CS364A: Problem Set #2

Due in class on Thursday, October 23, 2008

Instructions:

(1) Students taking the course for a letter grade should attempt all of the following 5 problems; those taking the course pass-fail should attempt the first 3.

(2) Some of these problems are difficult. I highly encourage you to start on them early and discuss them extensively with your fellow students. If you don't solve a problem to completion, write up what you've got: partial proofs, lemmas, high-level ideas, counterexamples, and so on. This is not an IQ test; we're just looking for evidence that you've thought long and hard about the material.

(3) You may refer to your course notes, and to the textbooks and research papers listed on the course Web page *only*. You cannot refer to textbooks, handouts, or research papers that are not listed on the course home page. Cite any sources that you use, and make sure that all your words are your own.

(4) Collaboration on this homework is *strongly encouraged*. However, your write-up must be your own, and you must list the names of your collaborators on the front page.

(5) No late assignments will be accepted.

Problem 6

In lecture we alluded to the fact that, for Bayesian-optimal mechanism design, "sufficient competition" can obviate the need for a reserve price. This problem demonstrates one way of making this statement precise.

(a) (3 points) Consider a distribution F that is regular in the sense of Lecture #4, and let φ denote the corresponding virtual valuation function. Prove that the expected virtual value $\varphi(v_i)$ of a valuation v_i drawn from F is zero.

(b) (7 points) Consider selling $k \geq 1$ identical items to bidders with valuations drawn i.i.d. from F. Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n + k$ bidders is at least that of the Bayesian-optimal auction for F with n bidders.

[Thus, modest additional competition is at least as valuable as knowing the distribution F and employing a corresponding optimal reserve price.]

[Hints: To develop intuition explore the case of $k = n = 1$ and $F(x) = x$ on $[0, 1]$. In general, use Myerson's characterization of the expected revenue of a truthful auction. Condition on the values of the first n bidders and then argue about the expected impact of the final k bidders on the revenue of the no-reserve Vickrey auction, using part (a).]

Problem 7

This problem explores variations on the VCG mechanism and some applications.

(a) (10 points) Recall that a VCG mechanism is parametrized by "pivot terms" $h_i(b_{-i})$ — a "shift" in player i's payment that depends only on the bids of the other players (and that therefore does not affect truthfulness). In lecture we showed how to choose pivot terms to simultaneously ensure nonnegative payments (from the players to the mechanism) and "individual rationality" (i.e., $p_i \leq b_i(\omega^*)$, where ω^* is the outcome chosen by the mechanism). We now consider dropping these two requirements and instead insisting on *budget-balance*, meaning that the sum of the payments (which can now be positive *or* negative) equals zero no matter what the players bid.

Formally, consider a general mechanism design problem with outcome set Ω, n players, and valuations (each a function from Ω to $\mathcal{R}^+$) drawn from a set $\mathcal{V}$. We will say that the pivot terms $\{h_i(\cdot)\}_{i=1}^n$ *budget-balance the VCG mechanism* if, for all possible bids $b_1(\cdot), \ldots, b_n(\cdot) \in \mathcal{V}$, the corresponding VCG payments (including the h_i's) sum to 0.

Some mechanism design problems admit such pivot terms, while others do not. Precisely, prove that there exist pivot terms that budget-balance the VCG mechanism *if and only if* the maximum-possible efficiency w.r.t. the bids can be represented as the sum of bid-independent functions — i.e., if and only if we can write

$$\max_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega) = \sum_{i=1}^{n} g_i(b_{-i}) \tag{1}$$

for every $b_1(\cdot), \ldots, b_n(\cdot) \in \mathcal{V}$, where each g_i is some function that does not depend on b_i.

[Hint: You might start with the standard VCG payments described in lecture and try summing them up. The sum should simplify so as to involve only a multiple of the left-hand side of (1) and bid-independent terms.]

(b) **(5 points)** Either directly or using part (a), prove that for a single-good auction with at least two bidders, there are no pivot terms that budget-balance the Vickrey auction.

(c) **(5 points)** Again consider a general mechanism design problem, with a set Ω of outcomes, and n players, where player i has a private real-valued valuation $v_i(\omega)$ for each outcome $\omega \in \Omega$. Suppose the function $f : \Omega \rightarrow \mathcal{R}$ has the form

$$f(\omega) = c(\omega) + \sum_{i=1}^{n} w_i v_i(\omega),$$

where c is a publicly known function of the outcome, and where each w_i is a nonnegative, public, player-specific weight. Such a function is called an *affine maximizer.*

Show that for every affine maximizer objective function f and every subset $\Omega' \subseteq \Omega$ of the outcomes, there is a truthful mechanism that optimizes f over Ω'.

