Ironing in the Dark

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This paper presents the first polynomial-time algorithm for position and matroid auction environments that learns, from samples from an unknown distribution, an auction with expected revenue arbitrarily close to the maximum possible. In contrast to most previous work, our results do not assume that the unknown distribution is regular, and require only that the distribution does not have an extremely heavy tail (a necessary assumption for any non-trivial results). Our performance guarantee is with respect to the strongest possible benchmark, the Myerson-optimal auction. Learning a near-optimal auction for an irregular distribution is technically challenging because it requires learning the appropriate "ironed intervals," a delicate global property of the distribution.

1. INTRODUCTION

The traditional economic approach to revenue-maximizing auction design, exemplified by Myerson [1981], posits a known prior distribution over what bidders are willing to pay, and then solves for the auction that maximizes the seller's expected revenue with respect to this distribution. Recently, there has been an explosion of work in computer science that strives to make the classical theory more "data-driven," replacing the assumption of a known prior distribution with that of access to relevant data, in the form of samples from an unknown distribution. In this paper, we study the problem of learning a near-optimal auction from samples, adopting the formalism of Cole and Roughgarden [2014]. The idea of the model, inspired by PAC-learning [Valiant 1984], is to parameterize through samples the "amount of knowledge" that the seller has about bidders' valuation distributions.

We consider single-parameter settings, where each of n bidders has a private valuation (i.e., willingness to pay) for "winning" and valuation 0 for "losing." Feasible outcomes correspond to subsets of bidders that can simultaneously win; the feasible subsets are known in advance.[1] We assume that bidders' valuations are drawn i.i.d. from a distribution F that is unknown to the seller. However, we assume that the seller has access to m i.i.d. samples from the distribution F — for example, bids that were observed in comparable auctions in the past. The goal is to design a polynomial-time algorithm $A(v_1, \ldots, v_m)$, mapping samples $v_i \sim F$ to truthful auctions, such that, for every distribution F, the expected revenue is at least $1 - \epsilon$ times the optimal ex-

[1]For example, in auction with k copies of an item, where each bidder only wants one copy, feasible outcomes correspond to subsets of at most k bidders.

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pected revenue.[2] The sample complexity of achieving a given approximation factor $1-\epsilon$ is then the minimum number of samples m such that there exists a learning algorithm A with the desired approximation. This model serves as a potential "sweet spot" between worst-case and average-case analysis, inheriting much of the robustness of the worst-case model (since we demand guarantees for every underlying distribution F) while allowing very good approximation guarantees.

1.1. Our Results

We give polynomial-time algorithms for learning a $(1-\epsilon)$-approximate auction from samples, for arbitrary matroid (or position) auction environments and valuation distributions that satisfy minimal assumptions.

For example, for valuation distributions with support in $[0, H]$, we provide a polynomial-time algorithm that, given a matroid environment with n bidders and m i.i.d. samples from an arbitrary distribution F, with probability $1-\delta$, approximates the maximum-possible expected revenue up to an additive loss of at most $3n\sqrt{\frac{\ln 2\delta^{-1}}{2m}} \cdot H$. Thus for every $\epsilon > 0$, the additive loss is at most ϵ (with probability at least $1-\delta$) provided $m = \Omega(n^2 H^2 \epsilon^{-2} \log \delta^{-1})$. Whenever the optimal expected revenue is bounded away from 0, this result immediately implies a comparable sample complexity bound for learning a $(1-\epsilon)$-(multiplicative) approximate auction. Our main result can also be used to give a no-regret guarantee in a stochastic learning setting (Section 5).

A lower bound of Cesa-Bianchi et al. [2013] implies that, already for simpler settings, the quadratic dependence of our sample complexity bound on $\frac{1}{\epsilon}$ is optimal. A lower bound of Huang et al. [2015] implies that, already with a single bidder, the sample complexity must depend polynomially on H. Whether or not the sample complexity needs to depend on n is an interesting open question.

Our technical approach is based on a "switching trick" (Proposition 2.1) and we believe it will lead to further applications. A key idea is to express the difference in expected revenue between an optimal and a learned auction in a matroid or position environment purely in terms of a difference in area under the true and estimated revenue curves. This "global" analysis avoids having to compare explicitly the true and estimated virtual valuations or the optimal and learned allocation rules. With this approach, there is clear motivation behind each of the steps of the learning algorithm, and the error analysis, while non-trivial, remains tractable in settings more general than those studied in most previous works.

The assumption of bounded valuations is not necessary for our results to hold. More generally, the only assumption required is that the optimal auction does not obtain a constant fraction of its expected revenue from valuation profiles with at least one extremely high-valued bidder (with valuation bigger than a parameter H). This assumption is trivially satisfied by any distribution with support in $[0, H]$, and is also satisfied (for a suitable H) by many (irregular) distributions with infinite support (this is demonstrated in the full version of the paper). Some assumption of this type is necessary to preclude pathological distributions that are impossible for any algorithm to learn.[3]

[2]By a truthful auction, we mean one in which truthful bidding is a dominant strategy for every bidder. The restriction to dominant strategies is natural given our assumption of an unknown distribution. Given this, the restriction to truthful auctions is without loss of generality (by the "Revelation Principle," see e.g. Nisan [2007]). Also, for the single-parameter problems that we study, there is always an optimal auction in which all bidders have dominant strategies [Myerson 1981].

[3]To appreciate this issue, consider a single-bidder problem and all distributions that take on a value M^2 with probability $\frac{1}{M}$ and 0 with probability $1-\frac{1}{M}$. The optimal auction for such a distribution earns expected revenue at least M. It is not difficult to prove that, for every m, there is no way to use m samples to achieve

1.2. Why Irregular Distributions Are Interesting

A majority of the literature on approximation guarantees for revenue maximization (via learning algorithms or otherwise) restricts attention to "regular" valuation distributions or subclasses thereof; see related work below for examples and exceptions. Formally, a distribution F with density f is *regular* if

$$\varphi(v) = v - \frac{1 - F(v)}{f(v)} \tag{1}$$

is a nondecreasing function of v. φ is also called the *virtual valuation* function. Intuitively, regularity is a logconcavity-type assumption that provides control over the tail of the distribution. While many important distributions are regular, plenty of natural distributions are not. For example, Sivan and Syrgkanis [2013] point out that mixtures of distributions (even of uniform distributions) tend to be irregular, and yet are obviously prevalent in the real world.

1.3. Why Irregular Distributions Are Hard

To understand why irregular distributions are so much more technically challenging than regular distributions, we need to review some classical optimal auction theory. We can illustrate the important points already in single-item auctions. Myerson [1981] proved that, for every regular distribution F, the optimal auction is simply a second-price auction supplemented with a reserve price of $\varphi^{-1}(0)$, where φ denotes the virtual valuation function in (1). (The winner, if any, pays the higher of the reserve price and the second-highest bid.) Thus, *learning the optimal auction reduces to learning the optimal reserve price*, a single statistic of the unknown distribution. And indeed, for an unknown regular distribution F, there is a polynomial-time learning algorithm that needs only poly($\frac{1}{\epsilon}$) samples to compute a $(1-\epsilon)$-approximate auction [Dhangwatnotai et al. 2010; Huang et al. 2015].

