define a notion of setting a limit on supply where the supply is heterogeneous rather than homogeneous. A multi-item auction *subject to supply limit* ℓ means that no more than ℓ goods may be assigned, with no limitation on which ℓ goods these shall be. Intuitively, this lets the market do the work of choosing which part of the supply to limit. This is in line with our robust approach, as a seller with no knowledge of how the values for the different goods are distributed cannot make this decision without risking a big loss in revenue. Notice that the simple supply-limiting mechanism we designed for multi-unit markets (Mechanism 1) is now well-defined for multi-item markets as well, and we can prove the following theorem:

Theorem 3 **(Supply-Limiting Mechanism for Matching Markets (Informal))**. *For every matching market with* $n \geq 2$ *bidders and* m *goods, the expected revenue of Mechanism 1 is at least a constant fraction of the optimal expected revenue.*

Qualitatively, Theorem 3 is interesting since it shows that a simple robust mechanism can achieve a fraction of the optimal expected revenue that is *independent of the size of the market*, as measured by parameters n and m. Moreover, the constant fractions we achieve are quite good in many cases, e.g., we achieve a fraction of 1/4 when the number of bidders equals the number of goods (Theorem 8). An interesting open problem is whether this is the best possible by any robust mechanism. The analysis of the approximation guarantees are via a general reduction (Reduction 3) along the lines of Reduction 2, instantiated with appropriate Bulow-Klemperer-type theorems.

Reduction 3 Approximation Guarantees via Bulow-Klemperer-Type Theorems

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0. **Start with original market with** n **buyers**

Denote the optimal expected revenue by OPT.

1. **Restrict to market with** $< n$ **buyers and supply limit** ℓ

The optimal expected revenue is a constant fraction of OPT, by subadditivity.

2. **Augment to get market with** n **buyers and supply limit** ℓ

The expected revenue of the Vickrey auction is a constant fraction of OPT, by a suitable Bulow-Klemperer-type theorem.

1.4. Organization

In Section 2 we discuss our approach to robustness and survey related literature. In Section 3 we formally present our model and preliminaries. Section 4 includes our analysis of competition enhancement for multi-unit markets. Sections 5 and 6 analyze multi-item matching markets where n is proportional to m and contain our main technical results for increasing demand and limiting supply, respectively. Extensions and generalizations can be found in Section 7. Section 8 concludes.

2. Prior-Independent Robustness and Related Work

In this section we discuss our approach to robust revenue guarantees, and present related work.

2.1. Definition

Robustness has been a long-time goal of mechanism and market design. Intuitively, robust mechanisms are mechanisms that "perform well" for a "large range" of economic environments. Their performance is insensitive to the environment's precise details and for this reason robustness is also referred to as *detail-freeness*. To formulate robustness one must specify what it means to perform well and for which range of environments should the performance guarantee hold. There are several alternative formulations in the literature, including robust optimization and others, and we discuss these in Section 2.3.

To define the *prior-independent* notion of robustness, we focus for simplicity on the single good case. Consider first a particular distribution F from which the buyers' i.i.d. values for the good are drawn. Let OPT_F be the optimal expected revenue that a truthful deterministic mechanism *with full knowledge of* F can achieve in this market. Let $\alpha \in (0,1]$ be an approximation factor. A mechanism is α*-optimal* with respect to F if its expected revenue is at least $\alpha\,\mathrm{OPT}_F$. This is similar to average-case approximation in combinatorial optimization, where an algorithm's approximation guarantee holds for inputs drawn from a known distribution; the difference is that the benchmark OPT_F is with respect to a truthful mechanism instead of an algorithm.

Now let $\mathcal{F}$ be a set of value distributions, called the *priors*. A mechanism is *robustly* α*-optimal with respect to* $\mathcal{F}$ if for every distribution F in this set, the mechanism is α-optimal with respect to F. In this case we also say that it gives an α-approximation to the optimal expected revenue. We have thus defined what it means for a robust mechanism to "perform well": it must achieve expected revenue that either exceeds or approximates the optimal expected revenue *simultaneously for every distribution* in a class of distributions $\mathcal{F}$.

In this paper we set the "large range" of distributions $\mathcal{F}$ to be all *regular* distributions. Roughly these are distributions whose tail is no heavier than that of a power law distribution. Regularity is a standard assumption in auction theory, analogous to that of downward-sloping marginal revenue in monopoly theory, and without it no good robust revenue guarantees are possible (see Bulow and Klemperer 1996, and Section 3 for a precise definition and negative examples).

The above definition of robustness is an interesting mixture of average- and worst-case guarantees. On one hand, performance is measured in expectation over the random input; on the other it is measured in the worst case over all distributions that belong to $\mathcal{F}$. Such robustness is referred to as prior-independence, since there is an underlying assumption that values are sampled from priors, and yet the robust mechanism must be independent of the priors as it must work well for all of them.

2.2. Rationales

What are the rationales behind prior-independent robustness? In particular, why measure whether a robust auction is performing well by comparing it to the optimal mechanism with access to the prior distribution? And why choose a large range of distributions with minimal assumptions (rather than incorporate partial information that the seller may have about the distributions in order to narrow it down)?

First, our main results show that for a large range of distributions on which little is assumed, we can get a constant approximation to the ambitious benchmark of OPT_F, even in challenging environments like multi-item markets for which the optimal mechanism remains elusive: Theorem 2 can be rephrased as stating that Vickrey with m additional bidders is robustly 1-optimal, and Theorem 3 can be rephrased as stating that Vickrey with supply limit $n/2$ is robustly α-optimal for some constant $1/\alpha$. The two choices above thus serve to strengthen our results.

An alternative approach to robustness could be, rather than to *approximate* the optimal mechanism for every distribution in $\mathcal{F}$, to design a mechanism that *maximizes* the minimum expected revenue where the minimum is taken over all distributions in the set $\mathcal{F}$. Such an approach would run into the open problem of finding the optimal mechanism for multi-item markets, even in the degenerate case where $\mathcal{F}$ contains only one distribution. Seeking approximation rather than maximization is what enables us to circumvent the open problem. In addition, choosing OPT_F as a benchmark allows the seller to make informed decisions regarding how much to invest in obtaining information on F.

The choice of setting $\mathcal{F}$ to be the class of all regular distributions has the advantage of capturing situations in which the seller has no information about the value distribution, as is the case for a new seller or new good on the market and for goods whose distribution is constantly shifting, as well as situations in which the seller's information is prohibitively expensive or highly noisy and thus too risky to rely upon. In other words, the same reason for avoiding dependence on prior distributions in mechanism design — lack of reliable, accessible information — also justifies avoiding dependence on prior information about the distributions. Moreover, assuming no partial information leads to simple and natural mechanisms, thus reinforcing our chosen robustness notion.

2.3. Robustness in the Literature

Robustness in mechanism design has been studied from three different perspectives — economics, operations research and computer science. The Wilson (1987) doctrine in economics calls for the development of mechanisms independent of the details of the economic environment, as far as these are not really common knowledge among the buyers and seller. Wilson writes that the importance of "repeated weakening of the common knowledge assumption" is that only in this way "will the theory approximate reality." In operations research, Scarf (1958) observed that "we may have reason to suspect that the future demand will come from a distribution that differs from that governing past history in an unpredictable way". In this context, Bertsimas and Thiele (2014) note in their essay on "Modern Decision-Making Under Uncertainty" that the need for a non-probabilistic theory has become pressing. In computer science, the dominant paradigm of worst-case analysis has been extended to mechanism design, reflecting the expectancy that mechanisms (like algorithms) should work well across a range of settings, as well as a general mistrust in designers' ability to accurately capture real-world distributions (Nisan 2014).

Given its importance, it is not surprising that there is a rich literature on robustness in mechanism design. Our approach contributes to this literature by simultaneously achieving robustness and simplicity while being applicable to multi-item environments.

Prior-Independence for Single-Parameter Markets For single-item and other single-parameter markets, Bulow and Klemperer (1996) were the first to study the effect of augmented demand. A simplified proof of their main result was given by Kirkegaard (2006). Dughmi et al. (2012) generalize this result to markets with matroid-based constraints, and use the generalized version to investigate conditions under which the Vickrey auction inadvertently yields approximately-optimal revenue. Hartline and Roughgarden (2009) develop a similar result for asymmetric buyers who do not share the same value distributions. Fu et al. (2015) study the tightness of the Bulow and Klemperer result. Using techniques related to an early version of this work (Roughgarden et al. 2012), Sivan and Syrgkanis (2013) develop a version of Bulow and Klemperer's result for *convex combinations* of distributions satisfying regularity.

A natural approach to prior-independent robustness is to instantiate the optimal mechanism developed by Myerson (1981) with an empirical rather than known distribution, where the samples come from the buyers' bids. This approach is explored by Segal (2003) and Baliga and Vohra (2003), and is asymptotically optimal as the size of the market goes to infinity. It also allows the seller to incorporate into the mechanism prior information about the class of possible value distributions (a "higher level" of prior information — not about the possible values but rather about their possible distributions). Dhangwatnotai et al. (2010) take the sampling approach further by designing simple mechanisms using only a single sample, which are nevertheless robustly α-optimal for small constant $1/\alpha$ parameters.

