Coordination Sequences, Planing Numbers, and Other Recent Sequences

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The OEIS Foundation,
Highland Park, NJ

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1 2 4 8 16 32 64 128 256 512

1024 2048 4096 8192 16384

32768 3 6 12 24 48 96 192

384 768 1536 3072 61 1 2 4 8

Periodic, easy - explain!

Outline

- Would not have become a mathematician w/o OEIS
- The succession question: need VP
- Claude Lenormand et le rabot
- “Choix de Bruxelles”
- Coordination sequences
- Knight’s move Ulam-Warburton cellular automaton
- Some recent sequences and unsolved problems

From XXX Mar 19 2018, Subject: Reminiscence from a young mathematician

Dear Neil, The other day, I had the occasion to use the OEIS, something I haven't done in nearly 15 years (as an algebraic geometer, I don't seem to get that many opportunities)! I was so happy to see it thriving.

I wanted to relay a bit of nostalgia and my heartfelt thanks. Back in the late 1990s, I was a high school in Oregon. While I was interested in mathematics, I had no significant mathematically creative outlet (working class family and subpar mathematics instruction) until I discovered the OEIS in the course of trying to invent some puzzles for myself. I remember becoming a quite active contributor through the early 2000s, and eventually at one point, an editor. My experience with the OEIS, and the eventual intervention of one of my high school teachers, catalyzed my interest in studying mathematics, which I eventually did at XXX College. I went on to a Ph.D. at the University of XXX, various postdocs, and am currently at XXX.

I wanted to thank you for seriously engaging with an 18 year old kid, even though I likely submitted my fair share of mathematically immature sequences.

I doubt I would have become a mathematician without the OEIS!

The Succession Question

Very Important!
Looking for suggestions for Vice-President

(1926-)

(1948-)

(1982-)

(Prince William rather than Prince Charles - Hilarie Orman)

Claude Lenormand

When OEIS reached 100,000 sequences in 2004 (also its 40th birthday), we had an e-party (see OEIS Wiki). 28 countries, 150 guests.

Claude Lenormand
St-Thibault, France
Aug 15, 2004
Longue vie à vous!

(Today, 2019, 15 years later, 320,000 sequences. 15,000/year.)

60 contributions from Lenormand, 2001-2003

Claude Lenormand, letter, November 2003

Deux transformations sur les mots

1. **RUNS transform**
2. **“Raboter”, to plane**

1. **RUNS:**

HHHTTHTTH… becomes 3212…

Kolakoski   A2 = 1,2,2,1,1,2,1,2,2,… is fixed   **(A mystery)**

Golomb A1462 = 1,2,2,3,3,4,4,4,5,5,5,6,6,6,6,7,… is fixed

a(n) = const.*n^(phi-1) + tiny, phi = golden ration

Are the two hybrids A156253 and A321020 analyzable?

Claude Lenormand (page 3)

2. "Raboter" or planing a word

Shorten each run of symbols by 1 term

un rabot

Golomb's 1 2 2 3 3 4 4 4 5 5 5 6 6 6 6 7 7 7 7 8 ...

becomes

2 3 4 4 5 5 6 6 6 7 7 7 8 ...

A319434. Formula?

12 = 1100_2 becomes 10_2 = 2

APPLY 'RABOTER' TO BINARY EXPANSION OF n

n	raboter(n)
0	ϵ
1	ϵ
10	ϵ
11	1
100	0
101	ϵ
110	1
111	3
1000	0
1001	0
1010	ϵ
1011	1
1100	2
1101	1
1110	3
1111	7
10000	0

ϵ = EMPTY STRING

raboter(n) = { ~~A318921~~ (ϵ = 0); A319419 (ϵ = −1) }

steps to reach ϵ : A319416
= "cuts-resistance of n"

Theorem ~~Conjecture~~: For $2^k \le n < 2^{k+1}$, average value of raboter(n) = $\left(\frac{3}{2}\right)^{k-1} - \frac{1}{2}$

(A27649)

Proof from D. Zeilberger and Chai Wah Wu

Conjecture 2: For $2^k \le n < 2^{k+1}$, ave. cuts-resistance of n is A189391(n) and $\sim \sqrt{\frac{8n}{\pi}}$

(see A319421)

Claude Lenormand (page 4)

The inverse operation: lengthen all runs by 1

12 = 1100 becomes 111000 = 56

A175046 says what happens to n (Leroy Quet, 2009)

3,12,7,24,51,...

This is an inverse to raboter.

Theorem: expand(n) <= (9 n^2 + 12 n)/5
with = iff n = 101010...10 in binary
(me, proved by Maximilian Hasler)

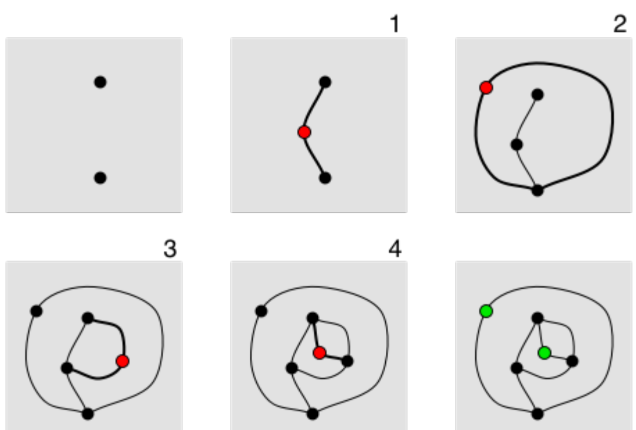
(Chai Wah Wu, arXiv, recent)

Conjecture: Average of expand(n) for n < 2^k is 2^k(4.3^k-1)

PLAY THE DIRGE

Choix de Bruxelles

Eric Angelini, Lars Blomberg, Remy Sigrist, NJAS



Game of sprouts

Choux de Bruxelles

Eric Angelini (Bruxelles)

Choix de Bruxelles (2)

A new operation on numbers (January 14 2019)

20218

Can halve if even or double

20218 goes to your choice of

10218
40218
20118
20418
20228
20214
202116 !