The technical challenge of irregular distributions is the need to *iron*. When the virtual valuation function φ of the distribution F is not nondecreasing, Myerson [1981] gave a recipe for transforming φ into a nondecreasing "ironed" virtual valuation function $\overline{\varphi}$ such that the optimal single-item auction awards the item to the bidder with the highest positive ironed virtual valuation (if any), breaking ties randomly (or lexicographically). Intuitively, this ironing procedure identifies intervals of non-monotonicity in φ and changes the value of the function to be constant on each of these intervals. (See also below and the exposition by Hartline [2014].)

The point is that the appropriate ironing intervals of a distribution are a *global* property of the distribution and its (unironed) virtual valuation function. Estimating the virtual valuation function at a single point — all that is needed in the regular case — would appear much easier than estimating the right intervals to iron in the irregular case.

We present two examples to drive this point home. The first, which is standard, shows that foregoing all ironing can lead to a constant-factor loss in expected revenue, even in single-item auctions. (Reserve prices are still worse in matroid environments, see Devanur et al. [2014].) The second example shows that tiny mistakes in the choice of ironing intervals can lead to a large loss of expected revenue.

Example 1.1 (*Ironing Is Necessary for Near-Optimal Revenue*). The distribution is as follows: with probability $1/H$ the value is H (for a large H) and it is 1 otherwise. The optimal auction irons the interval $[1, H)$ for expected revenue of $2 - \frac{1}{n}$ [Hartline 2014],

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near-optimal revenue for every such distribution — for sufficiently large M, all m samples are 0 w.h.p. and the algorithm has to resort to an uneducated guess for M.

which approaches 2 with many bidders n. Auctions that do not implicitly or explicitly iron obtain expected revenue only 1.

Example 1.2 (*Small Mistakes Matter*). Let F be 5 with probability $1/10$ and 1 otherwise, and consider a single-item auction with 10 bidders. The optimal auction irons the interval $[1, 5)$ and has no reserve price. If there are at least two bidders with value 5 one of them will get the item at price 5; if all bidders have value 1, one of them will receive it at price 1. If there is exactly one bidder with value 5, then her price is $\frac{1}{10} \cdot 1 + \frac{9}{10} \cdot 5 = \frac{46}{10}$.

Now consider an algorithm that slightly *overestimated* the end of the ironing interval to be $[1, 5 + \epsilon)$ with $\epsilon > 0$. (Imagine F actually has small but non-zero density above 5, so that this mistake could conceivably occur.) Now all bids always fall in the ironing interval and therefore the item is always awarded to one of the players at price 1. Not only do we lose revenue when there is exactly one high bidder, but additionally we lose revenue for auctions with at least two bidders with value 5. This auction has even worse revenue than a Vickrey (second-price) auction, so the attempt to iron did more harm than good.

Now consider the same setting as Example 1.2, with the exception that we slightly underestimate the ironing interval as $[1, 5 - \epsilon)$, instead of overestimating. We still lose some revenue compared to the optimal ironing interval —namely when there is one high bidder, she pays $\frac{46}{50} - \frac{9}{10}\epsilon$ instead of $\frac{46}{50}$— but the revenue is much closer to optimal than when the end point of the ironing interval was too high. This phenomenon, that underestimation is better than overestimation, is true more generally. Our learning algorithm deliberately reports ironing intervals (and a reserve price) that are slightly lower than the data would suggest, to guarantee that with high probability the start and end points of ironing intervals do not exceed the optimal such points.

1.4. Related Work

Elkind [2007] gives a polynomial-time learning algorithm for the restricted case of single-item auctions with discrete distributions with known finite supports but with unknown probabilities. Learning is done using an oracle that compares the expected revenue of pairs of auctions, and $O(n^2K^2)$ oracle calls suffice to determine the optimal auction (where n is the number of bidders and K is the support size of the distributions). Elkind [2007] notes that such oracle calls can be implemented approximately by sampling (with high probability), but no specific sample complexity bounds are stated.

Cole and Roughgarden [2014] also give a polynomial-time algorithm for learning a $(1 - \epsilon)$-approximate auction for single-item auctions with non-identical bidders, under incomparable assumptions to Elkind [2007]: valuation distributions that can be unbounded but must be regular. It is necessary and sufficient to have $m = \text{poly}(n, \frac{1}{\epsilon})$ samples, however in the analysis in Cole and Roughgarden [2014] the exponent in the upper bound is large (currently, 10). These sample complexity results were recently generalized and improved dramatically by Devanur et al. [2016], although all of their results still require the valuation distributions to be regular.

The papers of Cesa-Bianchi et al. [2015] and Medina and Mohri [2014] give algorithms for learning the optimal reserve-price-based single-item auction. Recall from Example 1.1 that, with irregular distributions, the expected revenue of the best reserve-price-based auction might be only half that of an optimal auction.

Dughmi et al. [2014] proved negative results (exponential sample complexity) for learning near-optimal mechanisms in multi-parameter settings that are much more complex than the single-parameter settings studied here. The paper also contains positive results for restricted classes of mechanisms.

Huang et al. [2015] give optimal sample complexity bounds for the special case of a single bidder under several different distributional assumptions, including for the case of bounded irregular distributions where they need $O(H \cdot \epsilon^{-2} \cdot \log(H\epsilon^{-1}))$ samples.

Morgenstern and Roughgarden [2015] recently gave general sample complexity upper bounds which are similar to ours and cover all single-parameter settings (matroid and otherwise), although their (brute-force) learning algorithms are not computationally efficient.

Our learning algorithm is in the spirit of the Random Sampling Empirical Myerson mechanism [Devanur et al. 2014] and its precursors, though different in a number of details. Previous work used the approach to construct a revenue curve from bidders in an auction and prove constant-factor approximations in prior-free settings. The present work seeks $(1-\epsilon)$-approximations in settings with an unknown distribution.

For previously studied models about revenue-maximization with an unknown distribution, which differ in various respects from the model of Cole and Roughgarden [2014], see Babaioff et al. [2011], Cesa-Bianchi et al. [2013], and Kleinberg and Leighton [2003]. For other ways to parameterize partial knowledge about valuations, see e.g. Azar et al. [2013] and Chiesa et al. [2012]. For other ways to parameterize a distribution by its "degree of irregularity" see Hartline [2014], Huang et al. [2015], and Sivan and Syrgkanis [2013]. For other uses of samples in auction design that differ from ours, see Fu et al. [2014], who use samples to extend the Crémer-McLean theorem [Crémer and McLean 1985] to partially known valuation distributions, and Chawla et al. [2014], which is discussed further below. For asymptotic optimality results in various symmetric settings (single-item auctions, digital goods), which identify conditions under which the expected revenue of some auction of interest (e.g., second-price) approaches the optimal with an increasing number of i.i.d. bidders, see Neeman [2003], Segal [2003], Baliga and Vohra [2003], and Goldberg et al. [2006]. For applications of learning theory concepts to prior-free auction design in unlimited-supply settings, see Balcan et al. [2008].

