Barriers to Near-Optimal Equilibria

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***Abstract*—This paper explains when and how communication and computational lower bounds for algorithms for an optimization problem translate to lower bounds on the worst-case quality of equilibria in games derived from the problem. We give three families of lower bounds on the quality of equilibria, each motivated by a different set of problems: congestion, scheduling, and distributed welfare games; welfare-maximization in combinatorial auctions with "black-box" bidder valuations; and welfare-maximization in combinatorial auctions with succinctly described valuations.**

The most straightforward use of our lower bound framework is to harness an existing computational or communication lower bound to derive a lower bound on the worst-case price of anarchy (POA) in a class of games. This is a new approach to POA lower bounds, which relies on reductions in lieu of explicit constructions.

More generally, the POA lower bounds implied by our framework apply to all classes of games that share the same underlying optimization problem, independent of the details of players' utility functions. For this reason, our lower bounds are particularly significant for problems of game design — ranging from the design of simple combinatorial auctions to the existence of effective tolls for routing networks — where the goal is to design a game that has only near-optimal equilibria. For example, our results imply that the simultaneous first-price auction format is optimal among all "simple combinatorial auctions" in several settings.

***Index Terms*—price of anarchy; mechanism design; complexity of equilibria**

I. Introduction

All of the classical equilibrium concepts in game theory, such as the Nash equilibrium, are defined without reference to any computational process. Indeed, some of the greatest hits of algorithmic game theory show that computing a Nash equilibrium is intractable in multiple senses [1]–[4], and this raises the possibility that Nash equilibria can solve problems that efficient algorithms cannot. The goal of this paper is to prove that game-theoretic equilibria are generally bound by the same limitations as algorithms that use polynomial computation or communication.

Our lower bound framework is quite general, and we give several example applications. The most straightforward use is to harness an existing computational or communication lower bound to derive a lower bound on the worst-case *price of anarchy (POA)* in a class of games — the worst-case approximation guarantee for equilibria with respect to some objective function on game outcomes. Sections I-A and I-B give two illustrative examples in well-studied classes of games — congestion games and simultaneous first-price auctions — and more examples appear in the full version of the paper. This is a new approach to POA lower bounds, which relies on reductions in lieu of explicit constructions. The latter can require a tour de force — see [5] and [6] for two recent examples — in part because it involves solving explicitly for the worst equilibrium.

Perhaps the strongest feature of our lower bound framework is that the consequent POA lower bounds apply not just to one particular class of games, but to *all* classes of games that share the same underlying optimization problem, independent of the details of players' utility functions. By contrast, small changes to players' utility functions can radically change the set of equilibria of a game, and for this reason explicit POA lower bounds tend to be brittle and need to be reworked on a case-by-case basis (e.g. [7]). Thus, our lower bounds are particularly significant for problems of game design, where the goal is to design a game that has only near-optimal equilibria. The full version of the paper explains how much recent and current work in algorithmic game theory, ranging from the design of simple combinatorial auctions to the existence of effective tolls for routing networks, boils down to such game design problems.

The next two sections are intended for readers interested in the gist of the paper's techniques or in concrete examples that motivate the more general formalism and results of Sections IV–VII. The reader interested primarily in the summary of results can skip to Section II.

A. Example: Cost-Minimization in Congestion Games

This section illustrates how near-optimal POA lower bounds for congestion games follow, conditional on $NP \neq coNP$, from NP-hardness results for the underlying optimization problem.

A *congestion game* [8] is defined by a ground set E of resources, a set of n players with action sets $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq 2^E$, and a non-negative cost function $c_e : \mathcal{Z}^+ \rightarrow \mathcal{R}^+$ for each resource $e \in E$. Given an action profile $\mathbf{a} \in \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, we define the load x_e as the number of players that use resource e in $\mathbf{a}$. The cost to player i is defined as

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$C_i(\mathbf{a}) = \sum_{e \in a_i} c_e(x_e)$. The cost of an outcome is defined as

$$\sum_{i=1}^{n} C_i(\mathbf{a}) = \sum_{e \in E} c_e(x_e) x_e, \tag{1}$$

where the equation follows from a reversal of sums.

For this example, we focus on *pure Nash equilibria (PNE)*, which are action profiles $\mathbf{a}$ such that $C_i(\mathbf{a}) \leq C_i(a'_i, \mathbf{a}_{-i})$ for every player i and unilateral deviation $a'_i \in \mathcal{A}_i$. The (pure) POA of a congestion game is the ratio between the cost (1) of the worst PNE and that of a minimum-cost outcome; Proposition I.2 below implies that this is well defined.

Simple examples show that the worst-case POA of congestion games is finite only if the set $\mathcal{C}$ of allowable cost functions is restricted [9]. The worst-case POA of congestion games, as a function of the set $\mathcal{C}$, is well understood [5], [10]–[12]. One simple and common parameterization is to consider the sets $\mathcal{C}_d$ of polynomials with nonnegative coefficients and degree at most d. There is a closed-form formula for the worst-case POA of congestion games with cost functions in $\mathcal{C}_d$ [5]; the dependence on d is roughly $(d/\ln d)^{d+1}$. The tight lower bounds in [5] are proved via non-trivial and interesting explicit constructions.

To bring complexity theory into the picture, define the Cost-Minimization(d) (CM(d)) problem as: given a description of a congestion game with cost functions in $\mathcal{C}_d$, compute the minimum cost of an action profile. We assume that players' strategies and the polynomial coefficients are provided explicitly as part of the input.

Theorem I.1 *Fix $d \geq 1$, and assume that:*

1) *There is a polynomial-time reduction R from an NP-complete problem Π that computes a parameter C^* and maps "yes" and "no" instances of Π to instances of CM(d) with cost at most C^* and at least ρC^*, respectively.*[1]
2) $NP \neq coNP$.[2]

Then, the worst-case pure POA in congestion games with cost functions in $\mathcal{C}_d$ is at least ρ.

The proof of Theorem I.1 relies on three simple facts. The first was proved by Rosenthal [8], the second follows from brute-force comparisons, and the third is obvious from (1).

Proposition I.2 ([8]) *Every congestion game has at least one PNE.*

Proposition I.3 *The problem of deciding whether or not a given action profile of a congestion game is a PNE can be solved in polynomial time.*

Proposition I.4 *The problem of computing the cost* (1) *of a given action profile of a congestion game can be solved in polynomial time.*

Proof: (of Theorem I.1) Let α denote the worst-case pure POA of congestion games with cost functions in $\mathcal{C}_d$, and consider the following nondeterministic algorithm $\mathcal{P}$ for the instances of the CM(d) problem produced by the assumed reduction R.