20428
2029 !
20236
10118
40418
20109
20436
40428
10109
40436

If a goes to b then also b goes to a

16 goes to any of

16, 26, 13, 112, 8, 32

Choix de Bruxelles (3)

1 — 2 — 4 — 8 — 16 — any of { 13, 26, 32, 112 }

A323286 = what n goes to in one step

Going from 1 to 3 takes 11 steps:

1 2 4 8 16 112 56 28 14 12 6 3

(Lorenzo Angelini)

A323454 = number of steps to reach n from 1 (or -1 if can't)

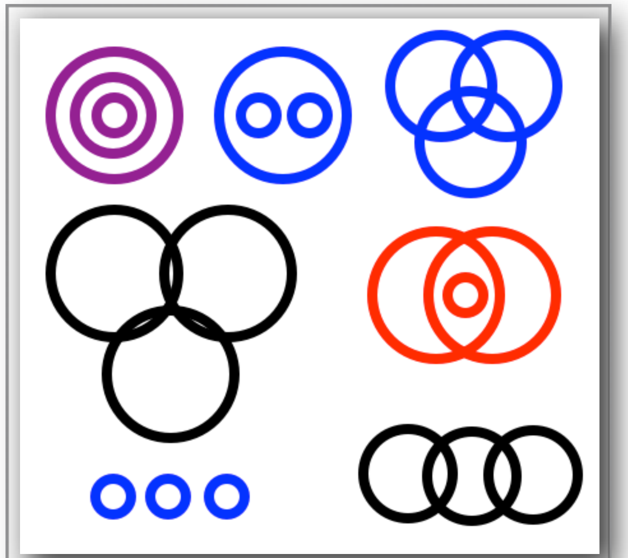
Theorem: Can reach n from 1 iff n does not end in 0 or 5

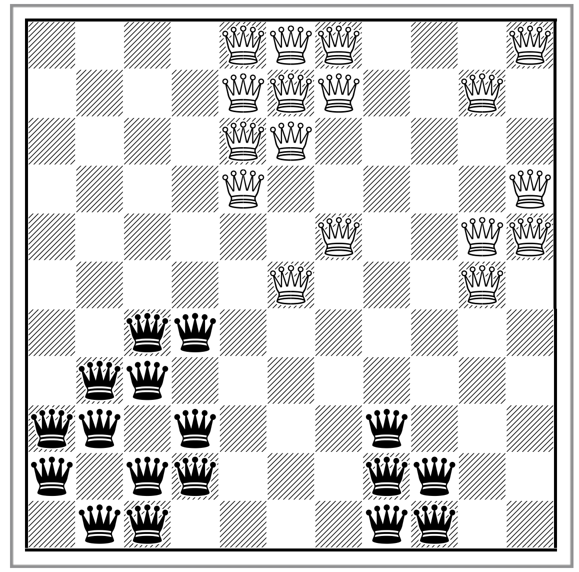
Coordination Sequences

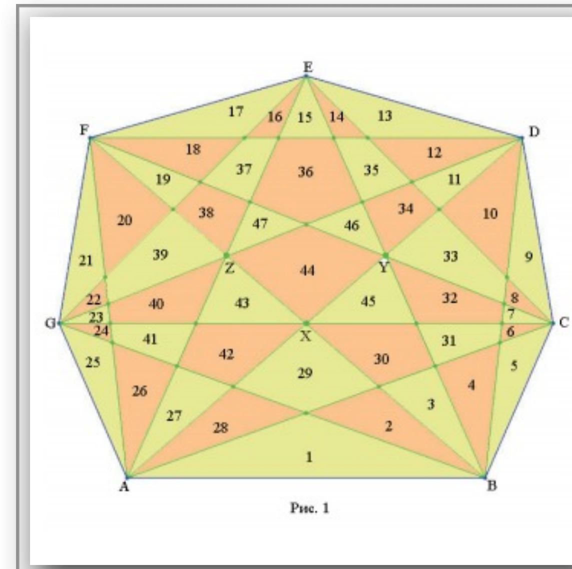
OEIS.org

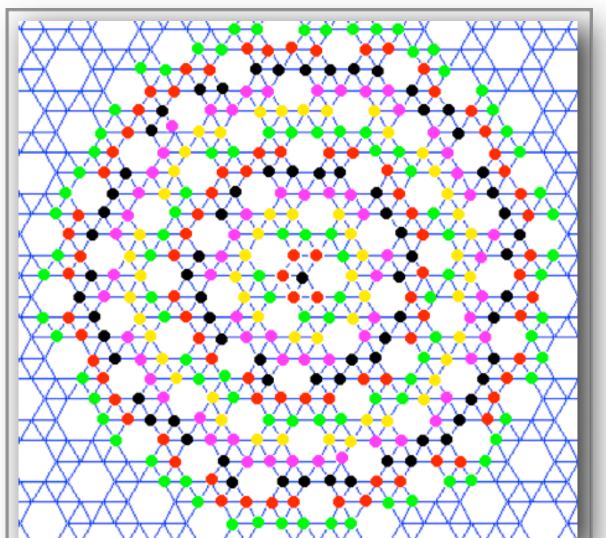
The poster, on the OEIS Foundation web site, http://oeisf.org