Other Robustness Notions for Single-Parameter Markets Value distributions are not used in the design of prior-independent auctions, but they are used in their analysis, namely in the definition of "robustly α-optimal" which is based on comparison to the optimal expected revenue. In prior-*free* auction design, distributions are not even used to evaluate the performance of an auction. This raises the question of which benchmark to use. Neeman (2003) studies the revenue performance of the English auction — in our setting an ascending-auction version of the Vickrey auction — comparing it to the benchmark of welfare, which is clearly an upper bound on revenue in auctions in which buyers have no better choice than to participate. A different approach is initiated by Goldberg et al. (2006), who define a notion of "reasonable" auctions to compete against. The relations between the different notions of robustness has also been studied (see, e.g., Hartline and Roughgarden 2014).

Multi-Parameter Markets In contemporaneous work with an early version of this paper, Devanur et al. independently consider a similar set of problems as us, but using different mechanisms and analyses (2011). Their mechanisms are arguably more complex and less natural since they are based on carefully-constructed price menus (rather than on enhanced competition). Following our early version, Azar et al. (2014) studied matching markets in which partial information about the value distributions is available to the seller in the form of a limited number of samples.

Closely related to our work, Bandi and Bertsimas (2014) apply the *robust optimization* approach to optimal mechanism design in multi-item markets. Their model differs from our model in several important aspects, including consideration of additive valuations rather than matching markets, and divisible rather than indivisible goods. A main goal of their paper is orthogonal to ours — to study the important issues of budgets and correlated values in mechanism design. They also address auctions without budget constraints but in their setting these reduce to single-item auctions, which is far from the case in our model.

It is also interesting to compare our robustness notion to theirs, where the latter is inspired by the robust optimization paradigm. Bandi and Bertsimas model the seller's knowledge about the values by an uncertainty set, thus accommodating for partial knowledge based on historical bidding data, and then optimally solve the related robust optimization problem. They use simulations to show that their robust optimization approach improves upon the revenue performance of Myerson's mechanism for single items, when the seller's knowledge of the prior distribution is inaccurate. We use a different notion of robustness inspired by approximation algorithms for combinatorial optimization, and our goal for single items is to surpass or approximate the performance of Myerson's mechanism tailored to the accurate distribution (which our mechanism is oblivious to).

Recently there have been significant advances on the problem of prior-*dependent* optimal mechanism characterization for multi-item markets. Cai et al. (2013) give a characterization for optimal mechanisms given access to the prior distributions, and with the relaxed requirement of *Bayesian*, rather than dominant strategy, truthfulness. The relaxed truthfulness notion requires that no buyer can do better *in expectation over the other buyers' valuations* than to participate and bid truthfully in the auction. It thus relies on common knowledge of the prior distributions among the buyers as well as for the seller (*cf.* Chung and Ely 2007).

Chawla et al. (2010a) give an upper bound on the optimal expected revenue for matching markets, and our techniques utilize one of their reductions. They achieve a prior-*dependent* 1/6.75-approximation for matching markets with multiple units and asymmetric buyers, and also a 3/32-approximation for an even more general environment (namely a graphical matroid with unit-demand buyers).

Simplicity in Mechanism Design Another mechanism design consideration that has drawn attention in recent years is simplicity. Chawla et al. (2007, 2010a) study (prior-dependent) posted-price mechanisms, where buyers simply choose from a menu of priced allocations. Hartline and Roughgarden (2009) seek conditions on single-parameter markets such that the simple Vickrey auction with (prior-dependent) reserves achieves near-optimal revenue. This simple auction format or a generalized version of it are common in online advertising and sponsored search, the main source of revenue for companies like `Google Inc.` or `Yahoo! Inc.` (Lahaie et al. 2007, Celis et al. 2014).

The extension of the Vickrey auction to multi-item markets, called the VCG mechanism (Vickrey 1961, Clarke 1971, Groves 1973), is arguably not as simple (Ausubel and Milgrom 2006, Rothkopf 2007). Yet in the matching markets we consider, many of the complications of VCG do not occur, namely, communicating the bids and running the auction are both computationally tractable, and our competition enhancement methods ensure that the revenue does not collapse. Chawla et al. (2013) analyze the VCG mechanism's performance in a job scheduling context, and some of our techniques are inspired by their analysis.

Another simple mechanism format that has been proposed recently (Babaioff et al. 2014) is a lottery between running Myerson's mechanism for the grand bundle of all goods, and between separate runs of Myerson's mechanism for every good.

3. Preliminaries

In Section 3.1 we describe our model including multi-unit and matching environments, in Section 3.2 we review the basics of optimal mechanism design and in Section 3.3 we discuss two technical tools (regularity and representative environments).

3.1. Model

An auction environment (or market) has as set $\{1,\dots,m\}$ of m goods (or items) for sale to a set $\{1,\dots,n\}$ of n bidders (or buyers). As a convention we use the index i for bidders and j for items. Throughout we make the distinction between items and units, where the latter are different copies of the same item so bidders have the same value for them.

We describe two main environments of interest; two extensions appear in Section 7.

Multi-Unit and Other Single-Parameter Environments Consider a general model of single-parameter environments: An environment is defined by a non-empty collection $\mathcal{I} \subseteq 2^{[n]}$ of bidder sets, each containing bidders who can *win* simultaneously. The sets in $\mathcal{I}$ are called *feasible allocations*. Every subset of a feasible allocation is also feasible (i.e., the set system $([n],\mathcal{I})$ is *downward-closed*). We assume that every bidder belongs to at least one feasible allocation.

Every bidder i has a private value $v_i \in [0,\infty)$ for winning, which is drawn independently at random from a distribution F_i with a density function f_i positive over a nonzero interval support. The density function is smooth with one exception — a constant amount of probability mass can concentrate on the highest point in the support. This exception is useful for Proposition 1. The described environment is called single-parameter since the value for winning is fully described by v_i.

Throughout we assume a risk-neutral quasi-linear utility model, in which a bidder's utility for winning is his value minus the payment he is charged, and bidders aim to maximize their expected utilities. We say that single-parameter bidders are *i.i.d.* (or symmetric) if their value distributions are identical. The environment is i.i.d. if the bidders are i.i.d.

We can now define a multi-unit (or *k-unit*) environment: It is a single-parameter environment in which a subset of bidders is a feasible allocation if and only if its size is at most k. This models k units for sale to $n \geq k$ *unit-demand* bidders who are interested in at most one unit. We will sometimes impose an additional supply limit of $\ell \leq k$, restricting feasible allocations to size at most ℓ.

A *matroid* environment is a single-parameter environment in which the set system $([n],\mathcal{I})$ of bidders and feasible allocations forms a *matroid* (Oxley 1992). Job scheduling markets (Example 3) are an example of matroid environments, and k-unit environments are a special case (corresponding to the k-uniform matroid). See Section 7.1 for further details.

Multi-Item Matching Environments In a matching environment there are m different items for sale with one unit available of each item. Feasible allocations are all matchings of items to bidders (each bidder wins at most one item and each item is allocated to at most one bidder). This models unit-demand bidders. We will sometimes impose an additional supply limit of $\ell \le m$, restricting the matchings to size at most ℓ. Every bidder i has a private value $v_{i,j} \in [0, \infty)$ for winning item j, which is drawn independently at random from a distribution $F_{i,j}$ with a smooth density function $f_{i,j}$ positive over a nonzero interval support. A matching environment is thus *multi-parameter*. We again assume a risk-neutral quasi-linear utility model.

We say the bidders are i.i.d. (or symmetric) if $F_{i,j}$ does not depend on the identity of bidder i, i.e., each item j has an associated distribution F_j and $F_{i,j} = F_j$. In other words, for every item j the values $\{v_{i,j}\}_{i\in[n]}$ are i.i.d. samples from F_j. Note that different items j, j' have different distributions $F_j, F_{j'}$, as necessary for applications such as the travel website in Section 1. Independence of the values is maintained across both bidders and items. For a treatment of *asymmetric* bidders see Sections 7.2 and 7.3.

3.2. Optimal Mechanism Design

Mechanisms By the revelation principle, without loss of generality we may restrict attention to direct mechanisms, which receive a vector of bids $\mathbf{b}$. In the single-parameter case $\mathbf{b} \in \mathbb{R}^n_{\ge 0}$ where b_i is bidder i's bid for winning, and in the matching case $\mathbf{b} \in \mathbb{R}^{nm}_{\ge 0}$ where $b_{i,j}$ is bidder i's bid for winning item j. We focus on deterministic mechanisms, comprised of:

1. An allocation rule $\mathbf{x} = \mathbf{x}(\mathbf{b})$, which maps a bid vector $\mathbf{b}$ to a feasible allocation; in the single-parameter case $\mathbf{x} \in \{0,1\}^n$, where $x_i = x_i(\mathbf{b})$ indicates whether bidder i wins, and in the matching case $\mathbf{x} \in \{0,1\}^{nm}$, where $x_{i,j} = x_{i,j}(\mathbf{b})$ indicates whether bidder i wins item j.

2. A payment rule $\mathbf{p} = \mathbf{p}(\mathbf{b})$, which maps a bid vector $\mathbf{b}$ to a payment vector. The payment vector $\mathbf{p}$ belongs to $\mathbb{R}^n_{\ge 0}$, where $p_i = p_i(\mathbf{b})$ is the payment charged to bidder i.