1) Given a description of a congestion game G produced by R, nondeterministically guess a PNE $\mathbf{a}$.
2) Verify that $\mathbf{a}$ is a PNE.
3) Compute the cost C of $\mathbf{a}$.
4) Output "yes" if and only if $C < \rho \cdot C^*$, where C^* is the parameter given by the reduction R.

Propositions I.2–I.4 imply that $\mathcal{P}$ is a well-defined, nondeterministic polynomial-time algorithm.

By the definition of the POA, the cost C of the PNE $\mathbf{a}$ is at most α times that of a minimum-cost outcome of G. Hence, if $\alpha < \rho$, then $\mathcal{P}$ outputs "yes" whenever there is an outcome with cost at most C^* and "no" whenever every outcome has cost at least $\rho \cdot C^*$. This would contradict our two assumptions, so we conclude that $\alpha \geq \rho$, as required. ∎

Theorem I.1 reduces proving lower bounds on the POA of congestion games to proving hardness of approximation results for the CM(d) problem. Perhaps surprisingly, this problem has not been well studied.[3] in the full version of the paper, we provide a new hardness of approximation result for the problem.

Theorem I.5 *There is a constant $\beta > 0$ such that, for all $d \geq 1$, it is NP-hard to approximate the CM(d) problem better than a $(\beta d)^{d/2}$ factor.*

Combining Theorems I.1 and I.5 proves that, assuming $NP \neq coNP$, the worst-case POA of congestion games with cost functions in $\mathcal{C}_d$ is at least $(\beta d)^{d/2}$. This lower bound is weaker than the tight lower bound of $\approx (d/\ln d)^{d+1}$ in [5] in two respects: it is conditional, and it is not quantitatively tight. On the other hand, the lower bound relies only on Propositions I.2–I.4, and therefore extends to many other classes of games for which the underlying optimization problem is CM(d) (see the full version for details).

B. Example: Welfare-Maximization in Combinatorial Auctions

No problem in algorithmic game theory has been studied more than that of welfare-maximization in combinatorial auctions (e.g. [17]). This section uses a representative special case of the problem to illustrate the additional ideas necessary to prove POA lower bounds in games with a super-polynomial number of player actions.

[1] Henceforth, we abbreviate such an assumption as "NP-hard to approximate better than a factor of ρ."

[2] This assumption can be weakened to $NP \not\subseteq PLS$, where PLS is defined in [13].

[3] In [14], several NP-hardness results are given for solving the problem exactly. For arbitrary nondecreasing cost functions, no finite approximation is possible (assuming $P \neq NP$) [14]. Hardness results for the more general model of congestion games with player-specific cost functions are considered in [15], [16].

In a combinatorial auction, there is a set U of m distinct items. There are n bidders. Each bidder i has a *valuation function* $v_i : 2^U \to \mathcal{R}^+$. Most of the results of this section apply to arbitrary classes of bounded valuation functions, but for concreteness we stick with valuation functions v_i that satisfy $v_i(\emptyset) = 0$, monotonicity (so $T \subseteq S$ implies $v_i(T) \leq v_i(S)$), and subadditivity (so $v_i(S \cup T) \leq v_i(S) + v_i(T)$ for every $S, T \subseteq U$). We also assume that every valuation $v_i(S)$ is integral and bounded above by a number $V_{\max}$ that is polynomial in n and m. An *allocation* is an assignment of each item of U to at most one bidder. The *welfare* $w(S_1, \ldots, S_n)$ of an allocation is $\sum_{i=1}^n v_i(S_i)$.

A simple approach to selling multiple items is to sell each seperately via a single-item auction [18]. This section focuses on the concrete case of *simultaneous first-price auctions (S1As)* [19], [20]. Every player i has the same action set $\mathcal{A}_i = \{0, 1, 2, \ldots, V_{\max}\}^m$, with each action representing an integral bid (between 0 and $V_{\max}$) on each of the m items.[4] Each item j is sold separately to the highest bidder — with ties between high bids broken lexicographically, say — and the winner of an item pays its bid. The utility $u_i(\mathbf{a})$ of a player i in an action profile $\mathbf{a}$ is defined as its value $v_i(S_i)$ for the items S_i that it wins, minus the sum of its bids on the items in S_i.

The underlying optimization problem in a S1A is that of determining an allocation with the maximum-possible welfare. For this example, we take a communication complexity approach to the problem. Our communication model is the standard one in this context [21]. Each of the n players holds its own subadditive valuation v_i and does not know the others $\mathbf{v}_{-i}$. A communication protocol specifies rules for exchanging bits of information between the players. In a round where player i sends a message, this message can depend only on v_i and the messages previously sent by all players. Nondeterministic protocols can additionally provide all players with a common advice string (which can depend on all inputs $\mathbf{v}$) before the protocol begins. A protocol for computing an allocation is typically deemed tractable if the resources required are polynomial in n and m. This is sublinear in the input size, which comprises $n \cdot 2^m$ numbers.

We next describe the analogs of Propositions I.2–I.4 appropriate for the current setting. First, Proposition I.2 does not hold for S1A's with subadditive bidder valuations — PNE need not exist [22].[5] We instead consider *mixed* Nash equilibria (MNE), where each player picks a probability distribution over its actions and no player can increase its expected utility by a unilateral deviation. Since S1A's have a finite number of players and strategies, Nash's theorem [23] implies that at least one MNE exists.

Second, the analog of Proposition I.3 also fails. Since the action space of each player has size $(V_{\max} + 1)^m$, which is exponential in m, an MNE of a S1A cannot generally be represented, let alone verified, using a number of bits that is polynomial in m. The next idea is to relax the equilibrium concept further, to ϵ*-approximate* mixed Nash equilibria (ϵ-MNE), where no player can increase its expected utility by more than ϵ by a unilateral deviation.[6]

We recall an important result of Lipton et al. [24]. By a t*-uniform* mixed strategy for a player i, we mean a uniform distribution over a multi-set of at most t actions from $\mathcal{A}_i$.

