CS364A: Problem Set #3

Due in class on Thursday, November 30, 2006

Instructions:

(1) You may refer to your course notes, general references (e.g., textbooks), and material on the course Web site, but *not* to additional specific research papers.

(2) Collaboration on this homework is actively encouraged. However, your write-up must be your own, and you must list the names of your collaborators on the front page.

(3) Grades will be assigned on a plus/check/minus scale.

Problem 0

Don't forget that a 1-3 page outline of your project report is due by Friday, November 17th at midnight. (Send it to the instructor via email.) See `http://theory.stanford.edu/~tim/f06/instr.html` for more detailed instructions.

Problem 1

Recall from class the Pigou bound $\alpha(\mathcal{C})$ for a set $\mathcal{C}$ of cost functions. Prove that if $\mathcal{C}$ is the set of nonnegative, nondecreasing, concave functions, then $\alpha(\mathcal{C}) = 4/3$.

[Hint: one approach is to first reduce the concave cost function case to the affine cost function case.]

Problem 2

Recall from class the AAE example, which showed that the price of anarchy can be as large as 5/2 in unweighted atomic selfish routing instances. (*Unweighted* means that all players control one unit of traffic; in a weighted instance, these traffic amounts can be arbitrary.)

(a) Modify the players' weights in the AAE example so that the price of anarchy in the resulting weighted atomic instance is precisely $(3 + \sqrt{5})/2 \approx 2.618$.

(b) Can you devise an unweighted atomic instance with 3 players, affine cost functions, and price of anarchy equal to 5/2? Can you achieve a price of anarchy of $(3 + \sqrt{5})/2$ using 3 players and variable weights?

(c) What is the largest price of anarchy in atomic instances with affine cost functions and only 2 players?

(d) **Extra Credit:** Prove an upper bound strictly smaller than $(3 + \sqrt{5})/2$ for the price of anarchy in unweighted atomic selfish routing networks with affine cost functions.

Problem 3

Prove that every atomic selfish routing game with affine cost functions ($c_e(x) = a_e x + b_e$ with $a_e, b_e \geq 0$) admits at least one equilibrium, even if different players control different amounts of traffic. Make use of the following potential function:

$$\Phi(f) = \sum_{e \in E} \left(c_e(f_e) f_e + \sum_{i \in S_e} c_e(r_i) r_i \right),$$

where S_e denotes the set of players that choose a path in f that includes the edge e.

Problem 4

In this problem we will investigate the convergence of best-response dynamics in variants of the load-balancing model studied in Problem 5 of HW#2. Recall that "best-response dynamics" means that you begin at an arbitrary assignment, and as long as the current assignment is not a (pure-strategy) Nash equilibrium, an arbitrary player who can benefit by deviating is allowed to switch machines. A deviating player is assumed to switch to its best option (e.g., in the most basic load-balancing model, a machine with the lightest load).

(a) Suppose that all players have the same (unit) weight but players can have different cost functions. Formally, each player j incurs a cost $c_i^j(k)$ on machine i if it is among k players assigned to i. Assume that for each fixed j and i, $c_i^j(k)$ is nondecreasing in k.

Solve two of the following four problems for full credit. You will receive extra credit for each problem after the first two that you solve.

(i) Prove that best-response dynamics need not converge.

(ii) Prove that, despite (i), a pure-strategy Nash equilibrium always exists.

(iii) Give a polynomial-time algorithm for computing a pure-strategy Nash equilibrium.

(iv) Show that if there are only two machines, then best-response dynamics converges to a pure-strategy Nash equilibrium.

(b) Now suppose that all players have the same cost function but can have different weights. Consider restricting best-response dynamics so that at each step, if there are several players who want to deviate, then only the player with largest weight is allowed to deviate (ties are broken arbitrarily). Prove that this process converges to a pure-strategy Nash equilibrium after at most n iterations.

(c) **Extra credit:** For the same model as in (b), does (unrestricted) best-response dynamics converge to a pure-strategy Nash equilibrium in time polynomial in the input (the number of players, the number of machines, and the logarithm of the maximum player weight)?