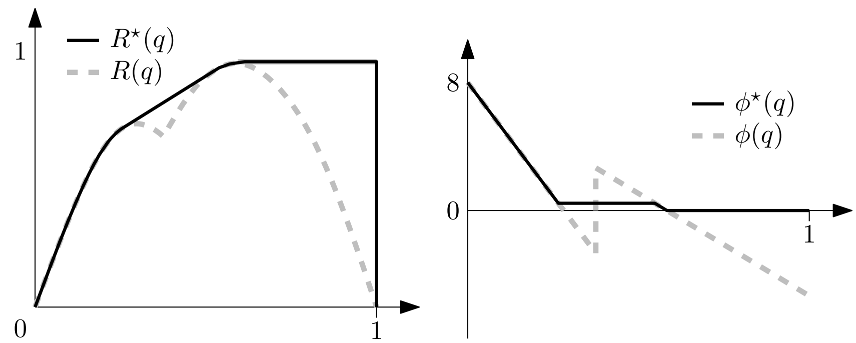
Finally, the technical issue of ironing from samples comes up also in Ha and Hartline [2013] and Chawla et al. [2014], in models incomparable to the one studied here. The setting of Ha and Hartline [2013] is constant-factor approximation guarantees for prior-free revenue maximization, where the goal is input-by-input rather than distribution-by-distribution guarantees. Chawla et al. [2014] study non-truthful auctions, where bidders' true valuations need to be inferred from equilibrium bids, and aim to learn the optimal "rank-based auction," which can have expected revenue a constant factor less than that of an optimal auction. Our goal of obtaining a $(1-\epsilon)$-approximation of the maximum revenue achieved by any auction is impossible in both of these settings.

Summarizing, this paper gives the first polynomial-time algorithm for position and matroid environments that learns, from samples from an unknown irregular valuation distribution, an auction with expected revenue arbitrarily close to the maximum possible.

2. PRELIMINARIES

2.1. The Empirical Cumulative Distribution Function and the DKW Inequality

Let $X = \{X_i\}_{i=1}^m$ be a set of m samples, and let $X^{(i)}$ be the i^{th} order statistic. We use the standard notion of the empirical cumulative distribution function (empirical CDF): $\widehat{F_m}(v) = \frac{1}{m} \cdot |\{X_i : X_i \leq v\}|$. The empirical CDF is an estimator for the quantile of a given value. The Dvoretzky-Kiefer-Wolfowitz (DKW) inequality [Dvoretzky et al. 1956; Massart 1990] states that the difference between the empirical CDF and the actual CDF decreases quickly in the number of samples. Let $\epsilon_{m,\delta} = \sqrt{\frac{\ln 2\delta^{-1}}{2m}}$, then



(a) Revenue curve $R(q)$ and the optimal induced revenue curve $R^\star(q)$.

(b) Virtual value function $\varphi(q)$ and the induced virtual value function $\varphi^\star(q)$.

Fig. 1: The dashed gray line is the revenue curve $R(q)$ and its derivative, the virtual value function $\varphi(q)$, for a irregular bimodal distribution. In black we have the curves φ^{opt} and R^{opt} corresponding to the optimal ironing interval, and optimal reserve price for this distribution. This example taken from [Hartline 2014, Chapter 3].

$\Pr\left[\sup_{v\in\mathbb{R}}\left|F(v)-\widehat{F_m}(v)\right|\leq\epsilon_{m,\delta}\right]\geq 1-\delta$. So the largest error in the empirical CDF shrinks in $O(m^{-1/2})$. For our purposes we will not need the CDF F, but rather its inverse F^{-1}. Define $\widehat{F_m^{-1}}(x)$ as $X^{(\max(1,\lceil x\cdot m\rceil))}$ for $x\in[0,1]$. (For convenience, define $\widehat{F_m^{-1}}(x)$ as 0 if $x<0$ and H if $x>1$.) By definition, for all $v\in[0,H]$:

$$\widehat{F_m^{-1}}\left(\widehat{F_m}(v)\right) \quad\leq\quad v \quad\leq\quad \widehat{F_m^{-1}}\left(\widehat{F_m}(v)+\frac{1}{m}\right). \tag{2}$$

In the remainder of this paper, we will use $\widehat{F}$, $\widehat{F^{-1}}$, and ϵ without explicitly referring to the number of samples m and confidence parameter δ.

2.2. Optimal Auctions using the Revenue Curve

Revenue Curve and Quantile Space. For a value v and a distribution F, we define the corresponding *quantile* as $1-F(v)$.[4] The quantile of v corresponds to the sale probability for a single bidder with valuation drawn from F who faces a posted price v. Note that higher values v correspond to lower quantiles. The *revenue curve* R of a valuation distribution F is the curve, in quantile space, defined by $R(q)=q\cdot F^{-1}(1-q)$ (Figure 1a). Myerson [1981] showed that the ex ante expected revenue for a bidder in a truthful auction is $\mathbb{E}_{q\sim U[0,1]}[-R(q)\cdot y'(q)]$, where y is the interim allocation function (defined formally below) for that auction. The derivative of R is the virtual value function $\varphi(.)$ — recall (1) — in quantile space, see Figure 1b. For regular distributions the revenue curve is concave and the virtual value function non-increasing, but for irregular distributions this is not the case.

Myerson [1981] showed that for any non-concave revenue curve R, one can create an allocation rule that will yield the same revenue as R's convex hull $\mathcal{CH}(R)$. This procedure is called *ironing*, and for each interval where R differs from $\mathcal{CH}(R)$, we define the *ironed virtual value* to be slope of the convex hull over this interval. This means the virtual values are equal in this interval, and hence any two bids in that range are

[4]This is for consistency with recent literature; "reversed quantile" might be a more accurate term.

interchangeable, and so an iron-based allocation function is constant on this interval.[5] The resulting *ironed* revenue curve $R^\star$ will be concave and the corresponding ironed virtual value function $\varphi^\star$ will be non-decreasing. It is also useful to think of a reserve price r (with corresponding quantile q_r) in a similar way, as effectively changing the virtual valuation function so that $\varphi^\star(q) = 0$ whenever $q \geq q_r$ (Figure 1b), with the corresponding revenue curve $R^\star$ constant in that region (Figure 1a).[6]

More generally, given any set of disjoint ironing intervals $\mathcal{I}$ and reserve price r, both in value space, we can imagine the effect on the revenue curve as follows. (For now this is a thought experiment; Proposition 2.1 connects these revenue curve modifications back to allocation rule modifications.) Let R be the revenue curve without ironing or a reserve price, and define $R^{(\mathcal{I},r)}$ as the revenue curve *induced* by a set $\mathcal{I}$ of ironing intervals and reserve price r. This curve is defined by

$$R^{(\mathcal{I},r)}(q) = \begin{cases} R(F(r)) & \text{if } q > F(r) \\ R(q_a) + \frac{q-q_a}{q_b-q_a}\left(R(q_b) - R(q_a)\right) & \text{if } q \in (q_a, q_b] \\ & \quad \text{with } [F^{-1}(1-q_b), F^{-1}(1-q_a)) \in \mathcal{I} \\ R(q) & \text{otherwise.} \end{cases} \tag{3}$$