Theorem I.6 (Small Support ϵ-MNE [24]) *Let G be a game with n players, each with at most N actions, and with all payoffs bounded between $-V_{\max}$ and $V_{\max}$. For every $\epsilon > 0$, G has a $(12n^2 \ln(n^2 N))/\epsilon^2$-uniform $\epsilon V_{\max}$-MNE.*

The proof of Theorem I.6 proceeds by drawing independent sample actions for every player according to some fixed MNE (one exists, by Nash's theorem), and proving that with positive probability the empirical distributions of the samples form an $\epsilon V_{\max}$-MNE.[7]

Since $N = (V_{\max} + 1)^m$ in a S1A and we are assuming that $V_{\max}$ is polynomial in n and m, we deduce an analog of Proposition I.2 as a corollary to Theorem I.6.

Proposition I.7 *For every $\epsilon > 0$, every S1A has a t-uniform $\epsilon V_{\max}$-MNE, where t is bounded above by a polynomial in n, m, and ϵ^{-1}.*

Since $V_{\max}$ is polynomially bounded, the number of bits needed to describe the approximate MNE in Proposition I.7 is polynomial in n, m, and t.

The analog of Proposition I.3 follows from the fact that, given a description of a mixed strategy profile $\mathbf{x}$, a player can privately compute both its expected utility in $\mathbf{x}$ and its expected utility in a best response to others' mixed strategies $\mathbf{x}_{-i}$.

Proposition I.8 *The problem of deciding whether or not a t-uniform mixed strategy profile of a S1A is an $\epsilon V_{\max}$-MNE can be solved with communication polynomial in n, m, and t.*

The analog of Proposition I.4 follows from the fact that, given a description of a mixed strategy profile, each player can privately compute its contribution to the social welfare.

Proposition I.9 *The problem of computing the expected welfare of a t-uniform mixed strategy profile of a S1A can be solved with communication polynomial in n, m, and t.*

We require one additional simple proposition.

Proposition I.10 *For every S1A, there is an action profile that induces an allocation with the maximum-possible welfare.*

[4] Alternatively, valuations and bids can be multiples of an arbitrarily small constant, like a penny, without affecting our results.

[5] Guaranteed equilibrium existence is clearly crucial to the proof of Theorem I.1, and more generally it is fundamental to the lower bound framework of this paper. See Section III-C for further discussion.

[6] Section III-A explains several senses in which POA lower bounds for ϵ-MNE with ϵ close to 0 are essentially as good as those for exact MNE.

[7] Hémon et al. [25] and Babichenko and Peretz [26] give quantitative improvements to Theorem I.6, but the original result is sufficient for our purposes.

Proof: If $(S_1^*, \ldots, S_n^*)$ is a welfare-maximizing allocation, have each player i bid 1 on every item of S_i^* and 0 on every other item. ∎

We now show how Propositions I.7–I.10 enable the translation of lower bounds for communication protocols to lower bounds on the the POA in S1A's. We define the POA of MNE (ϵ-MNE) in a S1A as the ratio between the welfare of an optimal allocation and the smallest expected welfare of a MNE (ϵ-MNE).

The worst-case POA of MNE in S1A's was recently determined to be precisely 2. Feldman et al. [27] provided the upper bound, while Christodoulou et al. [6] devised an intricate explicit construction to prove a matching lower bound. We next show how essentially the same lower bound follows from known lower bounds for communication protocols.

Theorem I.11 *Let $\mathcal{V}$ denote a set of valuation profiles with all valuations bounded above by $V_{\max}$. Assume that every nondeterministic communication protocol that distinguishes between valuation profiles $\mathbf{v} \in \mathcal{V}$ with maximum-possible welfare at least W^* and those with maximum-possible welfare at most W^*/ρ uses communication exponential in m, for all sufficiently large n and m.*

Then, for every polynomial function $p(n,m)$ of n and m, the worst-case POA of $p(n,m)^{-1}V_{\max}$-MNE in S1A's with valuation profiles in $\mathcal{V}$ is at least ρ.

Proof: Fix a polynomial function $p(n,m)$ and consider the following nondeterministic protocol $\mathcal{P}$:

1) Given a valuation profile $\mathbf{v} \in \mathcal{V}$, inducing a S1A G, nondeterministically compute a t-uniform $p(n,m)^{-1}V_{\max}$-MNE $\mathbf{x}$ of G, where $t = (12n^2 \ln(n^2 N))/p(n,m)^2$ as in Theorem I.6 and $N = (V_{\max}+1)^m$.
2) Verify that $\mathbf{x}$ is an $p(n,m)^{-1}V_{\max}$-MNE.
3) Compute the expected welfare W of $\mathbf{x}$.
4) Output "yes" if and only if $W > W^*/\rho$.

Propositions I.7–I.9 imply that $\mathcal{P}$ is a well-defined nondeterministic protocol that uses communication polynomial in n and m (provided $V_{\max}$ is polynomial in n and m). By Proposition I.10 and the definition of the POA, the expected welfare W of the $p(n,m)^{-1}V_{\max}$-MNE $\mathbf{x}$ is at least an α fraction of the welfare of an optimal allocation, where α is the worst-case POA of $p(n,m)^{-1}V_{\max}$-MNE in S1A's with valuation profiles in $\mathcal{V}$. Hence, if $\alpha < \rho$, then $\mathcal{P}$ outputs "yes" whenever there is an allocation with welfare at least W^* and "no" whenever every allocation has welfare at most W^*/ρ. This would contradict our assumption for sufficiently large m, so $\alpha \geq \rho$. ∎

Theorem I.11 reduces proving lower bounds on the POA of S1A's to proving exponential lower bounds for nondeterministic communication protocols. Many such lower bounds are known (see Section VI); the following is one example.

Theorem I.12 ([28, Theorem 4.1]) *Let $\delta > 0$ be an arbitrarily small constant. For subadditive bidder valuations and every $n \leq m^{(1/2)-\delta}$, every nondeterministic communication protocol that distinguishes between instances with optimal welfare $2n$ and instances with optimal welfare $n+1$ requires an exponential number (in m) of bits in the worst case.*

Theorem I.12 holds even when all bidder valuations are either 1 or 2. Combining this lower bound with Theorem I.11 shows that the worst-case POA of ϵ-MNE in S1A's with subadditive bidder valuations is at least 2 (as $n, m \to \infty$), even when ϵ tends to 0 inverse polynomially in n and m. This essentially reproduces the recent lower bound in [6] — which is for exact MNE, see Section III-A — and matches the upper bound in [27].