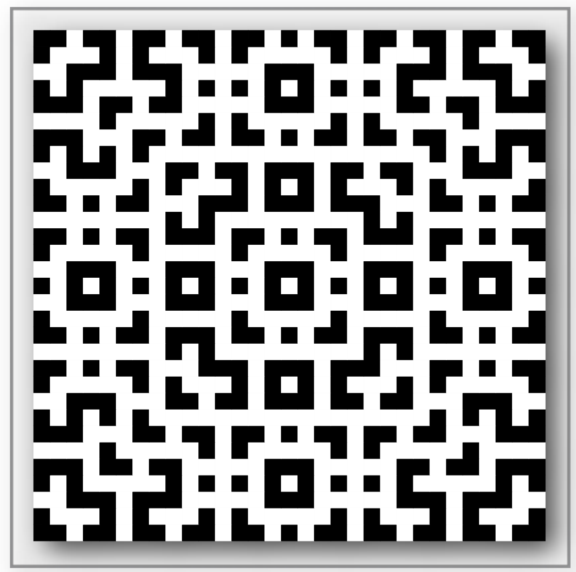
A250120

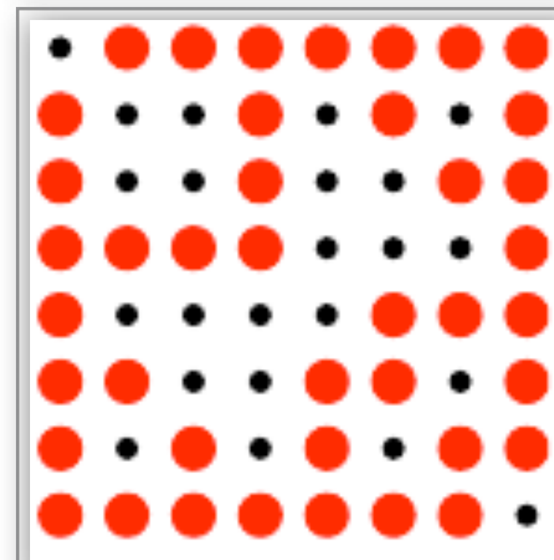


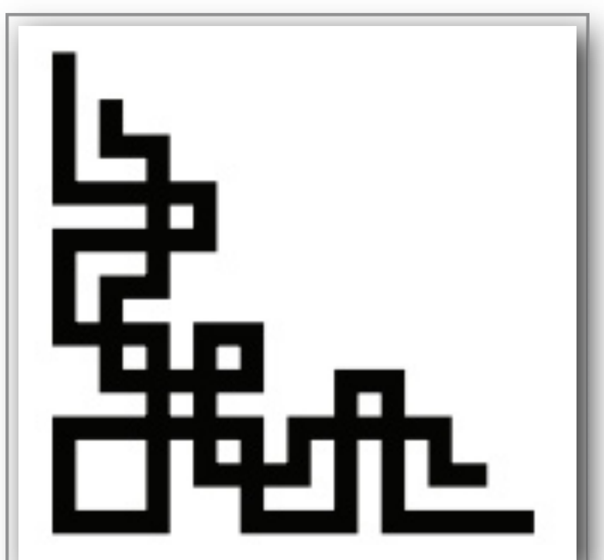


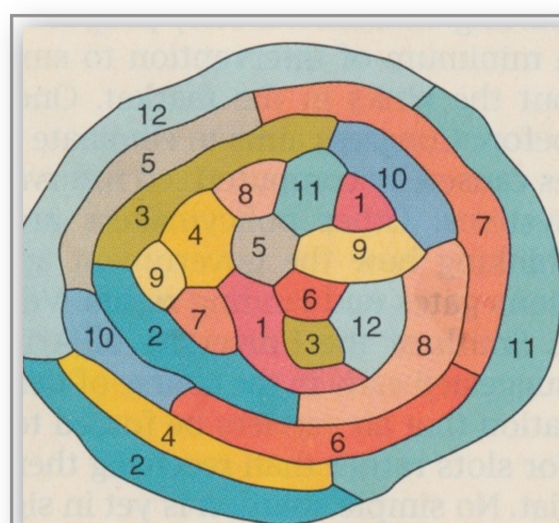


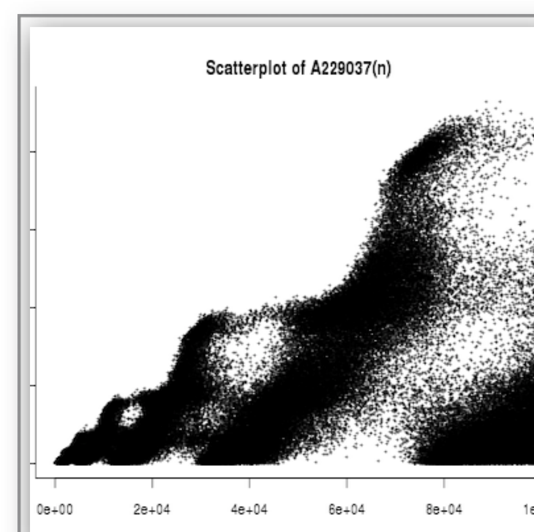




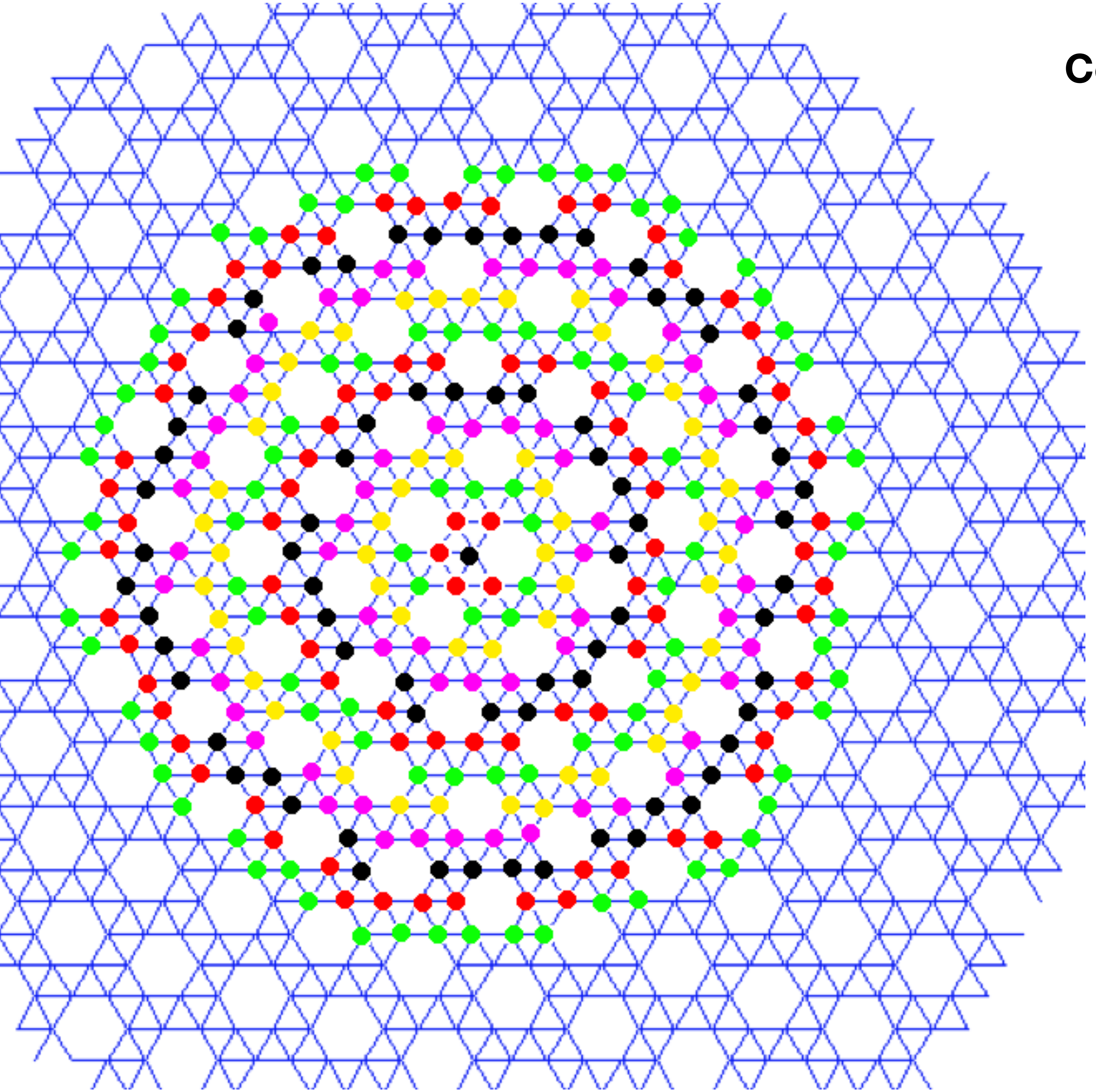








The 3.3.3.3.6 uniform tiling (A250120)



Coordination sequence
1,5,9,15,19,...

Conjecture