Fixing a bid vector $\mathbf{b}$, the mechanism's *welfare* in the single-parameter case is $\sum_i x_i v_i$, and in the matching case $\sum_{i,j} x_{i,j} v_{i,j}$. The mechanism's *revenue* is $\sum_i p_i$. Bidder i's utility in the single-parameter case is $x_i v_i - p_i$, and in the matching case $\sum_j x_{i,j} v_{i,j} - p_i$. A mechanism is (dominant strategy) *truthful* if for every bidder i and bid profile $\mathbf{b}_{-i}$ of the other bidders, i maximizes his utility by participating and bidding truthfully, i.e., bidding $b_i = v_i$ in the single-parameter case and $b_{i,j} = v_{i,j}$ for all j in the matching case. All the mechanisms we study are truthful, so from now on we no longer distinguish between bids and values and use v_i or $v_{i,j}$ to denote both. We will mainly be interested in a mechanism's *expected* revenue $\mathbb{E}_{\mathbf{v}}[\sum_i p_i]$, where $\mathbf{p} = \mathbf{p}(\mathbf{v})$ and the expectation is taken over i.i.d. values drawn from the value distributions.

The revenue benchmark against which we measure the performance of our mechanisms is the following: By optimal expected revenue we mean the maximum expected revenue over all (dominant strategy) truthful, deterministic mechanisms. Chawla et al. (2010b) show that for matching environments, the expected revenue from the optimal deterministic mechanism is within a constant factor of the expected revenue from the optimal randomized mechanism. Thus our results for deterministic mechanisms apply to randomized mechanisms up to a constant factor.

Maximizing Welfare and the Vickrey Auction The general form of the Vickrey auction is called the VCG mechanism, and it works for any market whether single-parameter or multi-item. VCG is remarkable in being both truthful and welfare-maximizing for every value profile $\mathbf{v}$. Its allocation rule chooses a feasible allocation that maximizes welfare; its payment rule charges every bidder i a payment equal to i's *externality* — the difference in the maximum welfare *of the other bidders* when i does not participate in the auction and when i does participate in it.

In the context of matching environments, the VCG allocation rule can be implemented as a maximum weighted matching over a bipartite graph, where vertices on one side are the bidders, vertices on the other side are the items, and the weight of every edge (i,j) is $v_{i,j}$ (Bertsekas 1991). The payment rule also solves bipartite matching problems to compute the payments. For single-parameter k-unit environments, Vickrey's allocation rule finds k bidders with highest values, and for matroid environments it uses a simple greedy algorithm to find a feasible allocation with highest welfare (and similarly for the Vickrey payment rules).

For our supply-limiting mechanisms, we add to the VCG or Vickrey mechanisms a supply limit ℓ and denote them by $\mathrm{VCG}^{\leq \ell}$ and $\mathrm{Vic}^{\leq \ell}$, respectively.

Maximizing Revenue and Myerson's Mechanism For single-parameter environments, Myerson (1981) characterized the optimal mechanism that maximizes expected revenue. (In fact, Myerson showed an even stronger result — his mechanism maximizes the expected revenue over all Bayesian truthful, randomized mechanisms!) Let F be a regular distribution with density f (see Section 3.3 for discussion of regularity). Define its *virtual value* function $\phi_F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ to be $\phi_F(v) = v - \frac{1-F(v)}{f(v)}$. Myerson showed the following.

LEMMA 1 **(Myerson).** *Given a single-parameter environment and a truthful mechanism* $(\mathbf{x}, \mathbf{p})$, *for every bidder i and value profile $\mathbf{v}_{-i}$ of the other bidders,* $\mathbb{E}_{v_i \sim F_i}[p_i(\mathbf{v})] = \mathbb{E}_{v_i \sim F_i}[x_i(\mathbf{v})\phi_{F_i}(v_i)]$.

Myerson's lemma says that in expectation over bidder i's value, his payment is equal to his virtual value when he is allocated. By summing over all bidders, this lemma implies that in expectation over the value profile, maximizing the revenue is equivalent to maximizing the total virtual value of allocated bidders, a quantity known as the *virtual surplus. Myerson's mechanism* maximizes expected revenue by finding the feasible allocation with maximum virtual surplus. For example, in a k-unit environment this will be the set of $\leq k$ bidders with the highest *positive* virtual values.

3.3. Technical Tools

Regularity We say that bidders are regular if their values are drawn from regular distributions.

Definition 1 (Regular Distribution). A distribution F is regular if its virtual value function is monotone non-decreasing.

Most commonly-studied distributions are regular, including the uniform, exponential and normal distributions, and distributions with log-concave densities. The assumption that bidders are regular is standard in optimal mechanism design and is necessary for designing good prior-independent mechanisms, as demonstrated by Dhangwatnotai et al. (2010): Fix a value z and a number n of bidders, and define an irregular, long-tailed value distribution F_z such that the probability for z is $1/n^2$ and otherwise the value is zero. Consider a single-item environment with n bidders whose values are drawn from F_z. The optimal auction has expected revenue at least z/n. But any prior-independent truthful auction essentially has to "guess" the value of z, since the probability that the non-winning bids provide information about z is small. Thus its expected revenue cannot be within a constant factor of z/n for every F_z.

Representative Environments A representative environment is the "single-parameter counterpart" of a matching environment. Consider a matching environment with m items, n symmetric bidders and value distributions $\{F_j\}_{j=1}^m$. The corresponding representative environment has the same m items, but nm single-parameter bidders — every bidder in the matching environment has m *representatives* in the representative environment. The jth representative of bidder i is only interested in item j and has a value $v_{i,j} \sim F_j$ for winning it. Every allocation in the representative environment can be translated to an allocation in the matching environment — if the jth representative of i wins, then item j is allocated to bidder i in the matching environment — and vice versa. An allocation in the representative environment is *feasible* if the corresponding allocation in the matching environment forms a matching, meaning that only one representative per bidder wins.

Intuitively, the representative environment is more competitive than the matching one, since representatives of the same bidder compete against each other on who will be the winner. Thus the expected revenue achievable in the representative environment should be at least the optimal expected revenue in the matching environment. Chawla et al. (2010a) formalize this intuition by showing that any truthful mechanism M for the matching environment translates to a truthful mechanism M^{rep} for the representative environment, such that the expected revenue of M^{rep} is only higher. Roughly this is by translating the allocation rule of M to an allocation rule in the representative environment as above, and viewing the payment rule of M as a price menu, whose prices are exceeded in the representative environment by charging every representative the minimum value it needs to bid in order to win.

Lemma 2 **(Chawla et al. (2010a))**. *The expected revenue of M^{rep} in the single-parameter representative environment is at least the expected revenue of M in the matching environment.*

4. Multi-Unit Markets

In this section we formally prove the results presented in Section 1.2, demonstrating our general framework. The approach of augmenting demand has been studied for multi-unit auctions, and a slightly generalized version of a result by Bulow and Klemperer (1996) is the following:

Theorem 4 **(Bulow-Klemperer-Type Theorem for Multi-Unit Markets)**. *For every k-unit environment with i.i.d. regular bidders and supply limit ℓ, the expected revenue of the Vickrey auction with* $\min\{k,\ell\}$ *additional bidders is at least the optimal expected revenue in the original market. In other words, Vickrey with* $\min\{k,\ell\}$ *additional bidders is robustly 1-optimal.*

As for limiting supply, we instantiate our general reduction (Reduction 3) with the above Bulow-Klemperer-type theorem to prove the following. For simplicity of presentation assume the number of bidders n is even.[5]

Theorem 5 **(Supply-Limiting Mechanism for Multi-Unit Markets)**. *For every k-unit environment with $n \geq 2$ i.i.d. regular bidders, the expected revenue of the supply-limiting mechanism* $\text{Vic}^{\leq n/2}$ *is at least a* $\max\{\frac{1}{2}, \frac{n-k}{n}\}$*-fraction of the optimal expected revenue. In other words, Vickrey with supply limit $n/2$ is robustly α-optimal for $\alpha = \max\{\frac{1}{2}, \frac{n-k}{n}\}$.*

In the above theorem, the supply limit of $n/2$ "kicks in" when the number of units k exceeds $n/2$, and in this case we get a $1/2$-approximation. If the supply k is limited to $n/2$ to begin with, the competition is inherently high and Vickrey with no supply limit provides an $\frac{n-k}{n}$-approximation.

Proof. We instantiate Reduction 3 as follows. To go from the original market to the restricted market, remove $\min\{\frac{n}{2}, k\}$ bidders from the original market, and if $k > \frac{n}{2}$ set a supply limit of $\ell = \frac{n}{2}$.

Analysis: We first claim that the restriction of the original market maintains at least a fraction of $\max\{\frac{1}{2}, \frac{n-k}{n}\}$ of the optimal expected revenue in the original market. This is because, as shown by Dughmi et al. (2012), the expected optimal revenue as a function of the bidder set is *submodular*.[6] Revenue submodularity means decreasing marginal returns to the expected revenue as more bidders are added, so the first $\max\{\frac{n}{2}, n-k\}$ bidders already capture at least a $\max\{\frac{1}{2}, \frac{n-k}{n}\}$-fraction of the optimal expected revenue. Limiting the supply to $\frac{n}{2}$ when $k > \frac{n}{2}$ has no effect since in this case the number of bidders remaining in the restricted environment is $\frac{n}{2}$.