Given $\mathcal{I}$ and r as above, we define the auction $A^{(\mathcal{I},r)}$ as follows: given a bid profile: (i) reject every bid with $b_i < r$; (ii) for each ironing interval $[a, b) \in \mathcal{I}$, treat all bids $\{b_j : a \leq b_j < b\}$ as identical (equal to some common number between a and b); (iii) among the remaining bidders, maximize the sum of the ironed bids of the winners; (iv) charge payments so that losers always pay 0 and so that truthful bidding is a dominant strategy for every player. This auction is well defined (i.e., independent of the choice of the common numbers in (ii)) in settings where the computation in (iii) depends only on the ordering of the ironed bids, and not on their numerical values. In this case, the payments in (iv) are uniquely defined (by standard mechanism design results). This is the case in every matroid environment[7] and also in position auctions.[8] In such a setting, we use $\mathcal{A}$ to denote the set of all auctions of the form $A^{(\mathcal{I},r)}$. We restrict attention to such settings in the remainder of the paper.

The Switching Trick. Given a distribution F, we explained two ways to use ironing intervals $\mathcal{I}$ and a reserve price r: (i) to define a modified revenue curve $R^{(\mathcal{I},r)}$ (and hence virtual valuations); or (ii) to define an auction $A^{(\mathcal{I},r)}$. The "switching trick" formalizes the connection between them: the expected maximum virtual welfare with the modified virtual valuations (corresponding to the derivative of $R^{(\mathcal{I},r)}$) equals the expected virtual welfare of the modified auction $A^{(\mathcal{I},r)}$ with the original virtual valuations.

More formally, let $x_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ be the ex-post allocation function of the welfare maximizing truthful auction that takes the bids $\mathbf{b}$ of all players and results in the allocation to bidder i. The *interim* allocation function $y_i : [0, 1] \to \mathbb{R}_+$ is the expected

[5]It takes some effort to show that keeping the allocation probability constant on an interval has exactly the effect we described here [Myerson 1981].

[6]Most of the existing literature would not consider the effect of the reserve price on the revenue curve, in which case the black and dashed lines would coincide after the second peak. However, by including its effect as we did, we'll be able to apply the Switching Trick described below.

[7]In a matroid environment, the set $\mathcal{F}$ of feasible outcomes satisfies: (i) (downward-closed) $T \in \mathcal{F}$ and $S \subseteq T$ implies $S \in F$; and (ii) (exchange property) whenever $S, T \in \mathcal{I}$ with $|T| < |S|$, there is some $i \in S \setminus T$ such that $T \cup \{i\} \in \mathcal{I}$.

[8]In a position auction, n bidders vie for k "slots," with at most one bidder assigned to each slot and at most one slot assigned to each bidder. Being assigned slot j corresponds to an allocation amount α_j, which historically corresponds to a "click-through rate." See Edelman et al. [2007] and Varian [2007] for details.

allocation to bidder i when her quantile is q, where the expectation is over the quantiles of the other bidders: $y_i(q_i) = \mathbb{E}_{\mathbf{q}_{-i}\sim U[0,1]^{n-1}}[x(F^{-1}(1-q_i), F^{-1}(\mathbf{1}-\mathbf{q}_{-i}))]$ where $F^{-1}(\mathbf{1}-\mathbf{q}_{-i})$ is $\mathbf{b}_{-i}$ for which each $b_j = F(1-q_j)$. For example, in the standard Vickrey (single-item) auction with n bidders, every bidder i has the interim allocation function $y_i(q) = (1-q)^{n-1}$.[9]

For every auction of the form $A^{(\mathcal{I},r)}$, the interim allocation function $y_i^{(\mathcal{I},r)}$ of a bidder i can be expressed in terms of the interim allocation function y_i without ironing and reserve price (see also Figure 1b):

$$y_i^{(\mathcal{I},r)}(q) = \begin{cases} 0 & \text{if } q > F(r) \\ \frac{1}{q_b-q_a}\int_{q_a}^{q_b} y(q)dq & \text{if } q \in [q_a, q_b) \text{ with } [F^{-1}(1-q_b), F^{-1}(1-q_a)) \in \mathcal{I} \\ y_i(q) & \text{otherwise.} \end{cases} \quad (4)$$

PROPOSITION 2.1 (SWITCHING TRICK). *Consider a matroid or position auction setting, as above. For every valuation distribution F, every reserve price r, every set $\mathcal{I}$ of disjoint ironing intervals, and every bidder i,*

$$\mathbb{E}_{q\sim U[0,1]}[R(q)\cdot (y_i^{(\mathcal{I},r)})'(q)] = \mathbb{E}_{q\sim U[0,1]}[R^{(\mathcal{I},r)}(q)\cdot y_i'(q)].$$

PROOF. Fix F, $\mathcal{I}$, r, and y. Let $y^{(\mathcal{I},r)}$ be the interim allocation rule from running auction $A^{(\mathcal{I},r)}$. Let R be the revenue curve of F and let $R^{(\mathcal{I},r)}$ denote the revenue curve induced by $\mathcal{I}$ and r.

— Define a distribution $F^{(\mathcal{I},r)}$ (which is not equal to F unless $\mathcal{I} = \emptyset$ and $r = 0$) that has the property that its revenue curve $q\cdot F^{(\mathcal{I},r)}(1-q)$ is $R^{(\mathcal{I},r)}$. To see that this is well-defined, observe the following. Any line ℓ through the origin only intersects $R^{(\mathcal{I},r)}$ once (if there are point masses in F then a line through the origin intersects R in a single interval). This means that we can use $R^{(\mathcal{I},r)}$ to construct $F^{(\mathcal{I},r)}$: $F^{(\mathcal{I},r)}(v)$ is the q for which $q\cdot v$ intersects with $R^{(\mathcal{I},r)}(q)$ (if there are any point masses then there will be a range of q for which this is the case; in that case take the largest such q). Alternatively, see Hartline and Roughgarden [2008] for an explicit formula for $F^{(\mathcal{I},r)}$.
— If we run the same auction $A^{(\mathcal{I},r)}$ on bidders with values drawn from $F^{(\mathcal{I},r)}$, the expected revenue is identical to the auction with bidder values drawn from F:

$$\mathbb{E}_{q\sim U[0,1]}[R(q)\cdot \left(y^{(\mathcal{I},r)}\right)'(q)] = \mathbb{E}_{q\sim U[0,1]}[R^{(\mathcal{I},r)}(q)\cdot \left(y^{(\mathcal{I},r)}\right)'(q)].$$

This can easily be seen by filling in the definitions from (3) and (4).
— If the bidders have distribution $F^{(\mathcal{I},r)}$, then we might as well not iron or have a reserve price at all; so

$$\mathbb{E}_{q\sim U[0,1]}[R^{(\mathcal{I},r)}(q)\cdot \left(y^{(\mathcal{I},r)}\right)'(q)] = \mathbb{E}_{q\sim U[0,1]}[R^{(\mathcal{I},r)}(q)\cdot y'(q)].$$

This is also easily seen by filling in the definitions.