More important than this POA lower bound for the specific auction format of S1A's, however, is the fact that this same lower bound applies to *every* sufficiently simple combinatorial auction. Here "simple" roughly means that players' action spaces have size sub-doubly-exponential in m; see Section VI for the precise statement. This general lower bound implied by Theorems I.11 and I.12 for all simple auctions has interesting implications for combinatorial auctions with subadditive bidder valuations:

1) Simultaneous first-price auctions minimize the worst-case POA (of approximate MNE) over all simple auction formats.
2) Complexity is an unavoidable property of every combinatorial auction with a worst-case equilibrium welfare guarantee that is better than 2.

C. Paper Organization

Examination of the proofs of Theorems I.1 and I.11 suggests that computational and communication lower bounds for optimization problems should apply to worst-case equilibria of associated games quite generally — intuitively, whenever the equilibrium concept is guaranteed to exist and easy to verify. The remainder of the paper develops this intuition into a general theory.

Section II summarizes our results. Section III discusses the limitations of our lower bound framework. Section IV describes our formalism for game design. Sections V–VII provide the formal statements of the results described in Section II. Due to space constraints, most proofs and applications of the results in Sections V–VII are omitted; these can be found in the full version of the paper.

II. Summary of Contributions

We offer three families of lower bounds for the minimum worst-case POA achievable by a "game plan" — a way of associating a game to each instance of an optimization problem (defined formally in Section IV). Each family is motivated by one or more well-studied problems in mechanism or network game design. In all cases, the goal is to understand when and how communication or complexity lower bounds for the underlying optimization problem translate to lower bounds for the worst-case POA. See also Tables I and II.

The first set of lower bounds is motivated by optimization problems in large networks and systems.For these problems,

TABLE I

WHEN DOES A HARDNESS OF APPROXIMATION LOWER BOUND TRANSLATE TO A POA LOWER BOUND? THE SECOND COLUMN DESCRIBES THE HARDNESS ASSUMPTION. "ND" STANDS FOR NONDETERMINISTIC, "RAND." FOR RANDOMIZED WITH ONE-SIDED ERROR, AND "ρ-HARD FOR P" MEANS THAT Π IS NP-HARD TO APPROXIMATE BETTER THAN A ρ FACTOR. THE THIRD COLUMN DESCRIBES THE SET OF GAME PLANS TO WHICH THE POA LOWER BOUND OF ρ APPLIES. m DENOTES THE NUMBER OF ITEMS IN A COMBINATORIAL AUCTION. "BR" STANDS FOR "BEST RESPONSE", "EU" FOR "EXPECTED UTILITY". THE FOURTH COLUMN INDICATES THE EQUILIBRIUM CONCEPT TO WHICH THE POA LOWER BOUND (OF ρ) APPLIES; SEE SECTION IV-A FOR DEFINITIONS.

Motivation	Assumptions on Π	Assumptions on Γ	Eq Concept	Refer to
Large Systems	ρ-hard for P and $NP \not\subseteq PPAD$	tractable EU	ϵ-MNE	Thm V.1
Large Systems	ρ-hard for P and $P \neq NP$	tractable EU	CE	Thm V.2
Simple Auctions	ρ-hard for ND protocols	$2^{2^{o(m)}}$ actions	ϵ-MNE	Thm VI.1
Simple Auctions	ρ-hard for rand. protocols	$2^{2^{o(m)}}$ actions	ϵ-CCE	full version
Tractable Auctions	ρ-hard for P and $coNP \not\subseteq MA$	tractable BR	ϵ-MNE	Thm VII.1

we show that NP-hardness of approximating the underlying optimization problem translates under mild conditions to lower bounds on the worst-case POA of all game plans — for approximate mixed Nash equilibria assuming that $NP \not\subseteq PPAD$ (via the $PPAD$ algorithm in [29]) and for (exact) correlated equilibria assuming that $P \neq NP$ (via the polynomial-time algorithm in [30], [31]). These lower bounds are the natural generalizations of the argument outlined in Section I-A for congestion games, and are relevant for many variants of congestion games, for scheduling games and coordination mechanisms [32], and for distributed welfare games [33] (details in the full version).

As mentioned in Section I-A, we also contribute a new hardness of approximation result for the underlying optimization problem in congestion games (Theorem I.5). The proof, which appears in the full version of this paper, builds on ideas used previously to establish strong inapproximability results for routing disjoint paths with congestion [34].

The second family of lower bounds are based on communication complexity and are unconditional, and are the natural generalization of the argument outlined in Section I-B. They are motivated by the problem of designing simple mechanisms for welfare-maximization in combinatorial auctions, where by "simple" we mean that the bid space of a player is strictly smaller than its valuation space (which generally has size doubly exponential in the number of items m). The gist of our main result here is: *the worst-case POA of simple mechanisms is bounded below by the communication hardness of approximating of the underlying optimization problem.* This lower bound applies even to mechanisms that are computationally unbounded. Combining our results with known communication hardness results for welfare-maximization in combinatorial auctions gives senses in which certain well-studied mechanisms, such as simultaneous first-price auctions, are optimal among all mechanisms with a sub-doubly-exponential-size action space.

The final family of lower bounds is motivated by special cases of welfare-maximization in combinatorial auctions with succinctly described valuation functions, where there are no non-trivial communication lower bounds. Here, we prove conditional lower bounds on the POA of mechanisms that are "tractable," where the most stringent requirement for tractability is that a bidder can compute in polynomial time an approximate best response given mixed strategies of the other players. Note this is arguably the minimal condition under which the mixed Nash equilibrium concept is computationally (and perhaps conceptually) plausible. The essence of our main result here is: *the worst-case POA of tractable mechanisms is bounded below by the NP-hardness of approximating the underlying optimization problem.* These lower bounds are for the POA of approximate mixed Nash equilibria, and are conditional on $coNP \not\subseteq MA$.[8] The main idea here is to use the guaranteed existence of near-optimal approximate equilibria to probabilistically certify a $coNP$-hard welfare-maximization problem. As an example application, these results show that, if $coNP \not\subseteq MA$, then greedy combinatorial auctions have essentially optimal worst-case POA for welfare-maximization with single-minded bidders among all tractable mechanisms.

III. Discussion and Limitations

A. Exact vs. Approximate Equilibria

Almost all of our POA lower bounds are for approximate equilibria; in principle, exact equilibria could be much closer to an optimal outcome. There are three reasons why our lower bounds for approximate equilibria are, for all practical purposes, as good as lower bounds for exact equilibria.