a(n+5)=a(n)+24
for n > 2

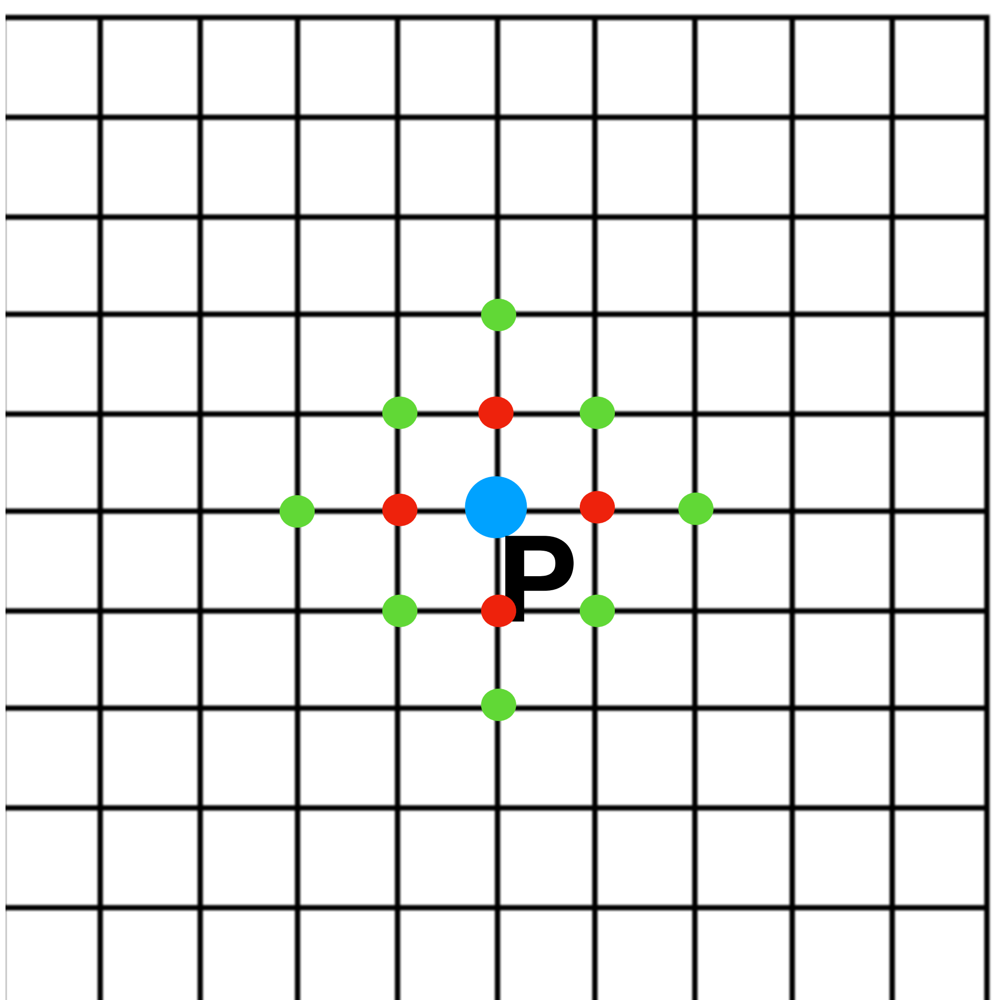
Coordination Sequences

Joint work with Chaim Goodman-Strauss

With thanks to Jean-Guillaume Eon, Brian Galebach, Joseph Myers, Davide Proserpio, Rémy Sigrist, Allan Wechsler, and others

Definition. G = graph, P = node,
the coordination sequence w.r.t P:
a(n) = number of nodes at edge-distance n from P