□

2.3. Notation

In the remainder of this paper, our analysis will rely on bounding the difference in revenue of an auction with respect to the optimal auction in terms of their revenue curves. We will use the following conventions, see Table I. The unaltered revenue curve

[9] In general matroid settings, different bidders can have different interim allocation functions (even though valuations are i.i.d.).

Revenue Curve	Description
R	$q \cdot F^{-1}(1-q)$
$\widehat{R}_{\min}$	$q \cdot \widehat{F^{-1}}(1-q-\epsilon)$
$\widehat{R}_{\max}$	$q \cdot \widehat{F^{-1}}(1-q+\epsilon+\frac{1}{m})$

Table I: Overview of notation for revenue curves.

for distribution F is denoted by $R(q) = q \cdot F^{-1}(1-q)$. To denote when we use an estimator for a revenue, i.e. a revenue curve that is constructed based on samples, we use a hat: $\widehat{R}(q) = q \cdot \widehat{F^{-1}}(1-q)$. Based on the available samples we construct high-probability upper and lower bounds for R, that are thus denoted as $\widehat{R}_{\max}(q) = q \cdot \widehat{F^{-1}}(1-q+\epsilon+\frac{1}{m})$ and $\widehat{R}_{\min}(q) = q \cdot \widehat{F^{-1}}(1-q-\epsilon)$.

We use a superscript to denote when a revenue curve is ironed and has a reserve price. For a general set of ironing intervals $\mathcal{I}$ and reserve price r, $R^{(\mathcal{I},r)}$ is the revenue curve induced by it, see (3). The superscript $\star$ denotes that the revenue curve is optimally ironed and reserved, i.e. $R^{\star}$ is the revenue curve of Myerson's auction using F, and $\widehat{R}^{\star}_{\max}$ is the revenue curve corresponding to the convex hull of $\widehat{R}_{\max}$ that additionally stays constant after the highest point. Finally, we use R^{alg} and R^{opt} to denote an algorithm ALG's revenue curve and the optimal revenue curve for F respectively (thus $R^{opt} = R^{\star}$, but when appropriate we use R^{opt} to emphasize its relation to R^{alg}).

For the ironing intervals $\mathcal{I}$ (and reserve price r) we use $\mathcal{I}_q$ (resp. r_q) when it is important that the ironing intervals are defined in *quantile space*. Finally, $\mathcal{I}_{opt}$, $\mathcal{I}_{alg}$, and $\mathcal{I}_{max}$ (and similarly for reserve price r) refer to the ironing intervals of the optimal auction, algorithm ALG and the optimal ironing intervals for $\widehat{R}_{\max}$ respectively.

3. ADDITIVE LOSS IN REVENUE FOR SINGLE-ITEM AUCTIONS

For ease of exposition, most of the technical sections focus on an unknown distribution with support in $[0, H]$. The full version of this paper explains how our results extend to all distributions for which the optimal auction does not obtain a constant fraction of its expected revenue from valuation profiles with at least one extremely high-valued bidder.

This section describes an algorithm that takes a set X of m samples, and a confidence parameter δ as input, and outputs a set $\mathcal{I}$ of ironing intervals and a reserve price r, both in value space. This section focuses on the case where $\mathcal{I}$ and r are used in a single-item auction $A^{(\mathcal{I},r)}_{(1)} \in \mathcal{A}$ (recall the notation in Section 2) and shows that the additive loss in revenue of $A^{(\mathcal{I},r)}_{(1)}$ with respect to the revenue of the optimal auction $A^{opt}_{(1)}$ for single-item auctions is $O(\epsilon \cdot n \cdot H)$, with $\epsilon = \sqrt{\frac{\ln 2\delta^{-1}}{2m}}$. In section 4 we extend the results to matroid and position auctions.

THEOREM 3.1 (MAIN THEOREM). *For a single-parameter environment with optimal auction of the form $A^{(\mathcal{I},r)}$ with n i.i.d. bidders with values from unknown irregular distribution F, with m i.i.d. samples from F, the additive loss in expected revenue of Algorithm 2 compared to the optimal expected revenue is at most $3 \cdot n \cdot H \cdot \sqrt{\frac{\ln 2\delta^{-1}}{2m}}$ with probability at least $1-\delta$.*

3.1. The Empirical Myerson Auction

We run a variant of the Empirical Myerson auction, which we have divided into two parts: the first is a learning algorithm ALG (Algorithm 1) that computes ironing in-

Algorithm 1 Compute the ironing intervals $\mathcal{I}$ and reserve price r.

COMPUTEAUCTION(X, δ)

1 Construct $\widehat{F^{-1}}$ from X; let $\epsilon = \sqrt{\frac{\ln 2 \cdot \delta^{-1}}{2|X|}}$.
2 Construct $\widehat{R}_{\min}(q) = q \cdot \widehat{F^{-1}}(1 - q - \epsilon)$.
3 Compute the convex hull $\mathcal{CH}(\widehat{R}_{\min})$, of $\widehat{R}_{\min}$.
4 Let $\mathcal{I}_q$ be the set of intervals where $\widehat{R}_{\min}$ and $\mathcal{CH}(\widehat{R}_{\min})$ differ.
5 **for each** quantile ironing interval $(a_i, b_i) \in \mathcal{I}_q$
6 Add $[\widehat{F^{-1}}(1 - b_i - \epsilon),\ \widehat{F^{-1}}(1 - a_i - \epsilon))$ to $\mathcal{I}$.
7 Let the reserve quantile be $r_q = \arg\max_q \widehat{R}_{\min}(q)$.
8 Let the reserve price be $r = \widehat{F^{-1}}(1 - r_q - \epsilon)$.
9 **return** $(\mathcal{I}, r)$

Algorithm 2 Empirical Myerson.

EMPIRICALMYERSON$(X, \delta, \mathbf{b})$

1 $\mathcal{I}, r \leftarrow$ COMPUTEAUCTION(X, δ)
2 **return** $A^{\mathcal{I},r}_{(1)}(\mathbf{b})$

tervals $\mathcal{I}$ and a reserve price r based on samples X and confidence parameter δ. The second step is to run the welfare-maximizing auction subject to ironing and reservation (Algorithm 2). In this section we focus on analyzing the single-item auction, but the only place this is used is in line 2 of Algorithm 2. Auctions for position auctions and matroid environments use the same learning algorithm (Algorithm 1) but then run the welfare-maximizing auction for position auctions or matroid environments respectively; we return to this in Section 4.

The Empirical Myerson auction takes an estimator for the quantile function $\widehat{F^{-1}}$ and constructs its revenue curve. From this, the convex hull $\mathcal{CH}(R)$ is computed and wherever $\mathcal{CH}(R)$ and R disagree, an ironing interval is placed. Then, the highest point on R is used to obtain the reserve price quantile $q_r = \arg\max_q R(q)$. Note that this is all done in quantile space, but we need to specify the reserve price and ironing intervals in value space. So the last step is to use the empirical CDF $\widehat{F}$ to obtain the values at which to place the reserve price and ironing intervals.