We can now apply the Bulow-Klemperer-type theorem for multi-unit markets (Theorem 4) to the restricted environment. In the first case, $k > \frac{n}{2}$ and the restricted environment has $\frac{n}{2}$ bidders, k units and supply limit $\ell = \frac{n}{2}$. In the second case, $k \leq \frac{n}{2}$ and the restricted environment has $n-k$ bidders, k units and no supply limit (i.e., $\ell = k$). In both cases, by Theorem 4 the expected revenue of

Vickrey with $\min\{\frac{n}{2}, k\}$ additional bidders is at least the optimal expected revenue in the restricted environment. So running Vickrey with $\min\{\frac{n}{2}, k\}$ additional bidders on the restricted environment is a $\max\{\frac{1}{2}, \frac{n-k}{n}\}$-approximation to the optimal expected revenue in the original environment. But this is equivalent to running the supply-limiting mechanism $\mathrm{Vic}^{\leq n/2}$ on the original environment, completing the proof. □

The approximation factor in Theorem 5 is asymptotically tight:

PROPOSITION 1. *For every $0 < \gamma < 1$, consider the supply-limiting mechanism $\mathrm{Vic}^{\leq \gamma n}$. There exists an n-unit environment with n i.i.d. regular bidders such that the expected revenue of $\mathrm{Vic}^{\leq \gamma n}$ is at most a $(\frac{1}{2} + o(1))$-fraction of the optimal expected revenue.*

Proof. Consider first the case that $1/n \leq \gamma \leq 1/2$, i.e., the supply limit is severe. Let the value distribution F be the uniform distribution over the support $[1, 1+\epsilon]$ for a sufficiently small parameter $\epsilon = \epsilon(n)$. The optimal expected revenue is roughly n, while $\mathrm{Vic}^{\leq \gamma n}$ can extract as revenue at most $\gamma n(1+\epsilon) \leq n/2 + o(1)$.

Now suppose $1/2 < \gamma \leq \frac{n-1}{n}$. For sufficiently large H, let the value distribution be $F(z) = \frac{z}{1+z}$ over the support $[0, H]$ with a point mass of $\frac{1}{1+H}$ at H. The optimal expected revenue is at least the expected revenue achieved by offering a posted price H to every one of the n bidders, which extracts $H(1 - \frac{H}{1+H}) = \frac{H}{1+H} \approx 1$ from every bidder in expectation. In comparison, the expected revenue in $\mathrm{Vic}^{\leq \gamma n}$ comes from the $(\gamma n + 1)$st highest bid. This bid is concentrated around $z = \frac{1-\gamma}{\gamma}$, the value of z such that $F(z) = 1 - \gamma$. So VCG achieves an expected revenue of roughly $\frac{1-\gamma}{\gamma}\gamma n = (1-\gamma)n < \frac{n}{2}$.
□

5. Matching Markets: Augmenting Demand

In this section we prove a Bulow-Klemperer-type theorem for matching environments — the first generalization of Bulow and Klemperer (1996) to a multi-item market. Recall what we mean by i.i.d. bidders in a matching environment: different items have different distributions, but independence is both across bidders and across items.

THEOREM 6 **(Bulow-Klemperer-Type Theorem for Matching Markets)**. *For every matching environment with i.i.d. regular bidders and m items, the expected revenue of the VCG mechanism with m additional bidders is at least the optimal expected revenue in the original market. In other words, VCG with m additional bidders is robustly optimal.*

Theorem 6 provides a simple handle on the unknown optimal expected revenue in matching markets. For example, in a market with two goods for sale, the best achievable revenue is at most what VCG can achive with two more bidders. Note that in markets with plentiful supply, i.e. markets in which $m \gg n$, the demand augmentation that is required is substantial. In Section

5.5 we present an alternative Bulow-Klemperer-type theorem with weaker requirements for this case.

5.1. Overview of the Proof

The proof is divided into two parts. In Section 5.2 we identify an upper bound on the optimal expected revenue in the original environment, and a lower bound on the revenue of the VCG mechanism in the augmented environment with m more bidders. These bounds are relatively simple to analyze and are already similar, though not identical, in form. In Section 5.3 we carefully relate the two bounds to establish the theorem.

Our proof is based on the following ideas. We first observe there is a simple upper bound on the optimal expected revenue in the matching environment — *the expected revenue from running m Vickrey auctions to sell each of the m goods to m separate sets of $n+1$ representatives*, who are single-parameter bidders only interested in one particular good (Lemma 3). Our goal is now to show that VCG with a total of m additional multi-parameter but unit-demand bidders does just as well in terms of revenue.

Recall that in the VCG mechanism, the winner of a certain good pays the externality he inflicts upon other bidders, which includes in particular the "damages" he causes the losing bidders who are not allocated any good by the mechanism. Thus, the payment for every good j is at least the highest value for j among the losers. In the augmented matching environment to which VCG is applied, it is guaranteed that there will be n losers, since there are m goods and $n+m$ bidders. The expected revenue from running VCG on the augmented environment is thus at least *the expected welfare from running m Vickrey auctions to allocate each of the m goods separately to the n losers* (Lemma 4). This lower bound is similar to the above upper bound.

The remaining challenge is a dependency issue — by definition, the losers are likely to have lower values for the goods than the $n+1$ representatives. We use the combinatorial structure of maximum weighted matchings to show that a bidder's values conditional on him losing in the VCG mechanism are, while lower, not likely to be *significantly* so compared to the unconditional case. Thus the losers' damages are enough to cover the expected revenue from the representatives.

On a technical level what we show is that, quite remarkably, the only thing that can be deduced about a bidder's value for an item j from his losing the auction completely is that it is lower than the value of the winner of item j. We establish this by introducing an auxiliary selling mechanism for item j, conceptually and revenue-wise half-way between selling the item separately and selling it as part of the VCG mechanism. The auxiliary mechanism runs a maximum weighted matching algorithm as in VCG, but defers the sale of item j until all other goods have been sold and exactly $n+1$ bidders remain unallocated. Thus, by construction, these bidders' values for item j are unaffected by the dependency issue described above.

5.2. Basic Upper and Lower Bounds

Upper bound. To upper bound the expected optimal revenue we use the following notation: let $\mathrm{Vic}_j(\eta)$ be the expected revenue from selling item j to η i.i.d. single-parameter representatives with value distribution F_j using the Vickrey auction. Then:

LEMMA 3 **(Upper Bound on Optimal Expected Revenue)**. *For every matching environment with n i.i.d. regular bidders and m items, $\sum_j \mathrm{Vic}_j(n+1)$ is at least the optimal expected revenue in the original market.*

Proof. Let $\{F_j\}_{j=1}^m$ be the regular value distributions of the matching environment. By Lemma 2, the optimal expected revenue in the matching environment is upper-bounded by the optimal expected revenue in its single-parameter counterpart, the corresponding representative environment. Recall that in the representative environment there are n single-parameter representatives per item j, whose values for j are i.i.d. draws from F_j. The representatives are grouped in n sets of size m corresponding to the original bidders in the matching environment, and feasibility constraints ensure that at most one representative from each set wins.

We now relax these feasibility constraints to get a new single-parameter environment in which the optimal expected revenue has only increased. Relaxing feasibility only increases the optimal expected revenue since by Meyerson's lemma (Lemma 1), it is equal in expectation to the optimal virtual surplus, and clearly the optimum subject to the constraints is bounded from above by the optimum when these are relaxed.

The new environment is equivalent in terms of revenue to a collection of m single-item environments, where in the j-th environment item j is auctioned to its n single-parameter representatives (values are i.i.d. draws from the regular distribution F_j). By Bulow and Klemperer's result (Theorem 1), the optimal expected revenue from the j-th environment is upper-bounded by $\mathrm{Vic}_j(n+1)$. Summing up over all items completes the proof. $\square$

Lower bound. We now turn to the VCG mechanism applied to the augmented environment, whose revenue is the sum of VCG payments for the items. The next lemma lower-bounds the VCG payment for item j.

LEMMA 4 **(Lower Bound on VCG Revenue)**. *For every matching environment, the VCG payment for item j is at least the value of any unallocated bidder for j.*

Proof. Say bidder i wins item j. The VCG payment for j is equal to the externality that i imposes on the rest of the bidders by winning j. In particular, i prevents an unallocated bidder i' from being allocated j. Thus the payment is at least the value of i' for j. $\square$

5.3. Relating the Upper and Lower Bounds via Deferred Allocation

The upper and lower bounds above share a similar form. On the one hand, by definition of the Vickrey auction, the upper bound $\mathrm{Vic}_j(n+1)$ on the expected revenue from separately auctioning off item j is equal to the expected *second-highest value for j among $n+1$ bidders with values drawn independently from F_j*. On the other hand, the lower bound on the VCG payment for item j in the augmented environment is equal to the *highest value for j among n unallocated bidders with values drawn independently from F_j*, where we use that in the augmented environment only m out of $n+m$ bidders are allocated. From this it may appear as if we have already shown that the lower bound exceeds the upper bound. However, a dependency issue arises — conditioned on the event that a bidder in the augmented environment is unallocated by VCG, his value for item j is no longer distributed like a random sample from F_j. We address this issue by introducing a *deferred allocation* selling procedure.

Algorithm 4 describes our selling procedure for item j.

Algorithm 4 Selling Item j by Deferred Allocation

Given a matching environment with $n+m$ bidders and m items, and an item j:

1. Find a maximum matching of all $m-1$ items other than j to the bidders. Let U be the set of $n+1$ bidders who remain unallocated.
2. Run the Vickrey auction to sell item j to bidder set U.