G

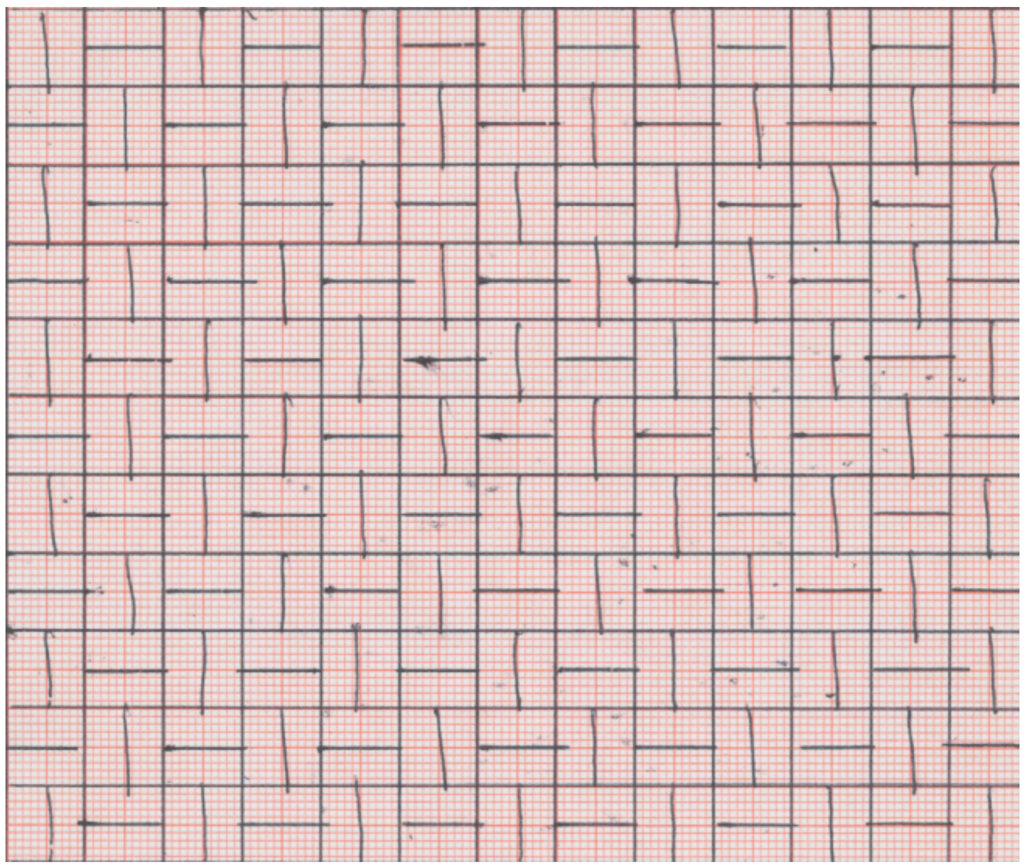


A8574

CS is 1, 4, 8, 12, 16, 20, 24, 28, …

G.f. = (1+2x+2x^2+2x^3+…)^2

Coordination sequences useful for identifying graphs, tilings, crystals, etc.



Brick pattern, Johnson Park, Piscataway, NJ
(near the zoo)

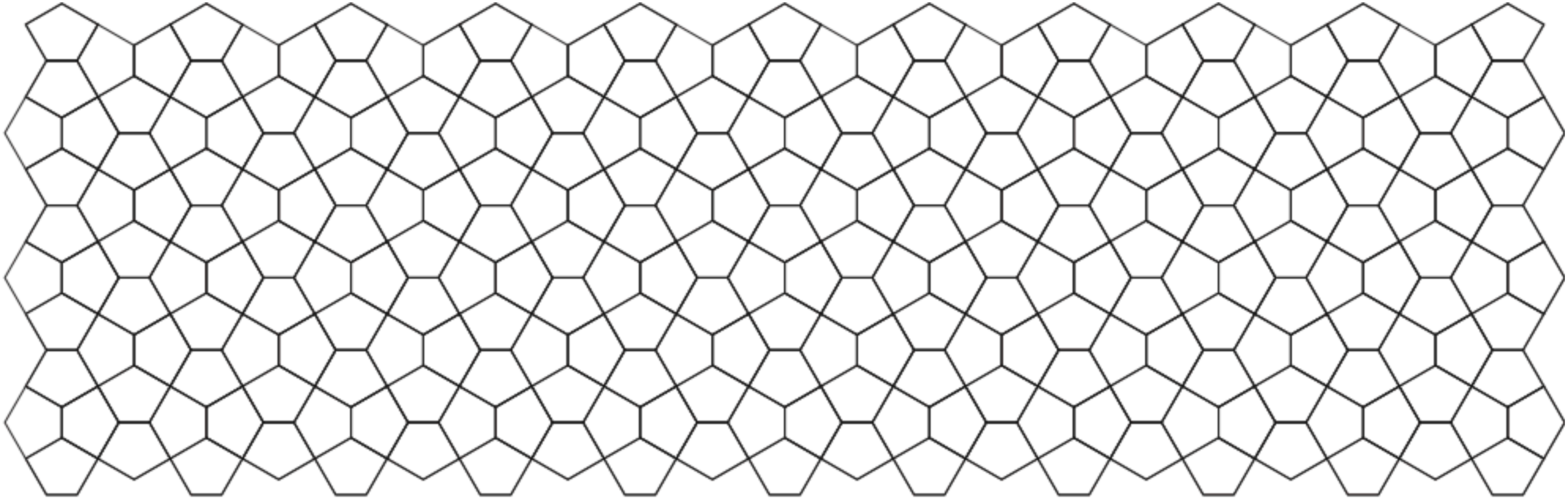
Two kinds of vertices:

Degree 4: 1, 4, 8, 12, 16, 20, 24, 28, ...

Degree 3: 1, 3, 8, 12, 15, 20, 25, 28, ...

and looking them up in the OEIS leads to ➜

The Cairo Tiling



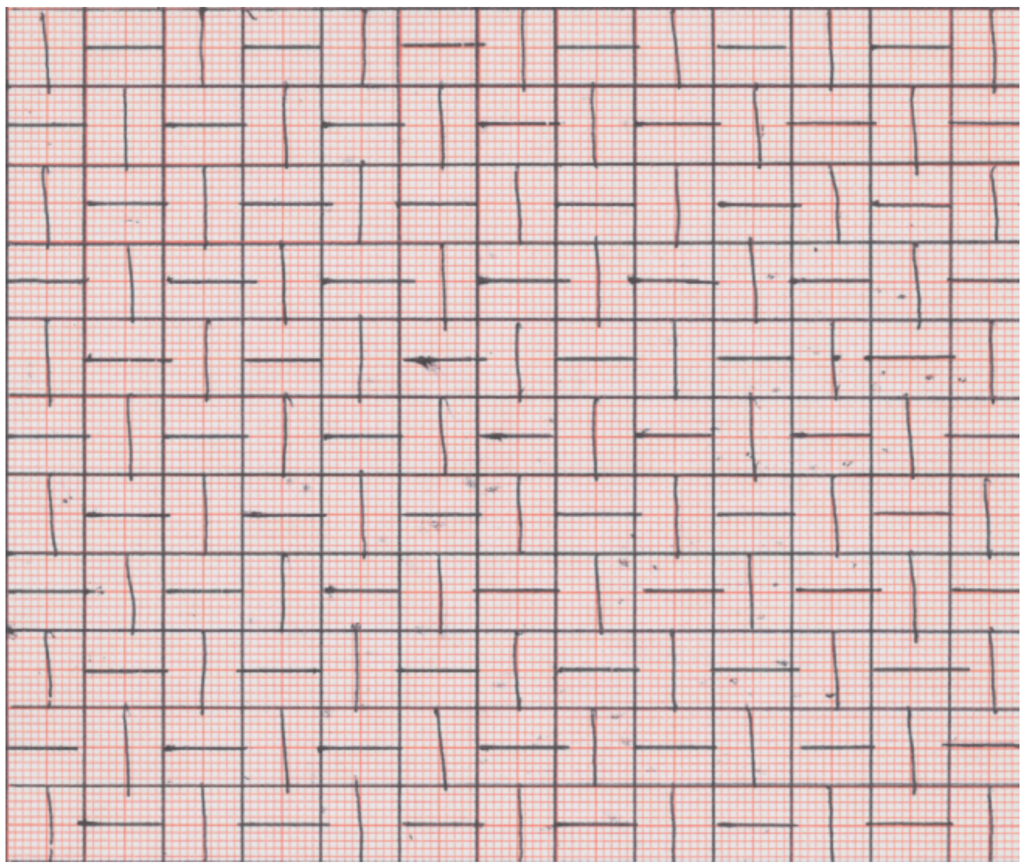
Two kinds of vertices:

Degree 4: 1, 4, 8, 12, 16, 20, 24, 28, …, same as square grid! Why? **A8574**

Degree 3: 1, 3, 8, 12, 15, 20, 25, 28, … **A296368**

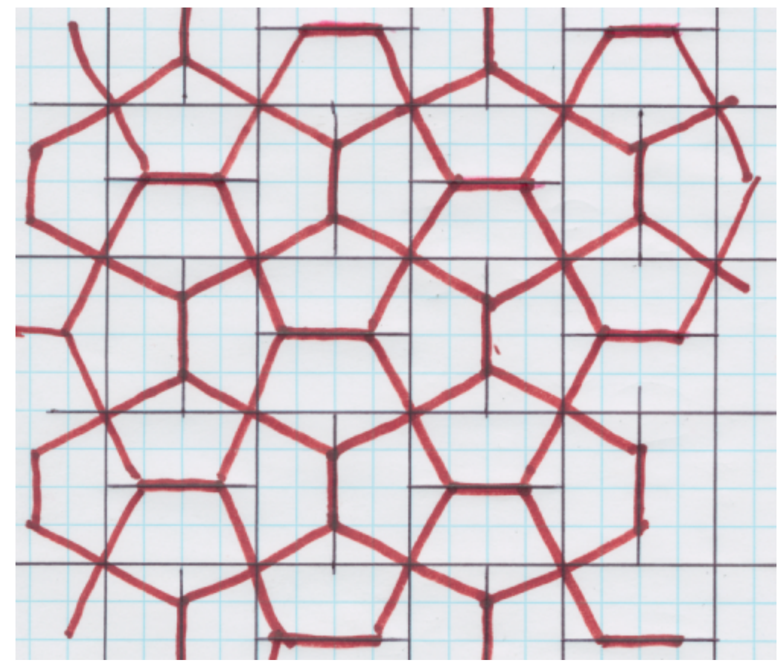
Such a simple fact should have a simple proof, which led Chaim Goodman-Strauss and me to ➜

Shorten all the bisecting lines by 50%



Brick pattern, Johnson Park, Piscataway, NJ

➜



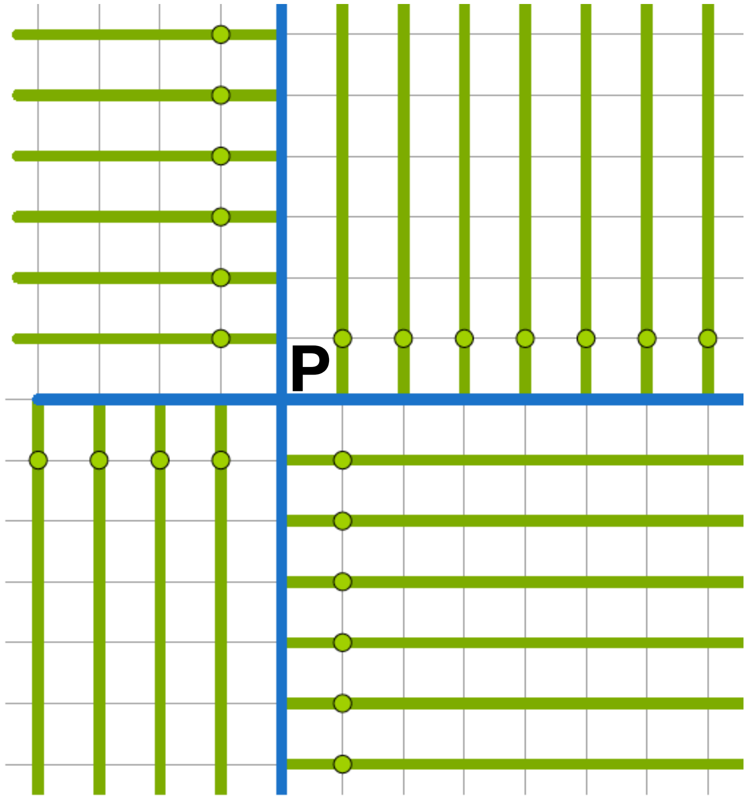
Same graph as Cairo tiling!

The Coloring Book Method for Finding Coordination Sequences

(C.G.-S. and NJAS, Acta Cryst. A75 (2019))