Our learning algorithm follows that approach, with the exception that in line 2 of COMPUTEAUCTION, we take the empirical quantile function to be $\widehat{F^{-1}}(1-q-\epsilon)$ rather than the arguably more natural choice of $\widehat{F^{-1}}(1 - q)$. The motivation here is to protect against overestimation — recall the cautionary tale of Example 1.2. From the DKW inequality we can derive that $\widehat{F^{-1}}(1-q-\epsilon) \leq F^{-1}(1-q)$ with probability $1-\delta$ (we prove this in Lemma 3.3), so we would hope that using this will sufficiently protect against overestimation, while incurring only a modest loss in revenue due to underestimation. That this approach indeed leads to good revenue guarantees is shown in the remainder of this section.

3.2. Additive Revenue Loss in Terms of Revenue Curves

We start with a technical lemma that reduces bounding the loss in revenue to bounding the estimation error due to using samples as opposed to the true distribution F.

LEMMA 3.2. *For a distribution F, let R^{alg} be the revenue curve induced by an algorithm $ALG \in \mathcal{A}$ and let R^{opt} be the optimal induced revenue curve. The additive revenue loss of ALG with respect to OPT is at most:*

$$n \cdot \max_{q \in [0,1]} \left(R^{opt}(q) - R^{alg}(q) \right).$$

PROOF. First, to calculate the ex ante expected revenue of a bidder i with interim allocation function y_i,[10] revenue curve R, ironing intervals $\mathcal{I}$ and reserve price r, we have by Myerson [1981]:

$$Rev[R, \mathcal{I}, r] = -\mathbb{E}_{q \sim U[0,1]}[R(q) \cdot (y^{(\mathcal{I},r)})'(q)]. \tag{5}$$

Next, we apply the Switching Trick of Proposition 2.1. Let $\mathcal{I}_{alg}, \mathcal{I}_{opt}$ be the sets of ironing intervals and r_{alg}, r_{opt} be the reserve prices of ALG and OPT respectively. This yields total revenues:

$$Rev[F, I_{alg}, r_{alg}] = \sum_{i=1}^{n} \int_0^1 -R^{alg}(q) y_i'(q) dq, \quad Rev[F, I_{opt}, r_{opt}] = \sum_{i=1}^{n} \int_0^1 -R^{opt}(q) y_i'(q) dq.$$

Note that the interim allocation function y_i for bidder i is the same one in both equations; the only difference between $y_i^{(\mathcal{I}_{opt}, r_{opt})}$ and $y_i^{(\mathcal{I}_{alg}, r_{alg})}$ was the ironing intervals and reserve price, so after applying the switching trick, y_i is simply the welfare-maximizing interim allocation rule for bidder i.[11] This is the key point in our analysis, and it allows us to compare the expected revenue of both auctions directly:

$$\begin{aligned}
Rev[F, \mathcal{I}_{opt}, r_{opt}] - Rev[F, \mathcal{I}_{alg}, r_{alg}] &= \sum_{i=1}^{n} \left(-\int_0^1 R^{opt}(q) y_i'(q) dq + \int_0^1 R^{alg}(q) y_i'(q) dq \right) \\
&= \sum_{i=1}^{n} \int_0^1 \left(R^{opt}(q) - R^{alg}(q) \right) \left(-y_i'(q) \right) dq \\
&\leq \max_{q \in [0,1]} \left(R^{opt}(q) - R^{alg}(q) \right) \cdot \sum_{i=1}^{n} \int_0^1 -y_i'(q) dq \\
&= \max_{q \in [0,1]} \left(R^{opt}(q) - R^{alg}(q) \right) \cdot \sum_{i=1}^{n} \left(-y_i(1) + y_i(0) \right) \\
&\leq n \cdot \max_{q \in [0,1]} \left(R^{opt}(q) - R^{alg}(q) \right).
\end{aligned}$$

The inequality holds as $-y_i'$ is non-negative. The last inequality holds because y_i always lies between 0 and 1. Rearranging the terms yields the claim. □

This is significant progress: the additive loss in revenue can be bounded in terms of the induced revenue curves of an algorithm ALG and the optimal algorithm, two objects that we have some hope of getting a handle on. Of course, we still need to show

[10] For single-item auctions with i.i.d. bidders, all bidders share the same interim allocation function. We write y_i to facilitate our extension to matroid environments in the next section.

[11] E.g., for single-item auctions, this is just the probability that bidder i has the largest valuation (given its quantile).

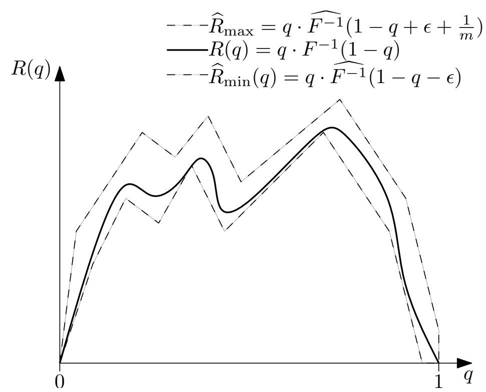


Fig. 2: We can sandwich R between two estimated revenue curves $\widehat{R}_{\text{min}}$ and $\widehat{R}_{\text{max}}$.

that the ironed revenue curve of Algorithm 1 is pointwise close to the ironed revenue curve induced by the optimal auction (Section 3.3).

3.3. Bounding the Error in the Revenue Curve

We implement the following steps to prove that the error in the learning algorithm's estimation of the revenue curve is small. The proofs of this section appear in the full version.

— (Lemma 3.3) We show that we can sandwich the actual revenue curve (without ironing or reserve price) R between two empirical revenue curves, $\widehat{R}_{\text{min}}$ and $\widehat{R}_{\text{max}}$ that are defined using the empirical quantile function.

— (Lemma 3.4 and Lemma 3.5) Let $\widehat{R}^{\star}_{\text{max}}$ (resp. $\widehat{R}^{\star}_{\text{min}}$) be the optimally induced revenue curve for $\widehat{R}_{\text{max}}$ (resp. $\widehat{R}_{\text{min}}$). The revenue curve induced by Algorithm 1, R^{alg}, is pointwise higher than the optimal induced revenue curve of the lower bound $\widehat{R}^{\star}_{\text{min}}$, and the optimal induced revenue curve for the upper bound, $\widehat{R}^{\star}_{\text{max}}$, is pointwise higher than R^{opt}.

— (Lemma 3.7) Finally, we show that $\widehat{R}^{\star}_{\text{max}}(q) - \widehat{R}^{\star}_{\text{min}}(q)$ is small for all q, and therefore the additive loss is small.