The following two claims show how deferred allocation resolves the dependency issue; namely, how the revenue from selling item j via the deferred allocation procedure bridges between the upper and lower bounds in Lemmas 3 and 4. The relation is also depicted in Figures 1a to 1c.

CLAIM 1 **(Deferred Allocation and Upper Bound)**. *The revenue from selling item j by deferred allocation (Algorithm 4) is equal in expectation to* $\mathrm{Vic}_j(n+1)$.

Proof. The revenue from selling item j to bidder set U by the Vickrey auction is the second-highest value of a bidder in U for j. Since we exclude item j in step (1) of the deferred allocation procedure and allocate it only in step (2), the allocation in step (1) does not depend on the bidders' values for j. Therefore, the values of the unallocated bidders in U for item j are independent random samples from F_j. The expected second-highest among $n+1$ values drawn independently from F_j is equal to $\mathrm{Vic}_j(n+1)$. □

To relate the revenue from deferred allocation to the lower bound in Lemma 4 we need the following stability property.

Lemma 5 **(Stability of Maximum Matching)**. *Consider an augmented matching environment with $n+m$ bidders and m items. Let bidder set U be as defined in Algorithm 4. If VCG is run on this environment, the set of bidders left unallocated is U with at most one bidder removed.*

Proof. First note that in the matching instances we consider, we may assume there is a unique maximum weighted matching. This holds with probability 1 since the weights are sampled from distributions as described in Section 3.1.[7]

The following is a well-known stability property of maximum weighted matchings (Lovász and Plummer 2009): In a complete weighted bipartite graph with $n+m$ nodes on one side and $m-1$ nodes on the other, consider the maximum weighted matching of size $m-1$. Now add a node to the short side of the graph and find the maximum weighted matching of size m. The set of matched nodes on the long side of the graph remains the same up to a single additional node.

The augmented matching environment corresponds to a complete bipartite graph with bidders on one side and items on the other, with the bidders' values for the items as edge weights. Algorithm 4 finds the maximum weighted matching of size $m-1$ in this graph with item j removed. VCG finds the maximum weighted matching of size m in this graph including item j. The lemma follows by applying the above stability property. $\square$

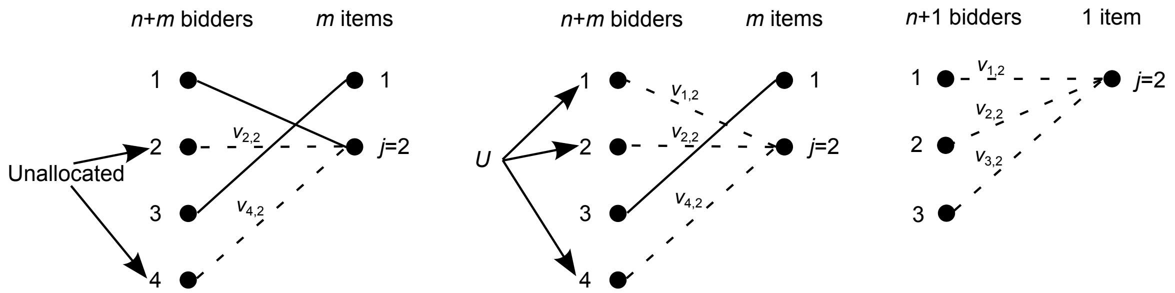
Claim 2 **(Deferred Allocation and Lower Bound)**. *Given an augmented matching environment with $n+m$ bidders and m items, the revenue from selling item j by deferred allocation (Algorithm 4) is at most the VCG payment for item j.*

Proof. The revenue from selling item j by deferred allocation is the second-highest value of a bidder in U for j. Let i_1, i_2 be the two bidders in U who value item j the most. By definition, these bidders are left unallocated by the deferred allocation procedure, and by Lemma 5, one of them (say i_1) is also unallocated by the VCG mechanism. Recall that an unallocated bidder's value for item j gives a lower bound on the VCG payment for j (Lemma 4). So the VCG payment for j is at least $v_{i_1,j}$, which in turn is at least the second-highest value of a bidder in U for item j. $\square$

5.4. Proof of Theorem 6

Putting everything together, we can now complete the proof of the Bulow-Klemperer-type theorem for matching markets.

Proof. We need to show that for every matching environment with n i.i.d. regular bidders and m items, the expected revenue of the VCG mechanism with m additional bidders is at least the optimal expected revneue. By Claim 2, the VCG payment for item j in the augmented environment is at least the revenue from selling item j by deferred allocation, which by Claim 1 is equal in expectation to $\mathrm{Vic}_j(n+1)$. Summing up over all items, the total expected VCG revenue in the



(a) VCG (b) Deferred allocation (c) Vickrey

Figure 1 **Relating bounds by deferred allocation**

Let $n = m = 2$ and item $j = 2$.

(a) **VCG**: Solid edges correspond to the maximum matching. The payment for j is $\geq \max\{v_{2,2}, v_{4,2}\}$ (Lemma 4), where $v_{2,2}, v_{4,2}$ are *not* i.i.d. samples from F_2 given that bidders 2 and 4 are unallocated.

(b) **Deferred allocation**: Solid edges correspond to the maximum matching excluding j. Bidders unallocated in (a) are a subset of the unallocated set U (Lemma 5). Since j is sold to U using Vickrey, the payment for j is the 2nd-highest among $v_{1,2}, v_{2,2}, v_{4,2}$, where these are i.i.d. samples from F_2.

(c) **Vickrey**: The payment for j is the 2nd-highest among $v_{1,2}, v_{2,2}, v_{3,2}$, where these are i.i.d. samples from F_2.

Comparing (a) to (b) and (b) to (c) shows that:

- the payment for j in (a) is at least the payment for j in (b) (Claim 2); and
- in expectation the payment for j in (b) equals the payment for j in (c) (Claim 1).

augmented environment is at least $\sum_j \mathrm{Vic}_j(n+1)$, and by Lemma 3 this upper-bounds the optimal expected revenue in the original environment. $\square$

5.5. The $m \geq n$ Case

In matching markets where items are more plentiful than bidders, the following Bulow-Klemperer-type theorem provides an alternative to Theorem 6, in which the required demand augmentation is n instead of m bidders.[8] Two additional differences in comparison to previous Bulow-Klemperer-type theorems are that the VCG mechanism is required to be supply-limiting, and the revenue guarantee is an approximation.

THEOREM 7 **(Bulow-Klemperer-Type Theorem for Matching with $m \geq n$).** *For every matching environment with n i.i.d. regular bidders and $m \geq n$ items, the expected revenue of the* $\mathrm{VCG}^{\leq n}$ *mechanism with n additional bidders is at least an n/m-fraction of the optimal expected revenue in the original market. In other words,* $\mathrm{VCG}^{\leq n}$ *with n additional bidders is robustly n/m-optimal.*

Proof. The proof is similar to that of Theorem 6, with the following adjustments.

Consider the bounds in Section 5.2 above. While the upper bound on the optimal expected revenue in Lemma 3 holds and is sufficient, the lower bound on VCG payments in Lemma 4 holds but needs to be strengthened. In the augmented environment, $\mathrm{VCG}^{\leq n}$ allocates n out of the m items to n out of the $2n$ bidders. Therefore the VCG payment for item j not only exceeds the value of any unallocated bidder for j, but also exceeds the value of any unallocated bidder for any unallocated item. We shall refer to the highest such value among unallocated bidders and items as the *global* lower bound on VCG payments, and denote it by G.

Now to relate the bounds as in Section 5.3, we use a slightly modified deferred allocation procedure (Algorithm 5).

Algorithm 5 Selling Item j by Deferred Allocation: The Case of $m \geq n$

Given a matching environment with $2n$ bidders and m items, and an item j:

1. Find a maximum matching of $n-1$ items other than j to the bidders. Let U be the set of $n+1$ bidders who remain unallocated.
2. Run the Vickrey auction to sell item j to bidder set U.

Observe that Claims 1 and 2 continue to hold when Algorithm 4 is replaced by Algorithm 5. For Theorem 6 these claims were sufficient to complete the proof, by the following chain of arguments: All items are allocated by VCG in the augmented environment (since it is welfare-maximizing and there are more bidders than items); the VCG payment for item j is at least the revenue from selling j by deferred allocation (by Claim 2); the deferred allocation revenue is equal in expectation to $\mathrm{Vic}_j(n+1)$ (by Claim 1); and $\sum_j \mathrm{Vic}_j(n+1)$ is at least the optimal expected revenue (by Lemma 3). For Theorem 7 we need an additional charging argument — and an approximation factor — since only n out of m items are allocated by $\mathrm{VCG}^{\leq n}$.

For every item $j \in [m]$ there are two cases:

1. If j is allocated, then the VCG payment for j is at least the revenue from selling j by deferred allocation (Claim 2).
2. If j is not allocated, then the VCG payment for any allocated item j' is at least the global lower bound G. A straightforward adaptation of the argument in Claim 2 shows that G is an upper bound on the revenue from selling j by deferred allocation (Algorithm 5).

To complete the proof, we charge the VCG payments for the n allocated items against the aggregate revenue from selling each of the m items by deferred allocation, where the latter is equal in expectation to $\sum_j \mathrm{Vic}_j(n+1)$. This leads to the approximation factor of $\frac{n}{m}$. $\square$

6. Matching Markets: Limiting Supply

In this section we present a supply-limiting mechanism for matching environments. For simplicity of presentation assume that the number of bidders n is even.