First, every known technique for proving POA bounds for an equilibrium concept also proves approximately the same POA bound for the approximate version of the equilibrium concept. This includes the smoothness technique in [12] and its variants [41], [45], the recent "mimicking" technique of [27], [42], and any other proof technique that is based solely on elementary manipulations of the best response condition. Thus, if nothing else, our lower bounds limit what is provable by all known POA upper bound methods.

Second, for many natural classes of games and equilibrium concepts, a lower bound of ρ for ϵ-approximate equilibria can be translated into a lower bound of $\rho - f(\epsilon)$ for exact equilibria, where $f(\epsilon) \to 0$ as $\epsilon \to 0$. For example, given a congestion game and a high-cost approximate equilibrium σ of it, minor modifications to the game transmute σ into an exact equilibrium while only slightly reducing the POA.

[8] Recall that if $coNP \subseteq MA$ then the polynomial hierarchy collapses (see e.g. [35]).

TABLE II

INSTANTIATIONS OF OUR LOWER BOUNDS FOR SPECIFIC PROBLEMS. THE FIRST PROBLEM IS TO COMPUTE THE OPTIMAL ACTION PROFILE IN A CONGESTION GAME WITH COST FUNCTIONS THAT ARE POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS AND DEGREE AT MOST d. $\beta \in (0,1)$ IS A CONSTANT INDEPENDENT OF d. THE SECOND PROBLEM IS MINIMIZING THE WEIGHTED SUM OF COMPLETION TIMES ON RELATED MACHINES. THESE TWO LOWER BOUNDS ARE CONDITIONAL ON $NP \not\subseteq PPAD$ (FOR ϵ-MNE) AND ON $P \neq NP$ (FOR CE). THE LAST FIVE PROBLEMS ARE WELFARE-MAXIMIZATION IN COMBINATORIAL AUCTIONS WITH GENERAL, SUBADDITIVE, XOS, SUBMODULAR, AND SINGLE-MINDED BIDDER VALUATIONS (SEE SECTION VI-A FOR DEFINITIONS). OF THESE, THE FIRST FOUR LOWER BOUNDS ARE UNCONDITIONAL; THE FIFTH IS CONDITIONAL ON $coNP \not\subseteq MA$.

Problem Π	POA Lower Bound	POA Upper Bound	Refer to
Routing	$(\beta d)^{d/2}$ (via Thm I.5)	$\approx (d/\ln d)^{d+1}$ [5]	Thms V.1 and V.2
Scheduling	> 1 (via [36])	32/15 [37]	Thms V.1 and V.2
General Vals	$\Omega(m^{(1/2)-\epsilon})$ (via [38])	$O(\sqrt{m})$ [39]–[41]	Section VI-D
Subadditive Vals	2 (via [28])	2 [27], [42]	Section I-B
XOS Vals	$e/(e-1)$ (via [28])	$e/(e-1)$ [41]	Section VI-D
Submodular Vals	$2e/(2e-1)$ (via [43])	$e/(e-1)$ [41]	Section VI-D
Single-Minded	$\Omega(m^{(1/2)-\epsilon})$ (via [44])	$O(\sqrt{m})$ [39]–[41]	Theorem VII.1

Finally, suppose there was a class of games with a POA phase transition between exact and approximate equilibria (we know of no such classes). The consequent non-robust POA bound for exact equilibria would not be a convincing guarantee for actual behavior, and approximate equilibria would likely be the more relevant solution concept.

B. Bayes-Nash Equilibria

Many of our results are motivated by combinatorial auctions, where bidders' valuations are usually considered private information, and yet we study only full-information games. The reason is our focus on lower bounds — restricting attention to full-information games only makes them stronger.

C. Equilibria without Guaranteed Existence

There are two genres of POA analyses that appear ill-suited for complexity-theoretic lower bounds. The first is POA bounds for equilibrium concepts that are not guaranteed to exist, such as pure Nash equilibria (PNE). The issue is that equilibria might exist only in instances where the underlying optimization problem happens to be easy, and as such a POA bound that is conditional on existence can bypass the (worst-case) lower bounds for the problem. Exhibit A of this phenomenon is PNE in simultaneous first-price auctions (Section I-B). Such equilibria are only guaranteed to exist for very special classes of bidder preferences [46], yet they are optimal whenever they exist, no matter how complex bidders' preferences are [19]. By contrast, the underlying welfare-maximization problem is generally intractable (cf., Theorem I.12).

D. POA Lower Bounds without Intractability

The second genre of POA analyses that appears immune to our approach concerns inefficiency caused by idiosyncratic details of a mechanism's implementation. The canonical example here is the generalized second price (GSP) auction for sponsored keyword search. Understanding the POA of the GSP auction, as done in [47], is important because the auction is widely used in practice. Because the underlying optimization problem in sponsored search is not difficult in any sense, we do not expect our lower bound framework to be relevant.

IV. FORMALISM

A. Preliminaries

We consider finite games, with n players and finite action sets $\mathcal{A}_1, \ldots, \mathcal{A}_n$. A vector $\mathbf{a} \in \mathcal{A}$, where $\mathcal{A} = \prod_{i=1}^n A_i$, is called an *action profile* or *outcome*. In a *cost-minimization game*, each player i has a cost function $C_i : \mathcal{A} \rightarrow \mathcal{R}^+$. In a *utility-maximization game*, each player i has a utility function $u_i : \mathcal{A} \rightarrow \mathcal{R}^+$. Sections I-A and I-B provide concrete examples.

Several equilibrium notions are relevant to the present work. We next review a well-known hierarchy of four equilibrium concepts, each more permissive than the previous one. We phrase the definitions for utility-maximization games; analogous definitions apply to cost-minimization games.

A *pure Nash equilibrium (PNE)* is an action profile $\mathbf{a}$ such that no player can increase its payoff via a unilateral deviation: $u_i(\mathbf{a}) \geq u_i(a_i', \mathbf{a}_{-i})$ for every player i and $a_i' \in \mathcal{A}_i$. In many games, PNE do not exist.

A *mixed Nash equilibrium (MNE)* is a profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ of mixed actions (with σ_i a distribution over $\mathcal{A}_i$) such that no player i can increase its expected utility (over $\prod_{j \neq i} \sigma_j$) via a unilateral deviation. Every finite game has at least one MNE [23].