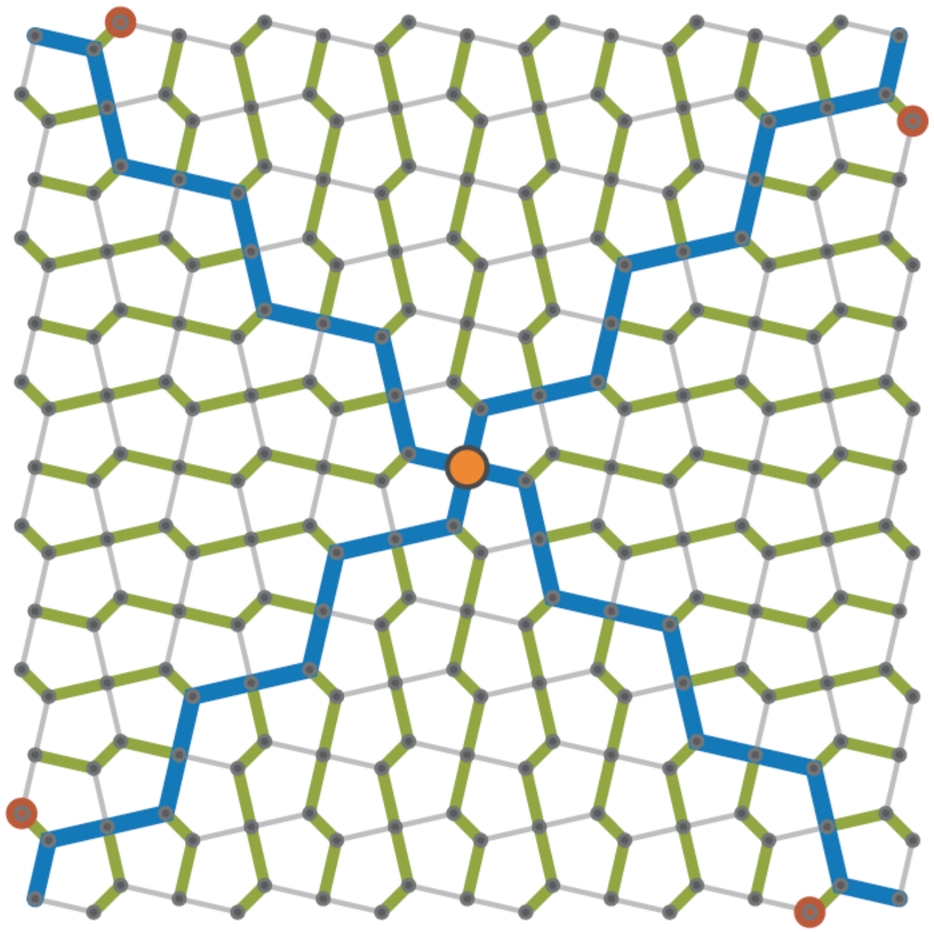
Find a subgraph H such that

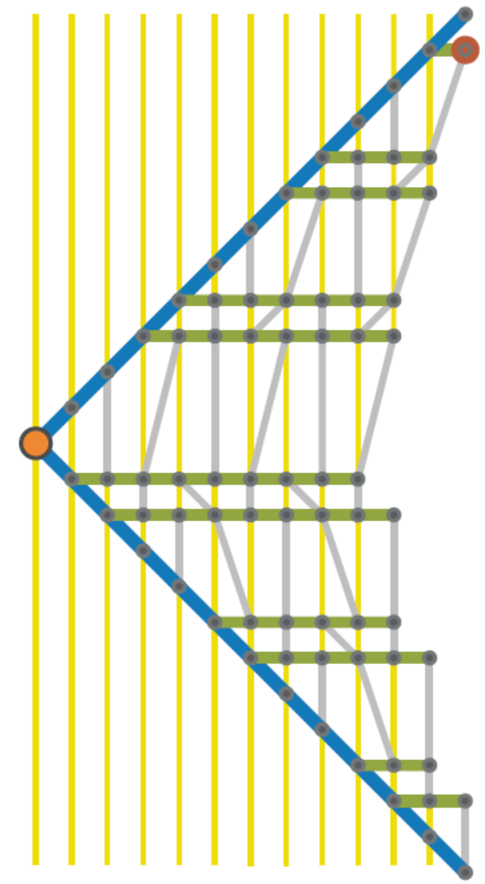
- H is connected, meets every node
- Paths in H from node to base P are minimal
- H consists of trunks, branches, and twigs
- It is should be easy to see that all paths are minimal
- and to count nodes at distance n from base P