THEOREM 8 **(Supply-Limiting Mechanism for Matching Markets)**. *For every matching environment with $n \geq 2$ i.i.d. regular bidders and m items, let $\alpha = \max\{\frac{n-m}{n}, \min\{\frac{1}{2}, \frac{n}{4m}\}\}$. Then the expected revenue of the supply-limiting mechanism* $\mathrm{VCG}^{\leq n/2}$ *is at least an α-fraction of the optimal expected revenue. In other words, VCG with supply limit $n/2$ is robustly α-optimal.*

Intuitively, achieving a good revenue guarantee becomes more difficult as the number of items m grows relatively to the number of bidders n, since the inherent competition among the bidders is split across different items. Accordingly, the fraction α in Theorem 8 depends on the parameters n and m of the environment as follows:

- If $m \leq \frac{n}{2}$ then $\alpha = \frac{n-m}{n}$, i.e., the approximation gets better as m becomes smaller, and for $m = \frac{n}{2}$ we get $\alpha = \frac{1}{2}$.
- If $m \geq \frac{n}{2}$ then $\alpha = \frac{n}{4m}$, and in particular for $m = n$ we get $\alpha = \frac{1}{4}$.

Also note that when $m \leq n/2$, the supply limit of $\mathrm{VCG}^{\leq n/2}$ has no effect, that is, the revenue guarantee is achieved by simply applying the VCG mechanism. For the case of $m \gg n$, Theorem 8 does not state a constant approximation. However it still holds in this case that VCG with a supply limit is robustly α'-optimal where $1/\alpha'$ is a constant, albeit with a more involved proof (Yan 2012). Theorem 8 also applies without change to *multi-unit* matching markets, in which there are k_j copies of every item j, and a total of $m = \sum_j k_j$ units overall.

6.1. Subadditivity

The following lemma is used to prove Theorem 8 and may also be of independent interest. It states that in any market environment, including one with asymmetric bidders whose values are drawn independently but not identically, the optimal expected revenue achievable from bidder sets S, T separately is at least the optimal expected revenue achievable from their union. Let $\mathrm{OPT}(\cdot)$ map a bidder set to its optimal expected revenue. Then:

LEMMA 6 **(Subadditivity of Optimal Expected Revenue in Bidder Set)**. *For every auction environment with bidder subsets S and T,* $\mathrm{OPT}(S) + \mathrm{OPT}(T) \geq \mathrm{OPT}(S \cup T)$.

Proof. It is not hard to see that $\mathrm{OPT}(\cdot)$ is monotone, so without loss of generality we can assume that S and T are disjoint. Let M be the optimal mechanism for $S \cup T$. For every value profile $\mathbf{v}_T$ of the bidders in T, we define the mechanism $M_{\mathbf{v}_T}$, which gets bids from the bidders in S and simulates M by using $\mathbf{v}_T$ as the bids of bidders in T. By an averaging argument, there exists a vector $\mathbf{v}_T$ such that mechanism $M_{\mathbf{v}_T}$'s expected revenue is at least the part of the optimal expected revenue of mechanism M that is charged to the bidders in S. On the other hand, the expected revenue of $M_{\mathbf{v}_T}$ is bounded above by $\mathrm{OPT}(S)$. Similarly, the part of the optimal expected revenue that is charged to the bidders in T is bounded above by $\mathrm{OPT}(T)$. This completes the proof. $\square$

A corollary of the subadditivity lemma is that removing bidders from an i.i.d. environment until an α-fraction of the original bidders remains maintains an α-fraction of the optimal expected revenue. By symmetry:

COROLLARY 1. *For every auction environment with n i.i.d. bidders and for every integer c that divides n,* $\mathrm{OPT}(n/c) \geq \frac{1}{c}\mathrm{OPT}(n)$.

6.2. Proof by Reduction

We prove Theorem 8 by instantiating our general reduction.

Proof of Theorem 8. Assume first that $m \leq n/2$. We instantiate Reduction 3 as follows: to go from the original market to the restricted market, remove m bidders from the original market. By Corollary 1, this restriction on the original market maintains at least an $\frac{n-m}{n}$-fraction of the original optimal expected revenue. We can now apply the Bulow-Klemperer-type theorem for matching markets (Theorem 6) to the restricted market, which has $n-m$ bidders and m items. The expected revenue of VCG with m additional bidders is at least the optimal expected revenue in the restricted market. But this is equivalent to running VCG on the original market, completing the proof for $m \leq n/2$.

Now assume that $m \geq n/2$. We instantiate Reduction 3 as follows: to go from the original market to the restricted market, remove $n/2$ bidders from the original market. By Corollary 1, this restriction on the original market maintains at least a $\frac{1}{2}$-fraction of the original optimal expected revenue. We can now apply the Bulow-Klemperer-type theorem for matching markets with more items than bidders (Theorem 7) to the restricted market, which has $n/2$ bidders and $m \geq n/2$ items. The expected revenue of $\mathrm{VCG}^{\leq n/2}$ with $n/2$ additional bidders is at least an $\frac{n}{2m}$-fraction of the optimal expected revenue in the restricted market, and so an $\frac{n}{4m}$-fraction of the original optimal expected revenue. But this is equivalent to running $\mathrm{VCG}^{\leq n/2}$ on the original market, completing the proof. $\square$

7. Extensions

In this section we extend our results to markets with a matroid constraint on who can win simultaneously and to asymmetric markets.

7.1. Matroid Markets

Recall that a matroid environment is a single-parameter environment in which the set system $([n], \mathcal{I})$ of bidders and feasible allocations forms a *matroid*. A matroid satisfies three axioms (Oxley 1992): (A1) $\emptyset \in \mathcal{I}$, (A2) $\mathcal{I}$ is downward-closed, and (A3) if $S, T \in \mathcal{I}$ and $|S| < |T|$ then there is a bidder $t \in T \setminus S$ that can be added to S such that $S \cup \{t\} \in \mathcal{I}$. The *rank* ρ of a matroid is the size of its maximal independent sets or *bases*, and the *packing number* κ of a matroid is its maximum number of disjoint bases. In the job scheduling example presented in Section 1.2, if there are four jobs arriving at time 0 of which two must be finished by time 1 and two must be finished by time 2, then the rank is $\rho = 2$ and the packing number is $\kappa = 2$. We will use the fact that an *intersection* of a matroid $([n], \mathcal{I})$ with a *u-uniform* matroid is a new matroid $([n], \mathcal{I}')$, in which a set of bidders S belongs to $\mathcal{I}'$ if and only if $S \in \mathcal{I}$ and $|S| \leq u$.

Dughmi et al. (2012) show a Bulow-Klemperer-type result for matroid environments:

THEOREM 9 **(Bulow-Klemperer-Type Theorem for Matroid Markets)**. *For every matroid environment with i.i.d. regular bidders, the expected revenue of the Vickrey auction with an additional basis of bidders is at least the optimal expected revenue in the original market. In other words, Vickrey with an additional basis of bidders is robustly optimal.*

We use Theorem 9 with our general reduction to prove the following result, in which the approximation depends on the inherent amount of competition in the market measured not by the number of bidders n but rather by their packing number κ. For simplicity of presentation assume the rank ρ is even. Then:

THEOREM 10 **(Supply-Limiting Mechanism for Matroid Markets)**. *For every matroid environment with $n \geq 2$ i.i.d. regular bidders, rank ρ and packing number κ, let $\ell = \rho/2$ if $\kappa = 1$ and $\ell = \rho$ otherwise. Then the expected revenue of the supply-limiting mechanism* $\mathrm{Vic}^{\leq \ell}$ *is at least a* $\max\{\frac{1}{4}, \frac{\kappa-1}{\kappa}\}$*-fraction of the optimal expected revenue. In other words,* $\mathrm{Vic}^{\leq \ell}$ *is robustly* $\max\{\frac{1}{4}, \frac{\kappa-1}{\kappa}\}$*-optimal.*

Proof. We instantiate Reduction 3 as follows: If $\kappa = 1$, intersect the original matroid with a $\frac{\ell}{2}$-uniform matroid to get a new matroid $([n], \mathcal{I}')$ with rank $\rho' = \frac{\ell}{2}$ and packing number $\kappa' \geq 2$. Otherwise, if $\kappa \geq 2$, simply set the new matroid $([n], \mathcal{I}')$ to be $([n], \mathcal{I})$. To go from the original market to the restricted market, remove from the original market a basis of bidders of size ρ' according to the matroid $([n], \mathcal{I}')$, and set the matroid of the restricted market to be $([n], \mathcal{I}')$.