LEMMA 3.3. *For a distribution F and m samples from F. Let $\widehat{R}_{min}(q) = q \cdot \widehat{F^{-1}}(1 - q - \epsilon)$ and $\widehat{R}_{max}(q) = q \cdot \widehat{F^{-1}}(1 - q + \epsilon + \frac{1}{m})$. With probability at least $1 - \delta$ for all $q \in [0, 1]$:*

$$\widehat{R}_{min}(q) \quad \leq \quad R(q) \quad \leq \quad \widehat{R}_{max}(q).$$

So while the the algorithm does not know the exact revenue curve R, it can be upper bounded by $\widehat{R}_{\text{max}}$ and lower bounded by $\widehat{R}_{\text{min}}$. We'll use $\widehat{R}_{\text{min}}$ to give a lower bound on the revenue curve R^{alg} induced by Algorithm 1, and $\widehat{R}_{\text{max}}$ to give an upper bound on the revenue curve R^{opt} induced by Myerson's optimal auction. We start with the latter.

LEMMA 3.4. *Let R^{opt} be the optimal induced revenue curve of R, and let $\widehat{R}^{\star}_{max}$ be the optimal induced revenue curve for $\widehat{R}_{max}$. Then with probability $1 - \delta$ for all $q \in [0, 1]$:*

$$R^{opt}(q) \quad \leq \quad \widehat{R}^{\star}_{max}(q).$$

So with high probability $\widehat{R}^{\star}_{\text{max}}$ is pointwise higher than R^{opt}. Proving that $\widehat{R}^{\star}_{\text{min}}$ is a lower bound for R^{alg} is slightly more involved since the ironing intervals and reserve

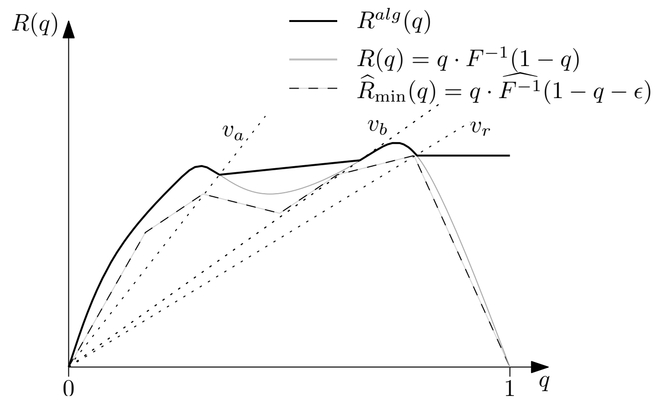


Fig. 3: When we pick a value range to iron based on $\widehat{R}_{\min}$, its effect on the actual revenue curve can be seen. The quantiles of the start and end point of the ironing procedure are given by the line that intersects the start and end point on $\widehat{R}_{\min}$ and the origin.

price are given by Algorithm 1, and may not be optimal. Therefore, the induced revenue curve R^{alg} is in general not concave, and the reserve quantile may not be at the highest point of the curve. In the following lemma, we use the fact that that the ironing intervals and reserve price of R^{alg} were chosen based on $\widehat{R}_{\min}$.

LEMMA 3.5. *Let R^{alg} be the revenue curve induced by Algorithm 1 and let $\widehat{R}^{\star}_{min}$ be the optimal induced revenue curve for $\widehat{R}_{min}$. Then with probability $1-\delta$ for all $q \in [0,1]$:*

$$\widehat{R}^{\star}_{min}(q) \quad \leq \quad R^{alg}(q).$$

First we show that to prove pointwise dominance of curve R^{alg} over $\widehat{R}_{\min}$, it is sufficient to show that any ray from the origin that intersects the revenue curves, first intersects with $\widehat{R}_{\min}$ and then R^{alg}.

PROPOSITION 3.6 (RAY DOMINANCE). *For two (potentially ironed) revenue curves R and R', if all rays from the origin that intersect with R or R' intersect R before R', then it must be that $R(q) \leq R'(q)$ for $q \in [0,1]$.*

PROOF. Fix R, R'. WLOG assume that R and R' are not ironed.[12] First observe that a revenue curve S consists of the set of points $\{(1-F(v), v \cdot (1-F(v))) : v \in \mathbb{R}^+\}$, where F is the CDF corresponding to S. Therefore a ray from the origin with slope v' intersects S on the set of points $\{(1-F(v), v \cdot (1-F(v))) : 1-F(v) = 1-F(v')\}$. Since $1-F(v)$ is non-increasing, this can only happen in either a single point, or single line segment.

We prove the contrapositive. If R is not dominated by R', then by continuity there is an interval (q_1, q_2) with $q_1 < q_2$ where R is above the graph of R'. Let $q' = \frac{q_1+q_2}{2}$ be the midpoint. Take the ray that starts at the origin and has slope $\frac{R'(q')}{q'}$. This ray intersects R' in the point $(q', R'(q'))$. Since $R'(q) > R(q)$ for $q \in (q_1, q_2)$, and R' is continuous, the ray must intersect with R' after it intersected with R. Since rays from the origin only intersect a revenue curve once, it could not have intersected R' before R either. $\square$

[12] Recall that from the Switching Trick, any ironed revenue curve can be written as the revenue curve for an unironed, different CDF.

To complete the proof of Lemma 3.5, We still need to show that every ray through the origin intersects $\widehat{R}_{\min}$ before R^{alg}. This part of the proof appears in the full version of the paper.

We now have our upper bound and lower bounds in terms of $\widehat{F^{-1}}$. Finally we show that the difference between the two is small.

LEMMA 3.7. *For a distribution F, let R^{alg} be the revenue curve induced by Algorithm 1 and R^{opt} the optimal induced revenue curve. Using m samples from F, with probability $1-\delta$ and $\epsilon = \sqrt{\frac{\ln 2\cdot\delta^{-1}}{2m}}$ for all q:*

$$R^{opt}(q) - R^{alg}(q) \le \widehat{R}^{\star}_{max}(q) - \widehat{R}^{\star}_{min}(q) \le \left(2\epsilon + \frac{1}{m}\right) H \le 3\epsilon \cdot H.$$

Theorem 3.1 now follows by combining Lemmas 3.2 and 3.7. The additive loss in expected revenue of Algorithm 2 is at most $3\epsilon \cdot n \cdot H$.

Proof of Theorem 3.1: By Lemma 3.2 we can express the total additive error of the expected revenue of an algorithm that yields ironing intervals $\mathcal{I}_{alg}$ and reserve price r_{alg} with respect to the optimal auction as:

$$\frac{Rev[F, \mathcal{I}_{opt}, r_{opt}] - Rev[F, \mathcal{I}_{alg}, r_{alg}]}{n} \le \max_{q\in[0,1]} \left(R^{opt}(q) - R^{alg}(q)\right).$$

By Lemma 3.7, Algorithm 2 yields

$$\max_{q\in[0,1]} \left(R^{opt}(q) - R^{alg}(q)\right) \le \left(2\epsilon + \frac{1}{m}\right) H.$$

The theorem follows. ■

When the optimal revenue is bounded away from zero, we get an analogous sample complexity bound for learning (efficiently) a $(1-\epsilon)$-(multiplicative) approximate auction.

4. MATROID AND POSITION ENVIRONMENTS

The results of the previous section extend to matroid and position auction environments.