A *correlated equilibrium (CE)* is a distribution σ over $\mathcal{A}$ such that no player can increase its expected payoff via a conditional deviation: for every player i and $a_i, a_i' \in \mathcal{A}_i$,

$$\mathbf{E}_{\mathbf{a} \sim \sigma}[u_i(\mathbf{a}) \,|\, a_i] \geq \mathbf{E}_{\mathbf{a} \sim \sigma}[u_i(a_i', \mathbf{a}_{-i}) \,|\, a_i] \,.$$

A *coarse correlated equilibrium (CCE)* is a distribution σ over $\mathcal{A}$ such that, for every player i and $a_i' \in \mathcal{A}_i$,

$$\mathbf{E}_{\mathbf{a} \sim \sigma}[u_i(\mathbf{a})] \geq \mathbf{E}_{\mathbf{a} \sim \sigma}[u_i(a_i', \mathbf{a}_{-i})] \,. \tag{2}$$

Last but not least, a distribution σ over $\mathcal{A}$ is an *ϵ-CCE* if (2) holds for every i and $a_i' \in \mathcal{A}_i$ with an extra "$-\epsilon$" on the right-hand side. Approximate versions of the other equilibrium concepts are defined in the same way.[9]

[9]Some previous papers study multiplicative notions of approximate equilibria; this additive definition is the appropriate one for the present work.

Given a game, a nonnegative minimization objective function, and an equilibrium concept, the *price of anarchy (POA)* is defined as the worst-case (over equilibria σ) ratio between the expected objective function value of an equilibrium and the optimal objective function value of the game. For a maximization objective function, the POA is the reciprocal of this (so that the POA is always at least 1). Holding the game fixed, the POA can only increase with the equilibrium set. For example, for the equilibrium concepts defined above, the POA is the smallest for the set of PNE and the largest for the set of ϵ-CCE.

B. Game Plan Design

The problem of *game design* is defined formally as follows. Given is an instance I of an optimization problem, consisting of a set $\mathcal{F}$ of feasible solutions and an objective function defined on them. Also fixed is some choice of an equilibrium concept. A *game G on I* consists of:

1) a finite set of n players;
2) an action space $\mathcal{A}_i$ for each player i;
3) a payoff $u_i(\mathbf{a})$ for each outcome $\mathbf{a} \in \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$;
4) a function $\Lambda : \mathcal{A} \to \mathcal{F}$ from outcomes of G to feasible solutions of I.

Such a game is *onto I* if Λ is onto $\mathcal{F}$. A congestion game (Section I-A) or a S1A (Section I-B and Proposition I.10) can be thought of as a game onto an instance of the underlying cost-minimization or welfare-maximization problem.

The game design problem for I and a class of games $\mathcal{G}$ on I is: determine the game $G \in \mathcal{G}$ whose worst-case equilibrium most closely approximates the optimal solution to I. When every game $G \in \mathcal{G}$ is onto I — so every game $G \in \mathcal{G}$ has an outcome mapping to the optimal solution of I — this game design problem is equivalent to minimizing the price of anarchy over $G \in \mathcal{G}$.[10]

While the game design problem is non-trivial even for a single instance I, we are generally interested in designing games with small POA across *all* instances of an optimization problem Π. By a *game plan* for an optimization problem Π, we mean a function Γ from the each instance I of Π to a game $\Gamma(I)$ on I. We call a game plan *onto* Π if $\Gamma(I)$ is onto I for every $I \in \Pi$, and *H-bounded* if the players' utilities in every game $\Gamma(I)$ are always between $-H$ and H.

We briefly mention two well-studied examples of game plan design and defer further examples and a detailed discussion to the full version of the paper. First, the literature on coordination mechanisms for scheduling, beginning with [32], provides particularly transparent examples of game plan design problems. The goal here is to define local machine scheduling policies (and thereby players' utility functions) to minimize the POA; each choice of a local policy can be thought of as a game plan in the above sense. Second, the literature on identifying simple auctions with good equilibria (e.g. [18], [40]) is implicitly seeking out near-optimal solutions to natural game plan design problems, with each choice of an auction format inducing a game plan.

[10]There is of course the trivial game with a unique outcome (or more generally, a unique outcome in strictly dominant strategies) that is mapped to the optimal solution of I. We will always impose natural restrictions on $\mathcal{G}$ that rule out uninteresting solutions of this sort.

V. POA Lower Bounds for Polynomial Game Plans

Our first set of lower bounds are for the relatively simple case of games of polynomial size. As mentioned in Section II and detailed in the full version, these lower bounds are relevant for several types of congestion games, scheduling games, and distributed welfare games.

Formally, we say that a game plan Γ for an optimization problem Π is *polynomial* if:

1) the function Γ can be implemented as an algorithm that, given a description of an instance $I \in \Pi$, outputs a description of the game $\Gamma(I)$ in time polynomial in the description length of I;
2) for every $I \in \Pi$, the size of every action set in $\Gamma(I)$ is polynomial in the description length of I.

Computational complexity lower bounds for approximating an optimization problem translate to conditional lower bounds on the worst-case POA in games induced by polynomial game plans. The strength of the complexity assumption required depends on the equilibrium concept and on the queries supported by the games produced by the game plan, although all of the complexity assumptions employed in this section are weaker than $NP \neq coNP$. We next discuss two conditional POA lower bounds that follow from previous work on the complexity of equilibria; we give further lower bounds of this type in the full version.

The *Expected Utility (EU)* problem for a game G is the following: given a mixed strategy x_i for each player i of G, presented as an explicit list of probabilities over the actions $\mathcal{A}_i$, compute the expected utility $\mathbf{E}_{\mathbf{a}\sim\mathbf{x}}[u_i(\mathbf{a})]$ of every player. The EU problem is trivial for games represented in normal form (where utilities are listed individually for each $\mathbf{a} \in \mathcal{A}$) but can be non-trivial for succinctly represented games. We say that the EU problem is *tractable* in a family $\mathcal{G}$ of games if it can be solved in time polynomial in the description of the game G and mixed strategies $\mathbf{x}$. We say that it is *strongly tractable* if it can be solved by a bounded division-free straight-line program of polynomial length (see [29] for more details). While there are classes of games for which the EU problem is $\#P$-hard, it is strongly tractable for almost all of the succinctly represented games with polynomial-size action sets that have been considered in the algorithmic game theory literature. Examples include congestion games, scheduling games, network design games, and facility location games; see [29]–[31] for details.

The *Expected Objective (EO)* problem for a game G on an instance I is the following: given a mixed strategy x_i for each player i of G, presented as an explicit list of probabilities, compute the expected objective function value $\mathbf{E}_{\mathbf{a}\sim\mathbf{x}}[\Gamma](\mathbf{a})$, where Γ is the mapping from action profiles of G to feasible solutions of I. The same techniques used to solve the EU problem for various games in [29]–[31] can also be used to solve the EO problem for those games.