Analysis: If $\kappa = 1$, intersecting with the uniform matroid maintains at least a $\frac{1}{2}$-fraction of the original optimal expected revenue. Removing a basis of bidders maintains at least a $\frac{\kappa'-1}{\kappa'}$-fraction. We can now apply the Bulow-Klemperer-type theorem for matroids (Theorem 9) to the restricted market with $n - \rho'$ bidders and matroid $([n], \mathcal{I}')$. The expected revenue of Vickrey with an additional basis of bidders is at least the optimal expected revenue in the restricted market, and so a $\max\{\frac{1}{4}, \frac{\kappa-1}{\kappa}\}$-fraction of the original optimal expected revenue. But this is equivalent to running $\mathrm{Vic}^{\leq \ell}$ on the original market, completing the proof. $\square$

7.2. Asymmetric Bidders: Augmenting Demand

An *attribute-based* environment is a k-unit environment with n bidders, each of whom has a publicly-observable attribute $a = a(i)$ that determines a non-publicly-known value distribution F_a (Dhangwatnotai et al. 2010). Bidders' values in an attribute-based environment are thus independently but not identically distributed. Attributes enable the incorporation of prior information into our model regarding which bidders are alike, while still avoiding assumptions about the value distributions themselves. In fact, our results can be interpreted as an encouragement to invest in this particular kind of prior information, which entails grouping similar bidders together rather than learning distributions. Examples of attributes are bidding styles such as "bargain-hunter" or "aggressive" on `eBay.com`, or in sponsored search and online advertising, advertiser features such as location. Throughout we assume *non-singular* attribute-based environments, where no bidder's attribute is unique. I.e., for every attribute a, let n_a denote the number of bidders in the environment with attribute a; then $n_a > 0 \implies n_a \geq 2$.

In this section we prove a Bulow-Klemperer-type theorem for attribute-based environments.[9] Let $\text{Vic}^{\leq \ell_a}$ be the Vickrey mechanism with a *local* supply limit ℓ_a for every a, which limits the number of bidders with attribute a who can win simultaneously. The proof of the following theorem uses a *commensuration* argument of Hartline and Roughgarden (2009), and applies the FKG inequality (Alon and Spencer 2008) to solve dependency issues.

Theorem 11 **(Bulow-Klemperer-Type Theorem for Asymmetric Markets)**. *For every attribute-based environment with n_a regular bidders per attribute and k units, the expected revenue of the* $\text{Vic}^{\leq n_a}$ *auction with* $\min\{n_a, k\}$ *additional bidders per attribute is at least a $\frac{1}{2}$-fraction of the optimal expected revenue in the original market. In other words,* $\text{Vic}^{\leq n_a}$ *with* $\min\{n_a, k\}$ *additional bidders per attribute is robustly $\frac{1}{2}$-optimal.*

Proof. Let $W^{\text{OPT}} = W^{\text{OPT}}(\mathbf{v}), W^{\text{Vic}} = W^{\text{Vic}}(\mathbf{v})$ denote the winning bidders chosen by the optimal mechanism in the original environment and by $\text{Vic}^{\leq n_a}$ in the augmented environment, respectively, given a value profile $\mathbf{v}$ of both original and augmenting bidders. Hartline and Roughgarden (2009) show that to prove a 1/2-approximation it suffices to establish two commensuration conditions among the two mechanisms:

(C1) $\mathbb{E}_{\mathbf{v}}[\sum_{i \in W^{\text{Vic}} \setminus W^{\text{OPT}}} \phi_i] \geq 0$,

(C2) $\mathbb{E}_{\mathbf{v}}[\sum_{i \in W^{\text{Vic}} \setminus W^{\text{OPT}}} p_i(\mathbf{v})] \geq \mathbb{E}_{\mathbf{v}}[\sum_{i \in W^{\text{OPT}} \setminus W^{\text{Vic}}} \phi_i]$,

where ϕ_i is the virtual value of bidder i.

The proof of (C2) in (Hartline and Roughgarden 2009, Lemma 4.5) holds in our setting. In contrast, proving (C1) in our setting turns out to be technically challenging due to dependencies among the random bidder sets W^{OPT} and W^{Vic}. We use an auxiliary allocation procedure (Algorithm 6), and rely on the fact that in our setting, $\mathrm{Vic}^{\leq n_a}$ applies a simple greedy algorithm: it rejects all but the top n_a bidders per attribute, and allocates the units to the $\leq k$ highest remaining bidders.

For the remainder of the proof, fix an attribute a, and let B_a be the set of $n_a + \min\{n_a, k\}$ bidders with attribute a in the augmented environment. Fix the values of the original bidders in B_a as well as the values of all bidders with different attributes, and let $\mathbf{v}_a$ denote the (random) value profile of the augmenting bidders in B_a. Let $W_a^{\mathrm{OPT}} = B_a \cap W^{\mathrm{OPT}}$ denote the bidders in B_a who win in the optimal mechanism, and let $W_a^{\mathrm{Vic}} = W_a^{\mathrm{Vic}}(\mathbf{v}_a) = B_a \cap W^{\mathrm{Vic}}$ denote the bidders in B_a who win in $\mathrm{Vic}^{\leq n_a}$. We can now define our auxiliary procedure in Algorithm 6. Let $W_a^{\mathrm{Aux}} = W_a^{\mathrm{Aux}}(\mathbf{v}_a)$ denote the bidders in B_a who win in this procedure, and observe $W_a^{\mathrm{OPT}} \subseteq W_a^{\mathrm{Aux}} \subseteq B_a$.

Algorithm 6 Auxiliary Allocation Procedure

In the augmented environment, given W_a^{OPT}, allocate the k units such that the welfare is maximized subject to the constraint that all bidders in W_a^{OPT} win.

We now establish two claims, the first of which relates the auxiliary procedure to $\mathrm{Vic}^{\leq n_a}$ and the second of which relates it to the optimal mechanism.

Claim 3. *For every a and every value profile $\mathbf{v}_a$ of the augmenting bidders with attribute a, the bidders in $W_a^{\mathrm{Vic}} \setminus W_a^{\mathrm{Aux}}$ have non-negative virtual values.*

Proof of Claim 3. Fix $\mathbf{v}_a$ and consider the allocation of $\mathrm{Vic}^{\leq n_a}$ in comparison to that of the auxiliary procedure. $\mathrm{Vic}^{\leq n_a}$ is free to replace bidders in W_a^{OPT}. Since $\mathrm{Vic}^{\leq n_a}$ is greedy, each replacement from B_a will have a higher value than the replaced bidder in W_a^{OPT}, and therefore (using regularity) also a higher virtual value. The proof follows by noticing that all bidders in W_a^{OPT} have non-negative virtual values (Lemma 1). □

Claim 4. *For every a, in expectation over the value profile $\mathbf{v}_a$ of the augmenting bidders with attribute a, summing over the highest $h \leq \min\{n_a, k\}$ virtual values of the bidders in $W_a^{\mathrm{Aux}} \setminus W_a^{\mathrm{OPT}}$ results in a non-negative total virtual value.*

Proof of Claim 4. Given $\mathbf{v}_a$, denote by $\psi_{(1)}, \psi_{(2)}, \ldots$ (where $\psi_{(i)} = \psi_{(i)}(\mathbf{v}_a)$) the virtual values of the bidders in $B_a \setminus W_a^{\mathrm{OPT}}$, sorted in decreasing order of both values and virtual values. Let $\mathbb{1}_{(i)} = \mathbb{1}_{(i)}(\mathbf{v}_a)$ indicate whether the i-th bidder in $B_a \setminus W_a^{\mathrm{OPT}}$ wins in the auxiliary procedure, and let $q_{(i)}$ be the probability that $\mathbb{1}_{(i)} = 1$ over a random choice of $\mathbf{v}_a$. The expected sum of the highest $\min\{n_a, k\}$ virtual values of the bidders in $W_a^{\mathrm{Aux}} \setminus W_a^{\mathrm{OPT}}$ can be written as $\sum_{i \leq \min\{n_a,k\}} \mathbb{E}_{\mathbf{v}_a}[\psi_{(i)} \cdot \mathbb{1}_{(i)}]$. To complete the proof of Claim 4 it is sufficient to show that this expression is non-negative.

Notice that like $\mathrm{Vic}^{\le n_a}$, the auxiliary procedure is greedy in nature. Thus if the i-th bidder in $B_a \setminus W_a^{\mathrm{OPT}}$ wins in the auxiliary procedure, and the values of bidders in $B_a \setminus W_a^{\mathrm{OPT}}$ increase, then the i-th highest bidder will still win. Formally, for two value profiles $\mathbf{v}_a \le \mathbf{v}'_a$ we have $\mathbb{1}_{(i)}(\mathbf{v}_a) \le \mathbb{1}_{(i)}(\mathbf{v}'_a)$ for every i. This positive correlation between how high the values and virtual values are and between the probability of winning in the auxiliary procedure allows us to apply the FKG inequality (Alon and Spencer 2008) as follows:

$$\begin{aligned}\sum_{i\le \min\{n_a,k\}} \mathbb{E}_{\mathbf{v}_a}[\psi_{(i)} \cdot \mathbb{1}_{(i)}] &\ge \sum_{i\le \min\{n_a,k\}} \mathbb{E}_{\mathbf{v}_a}[\psi_{(i)}] \cdot \mathbb{E}_{\mathbf{v}_a}[\mathbb{1}_{(i)}] \\ &= \sum_{i\le \min\{n_a,k\}} \mathbb{E}_{\mathbf{v}_a}[\psi_{(i)}] \cdot q_{(i)} \\ &= \sum_{i\le \min\{n_a,k\}} \left(\mathbb{E}_{\mathbf{v}_a}\left[\sum_{i'=1}^{i} \psi_{(i')}\right] \cdot \big(q_{(i)} - q_{(i+1)}\big) \right),\end{aligned}$$

where the first inequality is by FKG, and where we set $q_{(i+1)}$ for $i = \min\{n_a, k\}$ to 0.