[Hint: modify the VCG mechanism. Don't worry about individual rationality.]

(d) **(3 points)** For the rest of this problem, consider a combinatorial auction with a set S of m goods and n bidders. Assume that the valuation $v_i(\cdot)$ of bidder i depends only on its bundle T_i of goods; that it is nondecreasing (so $T_1 \subseteq T_2$ implies that $v_i(T_1) \leq v_i(T_2)$); that $v_i(\emptyset) = 0$; and that v_i is *subadditive*, meaning that $v_i(T_1) + v_i(T_2) \geq v_i(T_1 \cup T_2)$ for every pair T_1, T_2 of disjoint subsets of goods.

In this and the next two parts, we consider only the winner determination problem (i.e., we don't worry about payments or truthfulness, just polynomial-time surplus maximization). Given S and $v_1, \ldots, v_n$, call the winner determination problem *lopsided* if there is an optimal allocation of goods in which at least half of the total surplus of the allocation is due to players that were allocated a bundle with at least $\sqrt{m}$ goods. (I.e., if $2\sum_{i \in A} v_i(T_i^*) \geq \sum_{i=1}^{n} v_i(T_i^*)$, where $\{T_i^*\}$ is the optimal allocation and A is the subset of bidders i with $|T_i^*| \geq \sqrt{m}$.)

Show that in a lopsided problem, there is an allocation that gives all of the goods to a single player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible surplus.

(e) **(4 points)** Show that in a problem that is not lopsided, there is an allocation that gives at most one good to each player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible surplus.

[Hint: use subadditivity.]

(f) (4 points) Give a polynomial-time $O(\sqrt{m})$-approximate winner determination algorithm for subadditive valuations.

[Hint: make use of a graph matching algorithm.]

(g) (4 points) Give a polynomial-time, $O(\sqrt{m})$-approximate, truthful combinatorial auction for subadditive valuations.

[Hint: use part (c).]

Problem 8

This problems considers auctions that provide revenue guarantees of various forms.

(a) (3 points) Consider a digital goods auction (n bidders, n identical goods) with a twist: the auctioneer incurs a fixed production cost of 1 if there is at least one winner; if no goods are sold, then no such cost is incurred. Call an auction for this problem *budget-balanced* if, whenever there is at least one winner, the prices charged to the winners sum to exactly the cost incurred (namely, 1). Define the *surplus* of an outcome with winners S to be 0 if $S = \emptyset$ and $-1 + \sum_{i \in S} v_i$ otherwise.

Note that the surplus can be truthfully maximized in this problem using the extension of the VCG mechanism described in Problem 7(c). Prove that with the standard VCG payments (in which losers pay 0), the VCG mechanism is not budget-balanced — in fact, it can generate 0 revenue even when the auctioneer incurs cost 1.

(b) (2 points) Explain how to instantiate the ProfitExtract subroutine from Lecture #5 to obtain a non-trivial truthful, budget-balanced mechanism for the above problem (in which losers pay 0 and winners pay at most their bids).

(c) (8 points) The mechanism in (b) does not generally maximize the surplus. Precisely, show that the largest-possible difference (over all possible valuation profiles $\mathbf{v}$) between the maximum surplus and the surplus achieved by this mechanism is exactly

$$-1 + \sum_{i=1}^{n} \frac{1}{i},$$

which is roughly $\ln n - 1$.

(d) (5 points) We can generalize the result in (b) as follows. Consider a digital goods auction with players N, in which the auctioneer incurs a (publicly known) cost of $C(S)$ when the set of winners is $S \subseteq N$. Assume that $C(\emptyset) = 0$, that C is nondecreasing (meaning $C(S) \leq C(T)$ whenever $S \subseteq T$), and that C is *submodular*, meaning that

$$C(T \cup \{i\}) - C(T) \leq C(S \cup \{i\}) - C(S)$$

whenever $S \subseteq T$ and $i \notin T$. (This is a set-theoretic type of "marginal returns". For example, when $C(S)$ depends only on $|S|$, submodularity becomes discrete concavity.)