The following result is a consequence of [29, Theorem 2] and a proof similar to that of Theorem I.1.

Theorem V.1 *Let Π be an optimization problem such that the optimal objective function value is NP-hard to approximate better than a ρ factor. Assume that $NP \not\subseteq PPAD$ and that Γ is a polynomial and H-bounded game plan onto Π that induces games for which the EU problem is strongly tractable and the EO problem can be solved in polynomial time. Then, for every constant $\epsilon > 0$, the worst-case POA of ϵH-MNE in games induced by Γ is at least ρ.*

Similarly, the following result is a consequence of [31, Theorem 4.1] and a proof similar to that of Theorem I.1. Its assumptions are weaker (relaxing $NP \not\subseteq PPAD$ to $P \neq NP$ and strong tractability of EU to tractability) and its conclusion is incomparable (a lower bound for exact CE rather than ϵH-MNE).

Theorem V.2 *Let Π be an optimization problem such that the optimal objective function value is NP-hard to approximate better than a ρ factor. Assume that $P \neq NP$ and that Γ is a polynomial game plan onto Π that induces games for which the EU problem is tractable and the EO problem can be solved in polynomial time. Then, the worst-case POA of CE in games induced by Γ is at least ρ.*

VI. The Price of Anarchy in Simple Auctions: Lower Bounds from Communication Complexity

This section uses communication complexity arguments to give unconditional lower bounds on the worst-case POA of approximate MNE.

A. Welfare Maximization in Combinatorial Auctions

Recall from Section I-B the problem of welfare-maximization in combinatorial auctions. It is conventional and useful to parameterize the problem by a set of allowable valuation functions; see also [48]. Four well-studied special cases, in decreasing order of generality, are:

1) *General.* The only assumptions here are that valuation functions are nonnegative, nondecreasing (i.e., $S \subseteq T$ implies $v_i(S) \leq v_i(T)$), with $v_i(\emptyset) = 0$.
2) *Subadditive.* Recall from Section I-B that a subadditive valuation function v_i satisfies: for every pair $S, T \subseteq U$ of bundles, $v_i(S \cup T) \leq v_i(S) + v_i(T)$.
3) *XOS.* An XOS valuation function v_i can be represented as the pointwise maximum of additive valuations: $v_i(S) = \max_{\ell=1}^{r} \sum_{j \in S} z_j^{\ell}$, where each $z^{\ell} : U \to \mathcal{R}^+$ is an additive valuation over U.
4) *Submodular.* A submodular valuation function v_i satisfies: for every pair $S \subseteq T$ and $j \notin T$, $v_i(T \cup \{j\}) - v_i(T) \leq v_i(S \cup \{j\}) - v_i(S)$.

As in Section I-B, we assume that every valuation is an integer and bounded above by a value $V_{\max}$ that is polynomial in n and m; the relevant communication and computational lower bounds continue to hold with these assumptions.

B. Mechanisms

Let $\Pi(n, m, \mathcal{V})$ denote the welfare-maximization problem with m items and n bidders with valuations in $\mathcal{V}$. We identify instances I of $\Pi(n, m, \mathcal{V})$ with valuation profiles $\mathbf{v} \in \mathcal{V}^n$. By a *mechanism* $\mathcal{M}$ for $\Pi(n, m, \mathcal{V})$, we mean a game plan that meets the following conditions.

1) For every game $\mathcal{M}(\mathbf{v})$ induced by an instance $\mathbf{v} \in \Pi(n, m, \mathcal{V})$, the players of $\mathcal{M}(\mathbf{v})$ correspond to the n bidders of $\mathbf{v}$.
2) For every player i, the action set A_i is finite and independent of $\mathbf{v}_{-i}$.
3) For every player i and instances $\mathbf{v} = (v_i, \mathbf{v}_{-i})$ and $\mathbf{v}' = (v_i, \mathbf{v}'_{-i})$, the utility functions $u_i : \mathcal{A} \to \mathcal{R}$ in $\mathcal{M}(\mathbf{v})$ and in $\mathcal{M}(\mathbf{v}')$ are the same. That is, each bidder i's utility function in $\mathcal{M}(\mathbf{v})$ is independent of $\mathbf{v}_{-i}$.
4) The map Λ from action profiles $\mathcal{A}$ to feasible solutions is the same for every induced game $\mathcal{M}(\mathbf{v})$. (Because n and m are fixed, every instance $\mathbf{v} \in \Pi(n, m, \mathcal{V})$ has the same set of feasible solutions, the allocations of the m items to the n bidders.)

Encoding a S1A (Section I-B), for example, as such a mechanism is straightforward. We are not aware of any reasonable auction formats for welfare-maximization in combinatorial auctions that do not satisfy the above properties. There are certainly unreasonable auction formats that do — for instance, we do not require that the map Λ or the players' utilities can be evaluated in a reasonable amount of time.

C. Low-Communication Protocols

Generalizing the proof of Theorem I.11 yields the following.

Theorem VI.1 *Let $\Pi(n, m, \mathcal{V})$ be a welfare-maximization problem with maximum valuation $V_{\max}$ and $\mathcal{M}$ a $V_{\max}$-bounded mechanism onto $\Pi(n, m, \mathcal{V})$ such that, in every game $\mathcal{M}(\mathbf{v})$ induced by $\mathcal{M}$, the POA of $\epsilon V_{\max}$-MNE is at most ρ. Let N denote the largest number of actions available to a player in $\mathcal{M}(\mathbf{v})$.*

Then, there is a nondeterministic protocol that uses communication polynomial in n, $\frac{1}{\epsilon}$, $\log N$, and $\log V_{\max}$, and for every W distinguishes between instances of $\Pi(n, m, \mathcal{V})$ with optimal welfare at least W and instances of $\Pi(n, m, \mathcal{V})$ with optimal welfare less than W/ρ.

We show in the full version of the paper that, under the stronger assumption that all ϵ-CCE of the induced games $\mathcal{M}(\mathbf{v})$ are near-optimal, we can replace the nondeterministic communication protocol of Theorem VI.1 with a randomized protocol with one-sided error. The proof involves a low-communication simulation of no-regret algorithms.