It is not hard to see that $q_{(i)}$ is decreasing in i. Therefore it suffices to prove that $\sum_{i'=1}^{i} \mathbb{E}_{\mathbf{v}_a}[\psi_{(i')}] \ge 0$ for every $i \le \min\{n_a, k\}$. This is the sum of the expected virtual values of the top i bidders in $B_a \setminus W_a^{\mathrm{OPT}}$. Observe that the sum of the expected virtual values of any i augmented bidders in $B_a \setminus W_a^{\mathrm{OPT}}$ equals 0, and there are at least $\min\{n_a, k\}$ such bidders. It follows that this sum for the top i bidders is nonnegative. □

We now use Claims 3 and 4 to complete the proof of (C1). Still holding attribute a fixed, rewrite $\mathbb{E}_{\mathbf{v}_a}[\sum_{i\in W_a^{\mathrm{Vic}}\setminus W_a^{\mathrm{OPT}}} \phi_i]$ as $\mathbb{E}_{\mathbf{v}_a}[\sum_{i\in W_a^{\mathrm{Vic}}\setminus W_a^{\mathrm{Aux}}} \phi_i] + \mathbb{E}_{\mathbf{v}_a}[\sum_{i\in W_a^{\mathrm{Vic}}\cap (W_a^{\mathrm{Aux}}\setminus W_a^{\mathrm{OPT}})} \phi_i]$. The left-hand side is non-negative by Claim 3. We consider two cases for the right-hand side, which by greediness of the auxiliary procedure and $\mathrm{Vic}^{\le n_a}$ are the only possible cases:

1. $(W_a^{\mathrm{Aux}} \setminus W_a^{\mathrm{OPT}}) \subseteq W_a^{\mathrm{Vic}}$: In this case $W_a^{\mathrm{Vic}} \cap (W_a^{\mathrm{Aux}} \setminus W_a^{\mathrm{OPT}}) = W_a^{\mathrm{Aux}} \setminus W_a^{\mathrm{OPT}}$, and so by Claim 4 the sum of virtual values is non-negative in expectation.
2. $W_a^{\mathrm{Vic}} \subseteq W_a^{\mathrm{Aux}}$: In this case, $W_a^{\mathrm{Vic}} \setminus W_a^{\mathrm{OPT}}$ is a subset of the highest $\min\{n_a, k\}$ bidders in $W_a^{\mathrm{Aux}} \setminus W_a^{\mathrm{OPT}}$, and so by Claim 4 the sum of virtual values over this subset is non-negative in expectation.

We have shown that $\mathbb{E}_{\mathbf{v}_a}[\sum_{i\in W_a^{\mathrm{Vic}}\setminus W_a^{\mathrm{OPT}}} \phi_i] \ge 0$. Taking expectation and summing over all attributes we get $\mathbb{E}_{\mathbf{v}}[\sum_{i\in W^{\mathrm{Vic}}\setminus W^{\mathrm{OPT}}} \phi_i] \ge 0$, completing the proof of (C1) and Theorem 11. □

7.3. Asymmetric Bidders: Limiting Supply

Consider an attribute-based environment as defined in Section 7.2. For simplicity of presentation assume that n_a, the number of bidders with attribute a, is even for every a. Recall that $\mathrm{Vic}^{\le n_a/2}$ is the Vickrey mechanism with a local supply limit $n_a/2$ for every a, meaning that no more than half the bidders with the same attribute can win simultaneously. We now show that $\mathrm{Vic}^{\leq n_a/2}$ is a good supply-limiting mechanism for attribute-based environments. Note that it is considerably simpler than Myerson's optimal mechanism for asymmetric markets, which requires computing different virtual value functions for different attributes.

Theorem 12 **(Supply-Limiting Mechanism for Asymmetric Markets)**. *For every attribute-based environment with n_a regular bidders per attribute, the expected revenue of the supply-limiting mechanism $\mathrm{Vic}^{\leq n_a/2}$ is at least a $\frac{1}{4}$-fraction of the optimal expected revenue. In other words, $\mathrm{Vic}^{\leq n_a/2}$ is robustly $\frac{1}{4}$-optimal.*

Proof. We instantiate Reduction 3 as follows: to go from the original market to the restricted market, remove $\min\{\frac{n_a}{2}, k\}$ bidders with attribute a from the original market. By submodularity (Dughmi et al. 2012), this restriction on the original market maintains at least a $\frac{1}{2}$-fraction of the original optimal expected revenue (since we removed at most half of the bidders). We can now apply the Bulow-Klemperer-type theorem for asymmetric markets (Theorem 11) to the restricted market, which has $\max\{\frac{n_a}{2}, n_a - k\}$ bidders per attribute and k units. The expected revenue of $\mathrm{Vic}^{\leq n_a/2}$ with $\min\{\frac{n_a}{2}, k\}$ additional bidders is at least a $\frac{1}{2}$-fraction of the optimal expected revenue in the restricted market. But this is equivalent to running $\mathrm{Vic}^{\leq n_a/2}$ on the original market, completing the proof. $\square$

8. Conclusion and Discussion

Robustness has long been recognized as an important design principle in optimization; here we apply it to optimal mechanism design. We study enhanced competition as a means for designing robust and simple auctions, whose revenue guarantees provably exceed or approximate those of the optimal auction, even when the latter is not well understood. Our main contributions are as follows:

- The problem of designing optimal auctions in matching markets is challenging even given full distributional information and regardless of robustness considerations. Yet the robust approach of prior-independence can help us get a better understanding of the optimal auction.
- Prior-independence is similar to the standard approach in computationally-hard optimization: let a polynomial-time algorithm approximate what can be achieved by an algorithm with unlimited running time. Here we let a prior-independent mechanism approximate the revenue that can be achieved by a mechanism with access to full distributional information.
- To achieve prior-independence we develop a framework of competition enhancement, which encompasses the two complementary approaches of increasing demand and limiting supply and

relates them via a general reduction. Within this framework we show that Vickrey with m more bidders, or with $n < m$ more bidders and a limit of n on the total number of allocations, guarantees good revenue in both single- and multi-parameter markets and for many value distributions. Even without adding bidders, limiting the number of allocations to half the size of the market and running Vickrey has good revenue guarantees in a wide range of settings. The mechanisms we design are computationally tractable and have constant approximation factors that often improve as the inherent competition in the market grows. They also take advantage of the combinatorial structure in the market — e.g., the properties of matching play a key role in our analysis.

- A revenue-driven seller may deliberate between acquiring information to carefully set prices for existing buyers, drawing more potential buyers, and driving prices up by making what he's selling harder to get. Our results quantify the trade-offs between these strategies by answering questions like the following: How many more buyers are needed to replace information acquisition? What is a good balance between offering enough supply and extracting more revenue for every good sold? And how does the approach of withholding supply measure up to the other two strategies?
- Our framework is flexible and extends to markets with allocation constraints and to markets with asymmetries among the buyers. This demonstrates that robustness and simplicity are achievable even in complicated settings, and in fact are advisable when approaching such settings.

There are two main future research directions arising from our work. First, it is an interesting and challenging direction to study the best possible revenue guarantee subject to robustness. One well-defined quesiton is: given x additional bidders, what is the optimal prior-independent mechanism? Second, we believe our techniques will be useful for even more general multi-parameter markets — for instance, markets with gross substitutes preferences have combinatorial structure that can possibly be utilized using our methods; or markets with positive correlation (*affiliation*) among the bidders and thus inherently more competition that can be utilized.

Endnotes

1. By the revelation principle, this requirement is without loss of generality for a seller seeking a dominant strategy implementation.
2. The Myerson characterization also extends to *asymmetric* environments, where there are multiple distributions $\{F_1^i\}_{i\in[n]}$ and bidder i's value for the good (good 1) is drawn from his distribution F_1^i. However, in this case, the informational burden on the seller is even heavier, as it needs to have full knowledge of the distributions of all buyers, and even getting only approximately close to optimal revenue requires many samples from every distribution (Cole and Roughgarden 2014).

3. Note that while we enhance competition among buyers in the market, we do not turn it into a "competitive market" in which the buyers become price-takers. The focus of our work is not on large markets and asymptotic results, rather we aim for our results to hold in markets of any size.

4. Multi-unit auctions are not to be confused with multi-*item* auctions in which there are multiple heterogeneous goods.

5. If n is odd, one can first remove a bidder from the environment, losing at most a $1/n$-fraction of the optimal expected revenue.

6. Recall that a function f from sets of bidders to $\mathbb{R}$ is *submodular* if for every two sets $S \subset T$ and every bidder $i \notin T$ it holds that $f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T)$. Equivalently, the marginal contribution of a bidder to the value of f is decreasing.

7. It is not hard to adapt the proof to the case in which there are multiple maximum weighted matchings, to show that one possible allocation of VCG run on the augmented matching environment leaves unallocated a set of bidders equal to U with at most one bidder removed.

8. Similarly, for every $\eta \in [n, m]$ there is a Bulow-Klemperer-type theorem with η additional bidders. This does not improve the guarantees in Section 6.

9. It is not hard to show that the Bulow-Klemperer-type theorem for asymmetric matroid environments of (Hartline and Roughgarden 2009, Theorem 4.4) applies to attribute-based environments. This theorem requires augmenting the demand with an additional "duplicate" bidder for every original bidder, and adding the constraint that at most one of each such pair wins simultaneously. Our version in Theorem 11 utilizes the fact that many of the bidders in an attribute-based environment are symmetric — namely all those with the same attribute — in order to avoid the pair constraints, and when k is relatively small requires less bidders to be added to the environment.

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