The *Shapley value of i in S*, denoted $\chi_{Sh}(i, S)$, is defined as follows. For an ordering π of the players of S, let T_π denote those preceding i in π. Then $\chi_{Sh}(i, S) := \mathbf{E}_\pi[C(T_\pi \cup \{i\}) - C(T_\pi)]$, where π is chosen uniformly at random. In other words, assuming that the players of S are added to the empty set 1-by-1 in a random order, $\chi_{Sh}(i, S)$ is the expected jump in cost caused by i's arrival.

Prove that, under the assumptions on C above, $\chi_{Sh}(i, S) \geq \chi_{Sh}(i, T)$ whenever $S \subseteq T$.

(e) (7 points) By using Shapley values as prices, generalize the budget-balanced truthful mechanism in (b) to a digital goods auction with an arbitrary nondecreasing, submodular cost function. Be sure to prove that your mechanism is truthful.

(f) (up to 8 points extra credit) Part (c) characterized the worst-case surplus loss of the mechanism in (b). Say whatever you can about an analogous characterization for the mechanisms in part (e), parametrized by the particular nondecreasing submodular cost function C.

Problem 9

Consider the following pricing problem. There is one consumer who wants at most one of n non-identical goods. Assume that the consumer's private valuations $v_1, \ldots, v_n$ for the n goods are i.i.d. draws from a known regular prior distribution F. Our goal is to set prices $p_1, \ldots, p_n$ for the n goods (which can depend on F but not the actual v_i's) to maximize expected revenue, assuming that the consumer responds to prices by picking the good that maximizes $v_i - p_i$ (or picking no good if $p_i > v_i$ for every i).

(a) (7 points) Prove that the maximum-achievable expected revenue is bounded above by the expected revenue of an optimal single-good auction with n bidders with valuations drawn i.i.d. from F.

(b) (8 points) Design a simple pricing algorithm that (for every regular distribution F) obtains expected revenue at least a constant fraction of that of an optimal single-good auction with n bidders with valuations drawn i.i.d. from F. Do your best to optimize the constant.

Problem 10

In a *procurement auction*, the tables are turned: there is a single buyer and multiple sellers. Each seller i has an intrinsic and private value for their good (which we will call a *cost* c_i), and the buyer wants to obtain a desired subset of goods. In the simplest case, the buyer needs to acquire one good — and, subject to this hard constraint, wants to pay as little as possible. The analog of the (truthful) second-price auction is to award the buyer the cheapest good and charge the buyer the second-lowest reported cost.

We consider a stylized but educational more complex example. Let $G = (V, E)$ be an undirected graph. Each edge e is a seller and has a private cost c_e, while the buyer is required to buy a spanning tree. Assume that G is 2-edge-connected (i.e., at least two edges cross every cut). You can also assume that all reported edge costs are distinct.

(a) (6 points) We consider only the efficient allocation rule, which in this context picks the MST of G (according to the reported costs of the edges). Prove that the unique truthful payments (from the buyer to the edges) that always pay zero to unpicked edges are the following: for every edge e of the chosen MST T, let f denote the cheapest edge other than e that crosses the (unique) cut induced by the two connected components of $T - \{e\}$; then the payment from the buyer to edge e is c_f.

(b) (8 points) Fix distinct edge costs for G, let T_1 be the corresponding MST, and T_2 the cheapest spanning tree edge-disjoint from T_2. (Assume that such a tree exists.) Construct a bipartite graph $H = (U, W)$ in which vertices of U and W correspond to the edges of T_1 and T_2, respectively. Include edge (e_1, e_2) in H if and only if $T_1 - \{e_1\} \cup \{e_2\}$ is a connected subgraph (and hence a spanning tree) of G. Prove that H has a perfect matching.

[Hint: You should assume and use *Hall's Theorem*, which states that a bipartite graph $H = (U, W)$ with $|U| = |W|$ has a perfect matching if and only if for every subset $S \subseteq U$ with neighbors $\Gamma(S) \subseteq W$ on the other side, $|S| \leq |\Gamma(S)|$. (The "only if" direction is trivial; the "if" direction is not.)]

(c) (4 points) Using parts (a) and (b), prove that the total payment charged by the mechanism in (a) to the buyer is at most that of the cheapest spanning tree that is edge-disjoint from the MST.

(d) (7 points) In contrast, suppose a buyer is required to buy a path from vertex s to vertex t in a directed graph G with n vertices (again in which edges are sellers with private costs). Prove that the (unique) mechanism that implements the efficient allocation (i.e., that always selects a shortest path P) might charge the buyer $\Omega(n)$ times the length of the shortest path that is disjoint from the chosen path P.