D. Implications for Simple Auctions

We saw in Section I-B how Theorem VI.1, in conjunction with known exponential lower bounds for nondeterministic communication protocols that approximately maximize welfare with subadditive bidder valuations, implies that there is no "simple" mechanism — meaning sub-doubly-exponential (in m) action spaces — that only induces games with worst-case POA (of approximate MNE) better than 2. Analogous results are known for several other valuation classes.

1) Combining Theorem VI.1 with [38, Theorem 3] proves a lower bound of $\min\{n, m^{(1/2)-\epsilon}\}$ on the worst-case POA of simple mechanisms with general bidder valuations, where $\epsilon > 0$ is an arbitrarily small constant.
2) Combining Theorem VI.1 with [28, Theorem 4.2] proves a lower bound of $e/(e-1)$ on the worst-case POA of simple mechanisms with XOS bidder valuations.
3) Combining Theorem VI.1 with [43, Theorem 1.1] proves a lower bound of $2e/(2e-1)$ on the worst-case POA of simple mechanisms with submodular bidder valuations.

As in Section I-B, these lower bounds apply to the worst-case POA of ϵ-MNE, where ϵ can be as small as inverse polynomial in n and m.

For general valuations, the POA upper bounds for simple greedy combinatorial auctions in [39]–[41] show that this lower bound is nearly tight.[11] For XOS valuations, the POA upper bound for S1A's in [27] shows that the lower bound of $e/(e-1)$ is tight,[12] and therefore S1A's minimize the POA over all simple mechanisms in this setting.

Sudmodular valuations pose an intriguing challenge. The lower bound of $e/(e-1)$ in [6] on the POA of S1A's applies also to bidders with submodular valuations. Our lower bound is incomparable: it is smaller (only $2e/(e-1)$) but applies to all simple mechanisms. Since Feige and Vondrak [49] prove that an approximation ratio (slightly) better than $e/(e-1)$ can be achieved for submodular bidder valuations with communication polynomial in n and m, Theorem VI.1 is incapable of proving a lower bound of $e/(e-1)$ for all simple mechanisms. Is there a simple mechanism with worst-case POA strictly less than $e/(e-1)$ when bidders have submodular valuations?

VII. The Price of Anarchy in Tractable Auctions: Lower Bounds from Computational Complexity

This section considers the welfare-maximization problem in combinatorial auctions with bidders that have succinctly described valuations. Communication lower bounds are not relevant for such problems. To derive (conditional) lower bounds for the worst-case POA of "simple" mechanisms, we need to impose computational restrictions on mechanisms.[13]

We call a mechanism $\mathcal{M}$ for $\Pi(n, m, \mathcal{V})$ with valuations bounded by $V_{\max}$ *tractable* if $u_i(\mathbf{a})$ and $\Lambda(a)$ can be evaluated in time polynomial in n, m, and $V_{\max}$ for every action profile $\mathbf{a}$ of every induced game $\mathcal{M}(\mathbf{v})$.[14] We say that $\mathcal{M}$ *supports approximate best responses* if in every game $\mathcal{M}(\mathbf{v})$ induced by $\mathcal{M}$, $\max_{a \in \mathcal{A}_i} \mathbf{E}_{\mathbf{a}_{-i} \sim \mathbf{y}_{-i}}[u_i(a_i, \mathbf{a}_{-i})]$ can be computed, with probability at least $1/n^2$ and up to additive loss $\epsilon V_{\max}$, in polynomial time (in $n, m, \frac{1}{\epsilon}, \log V_{\max}$) whenever $\mathbf{y}_{-i}$ is a product distribution $\prod_{j \neq i} y_j$ with marginals y_j that can be sampled from in polynomial time.[15] Supporting best responses is arguably a minimal condition for the MNE equilibrium concept to be meaningful in the game induced by a mechanism — otherwise, players can't efficiently realize that they are playing an equilibrium.

We now prove a conditional lower bound on the POA of $\epsilon V_{\max}$-MNE of games induced by tractable mechanisms that support approximate best responses. The idea is to use the guaranteed existence of near-optimal $\epsilon V_{\max}$-MNE to decide nondeterministically a $coNP$-complete problem.

Theorem VII.1 *Let $\Pi(n, m, \mathcal{V})$ denote a welfare-maximization problem with maximum valuation $V_{\max}$ such that, for some $W \geq V_{\max}$ and ρ, it is NP-hard to decide whether or not an instance $\mathbf{v}$ has optimal welfare at least W (a "yes" instance) or optimal welfare at most W/ρ (a "no" instance). Assuming $coNP \not\subseteq MA$, for every tractable $V_{\max}$-bounded mechanism $\mathcal{M}$ that supports approximate best responses and is onto $\Pi(n, m, \mathcal{V})$, for all constants $\epsilon, \delta > 0$, the worst-case POA of $\epsilon V_{\max}$-MNE in games induced by $\mathcal{M}$ is at least $\rho - \delta$.*

Theorem VII.1 transfers known NP-hardness results for welfare-maximization with bidders with succinctly described valuations — examples include single-minded, coverage, and budgeted additive valuations — to lower bounds on the POA of ϵ-MNE of tractable mechanisms that support approximate best responses, assuming that $coNP \not\subseteq MA$.

VIII. Conclusion

This paper described several senses in which communication and computational lower bounds for algorithms for optimization problems translate to lower bounds on the price of anarchy in games derived from these problems. The most straightforward use of our lower bound framework is to prove POA lower bounds using reductions instead of explicit constructions. Such lower bounds can apply both to specific classes of games of interest, and more generally to all game plans defined on a hard optimization problem. Knowing such a fundamental limit on what is possible draws a "line in the sand" for positive results. Just as the holy grail in polynomial-time approximation algorithm design is to match the best-known inapproximability result, we can now propose that the natural goal in game design is to achieve a worst-case POA that matches the corresponding (conditional or unconditional) lower bound identified in this paper.

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[11] Similar lower bounds for the special cases of S1A's and greedy auctions are given in [20] and [40], respectively.

[12] For the special case of S1A's, this lower bound was proved previously by Christodoulou et al. [6] using an explicit construction.

[13] Otherwise, we could take $\mathcal{M}$ to be a direct-revelation mechanism with an NP-hard map Λ, like the VCG mechanism.

[14] Formally, throughout this section we should speak of problems $\Pi(n, m, \mathcal{V})$ and families of mechanisms $\mathcal{M}$ parameterized by n, m, and $\mathcal{V}$, with $n, m \to \infty$.

[15] For example, in an S1A, computing a best response corresponds to a randomized generalization of a demand query.

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