THEOREM 4.1. *For position and matroid auctions with n i.i.d. bidders with values from unknown distribution F, m i.i.d. samples from F, with probability $1-\delta$, the additive loss in expected revenue of running the welfare-maximizing auction using ironing intervals and reserve price from Algorithm 1 compared to the optimal expected revenue is at most $3 \cdot n \cdot H \cdot \sqrt{\frac{\ln 2\delta^{-1}}{2m}}$.*

The learning algorithm uses the same subroutine (Algorithm 1) to learn ironed intervals and a reserve price, and returns the auction that first deletes all bidders not meeting the reserve and then chooses the feasible outcome maximizing the ironed virtual welfare. The proof follows from Proposition 4.2 and Proposition 4.3, which show that the optimal auctions for matroid and position auction environments are in $\mathcal{A}$ (i.e., have the form $A^{(\mathcal{I},r)}$ for a suitable choice of ironed intervals $\mathcal{I}$ and reserve price r), and from Lemmas 3.2–3.7, which rely only on this property.

4.1. Position Auctions

A position auction [Varian 2007] is one where the winners are given a position, and position i comes with a certain quantity x_i of the good. The canonical example is that of ad slot auctions for sponsored search, where the best slot has the highest click-through-rate, and subsequent slots have lower and lower click-through-rates. In an optimal auction, the bidder with the highest ironed virtual value gets the best slot, the second highest ironed virtual value the second slot, and so on.

PROPOSITION 4.2. *The optimal auction $A^{opt}_{(pos)}$ for position auctions can be expressed as an auction with ironing and reserve price in value space: $A^{opt}_{(pos)} \in \mathcal{A}$.*

PROOF. In the optimal auction, the bidder with the highest ironed virtual value is awarded the first position (with allocation x_1), the bidder with the second highest ironed virtual value the second position with x_2, and so on. Since the ironed virtual value is monotonically non-decreasing in the value of a bidder, and identical in ironing intervals, this can equivalently be described by an auction in $\mathcal{A}$. $\square$

4.2. Matroid Environments

In a matroid environment, the feasible allocations are given by matroid $\mathcal{M} = (E, I)$, where E are the players and I are independent sets. The auction can simultaneously serve only sets S of players that form an independent set of the matroid $S \in I$. A special case of this is the rank k uniform matroid, which accepts all subsets of size at most k, i.e. it is a k-unit auction environment.

In matroid environments, the ex-post allocation function $x_i(\mathbf{b})$ and interim allocation function $y_i(q)$ are no longer the same for each player, e.g. imagine a player i who is not part of any independent set, then $y(q) = 0$ everywhere. However, the optimal allocation can still be expressed in terms of an auction $A \in \mathcal{A}$.

PROPOSITION 4.3. *The optimal auction $A^{opt}_{(mat)}$ for matroid auctions can be expressed as an auction with ironing and reserve price in value space: $A^{opt}_{(mat)} \in \mathcal{A}$.*

Proof of Proposition 4.3: A property of matroids is that the following simple greedy algorithm yields the optimal solution:

GREEDY(E, I)

$S \leftarrow \emptyset$
while $\{i : i \notin S \wedge S \cup \{i\} \in I\} \neq \emptyset$
  add $\arg\max\{v_i : i \notin S \wedge S \cup \{i\} \in I\}$ to S
return S

where v_i is the (ironed virtual) value associated with bidder i. Since the order of largest values is the same for both virtual values and bids (up to ties), the allocation of the optimal auction is identical to the auction that irons on $\mathcal{I}_{opt}$ and has reserve price r_{opt} (up to tie-braking); hence $A^{opt}_{(mat)} \in \mathcal{A}$. ■

5. NO-REGRET ALGORITHM

So far we assumed access to a batch of samples before having to choose an auction. In this section we show that running the algorithm in a repeated setting, using past bidding behavior as the samples, leads to a no regret algorithm. The goal here is to achieve total additive error $o(T) \cdot O(\text{poly}(n, H, \delta))$ — the error can be polynomial in all parameters except the time horizon T, for which it should be sublinear. We show that for Algorithm 3 the total loss grows as $\widetilde{O}(\sqrt{T}\sqrt{n}\sqrt{\log(\delta^{-1})}H)$ and hence results in a no-regret algorithm.

We run Algorithm 3. Invoking Theorem 3.1 with confidence parameter δ/T and taking a union bound over the rounds, we have the following fact.

Algorithm 3 A no-regret algorithm for optimal auctions.

```
NO-REGRET-AUCTION(δ, T)
▷ Round 0:
Collect a set of bids b, run an arbitrary mechanism
X ← b
for round t = 1...T
      Collect a set of bids b
      EMPIRICALMYERSON(X, δ/T, b)
      X ← X ∪ b
```

PROPOSITION 5.1. *With probability* $1-\delta$*, for all rounds simultaneously, each round* $t \in [1, T]$ *of Algorithm 3 has additive loss at most* $3\sqrt{\frac{\ln(2T\delta^{-1})}{2nt}} \cdot n \cdot H$.

This leads to the following no-regret bound.

THEOREM 5.2. *With probability* $1 - \delta$*, the total additive loss of Algorithm 3 is* $O(\sqrt{n}\sqrt{T \log T}\sqrt{\log \delta^{-1}} \cdot H)$*, which is* $\widetilde{O}(T^{1/2})$ *with respect to* T.

PROOF. By Proposition 5.1 with probability $1 - \delta$ for all rounds simultaneously, the additive loss for round t is bounded by $3\sqrt{\frac{\ln(2T\delta^{-1})}{2nt}} \cdot n \cdot H$. The loss of day 0 is at most $H \cdot n$. The total loss can then be bounded by:

$$(n \cdot H) \cdot \left(1 + 3\sum_{t=1}^{T} \sqrt{\frac{\ln(2T\delta^{-1})}{2nt}}\right)$$

We can rewrite the sum:

$$\begin{aligned}
\sum_{t=1}^{T} \sqrt{\frac{\ln(2T/\delta)}{2nt}} &= \sqrt{\frac{\ln(2T/\delta)}{2n}} \cdot \sum_{t=1}^{T} \sqrt{\frac{1}{t}} \\
&\leq \sqrt{\frac{\ln(2T/\delta)}{2n}} \cdot 2\sqrt{T} \\
&= \sqrt{\frac{2T\ln(2T/\delta)}{n}}
\end{aligned}$$

Hence the total loss is

$$(n \cdot H) \cdot \left(1 + 3\sqrt{\frac{2T\ln(4T/\delta)}{n}}\right) = O(\sqrt{n}\sqrt{T \log T}\sqrt{\log \delta^{-1}} \cdot H);$$

the dependence on T is $O(\sqrt{T \log T}) = \widetilde{O}(\sqrt{T})$. □

The bound of $O(\sqrt{T \log T})$ is almost tight, as there is a lower bound of $\Omega(\sqrt{T})$ given by [Cesa-Bianchi et al. 2013]. Also note that if we do not know T a priori, we can use a standard doubling argument to obtain the same asymptotic guarantee.

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