Square grid: a(n) = 4n

Trunks and Branches for Cairo tiling, tetravalent node



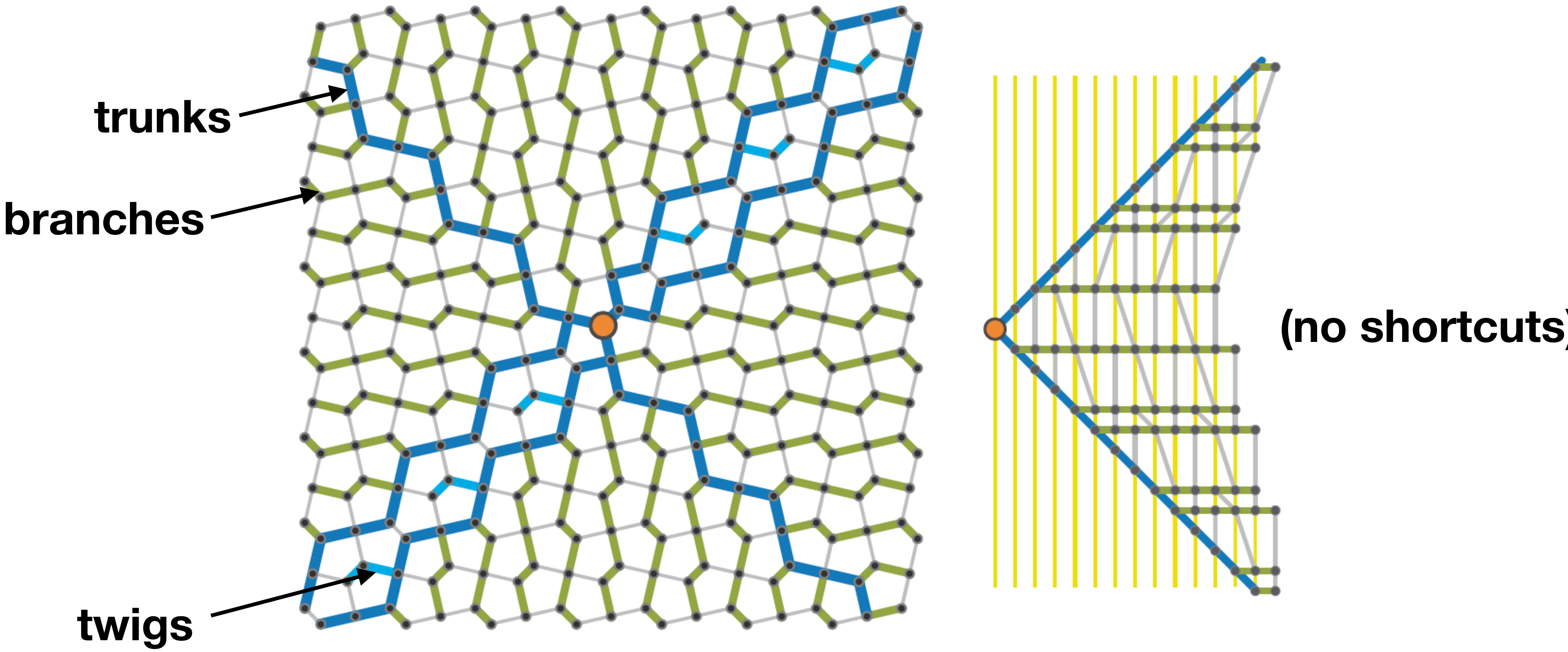


Sector redrawn to show that there are no shortcuts

and number of nodes goes up by 1 each step

So a(n) = 4n, same as for square grid

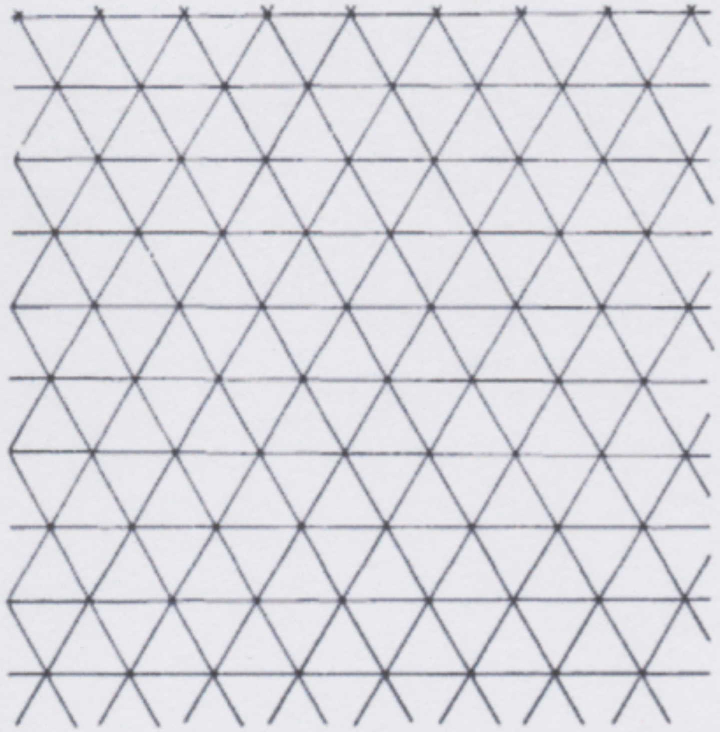
Trunks and Branches for Cairo tiling, trivalent node



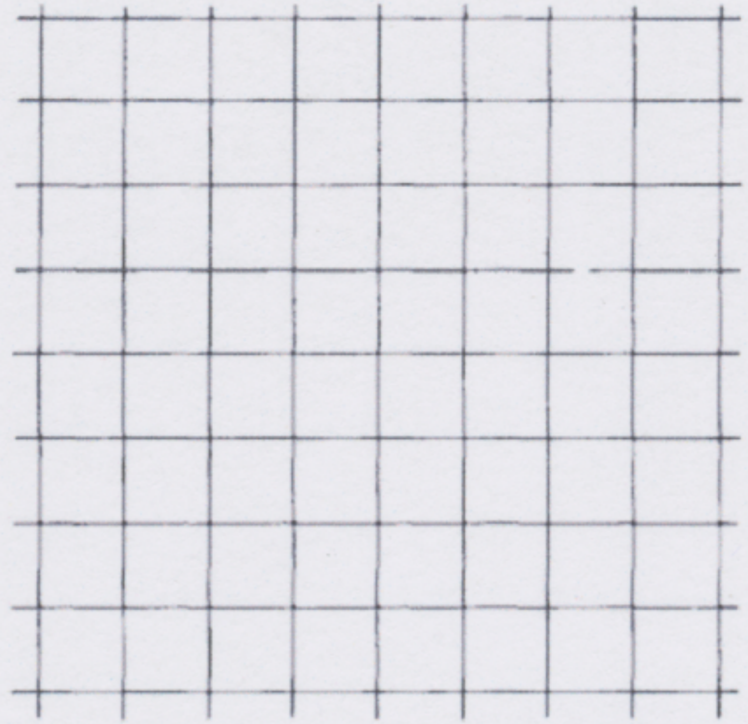
Theorem: a(0)=1, a(1)=3, a(2)=8, then

a(n)=4n (n odd), 4n-1 (n=0 mod 4), 4n+1 (n=2 mod 4) A296368

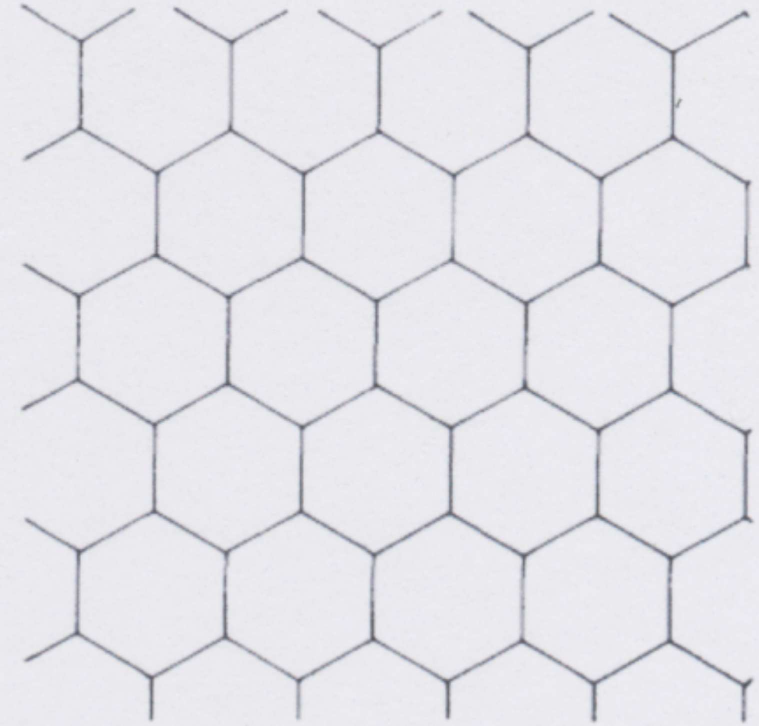
The 11 uniform or Archimedean tilings (part 1)



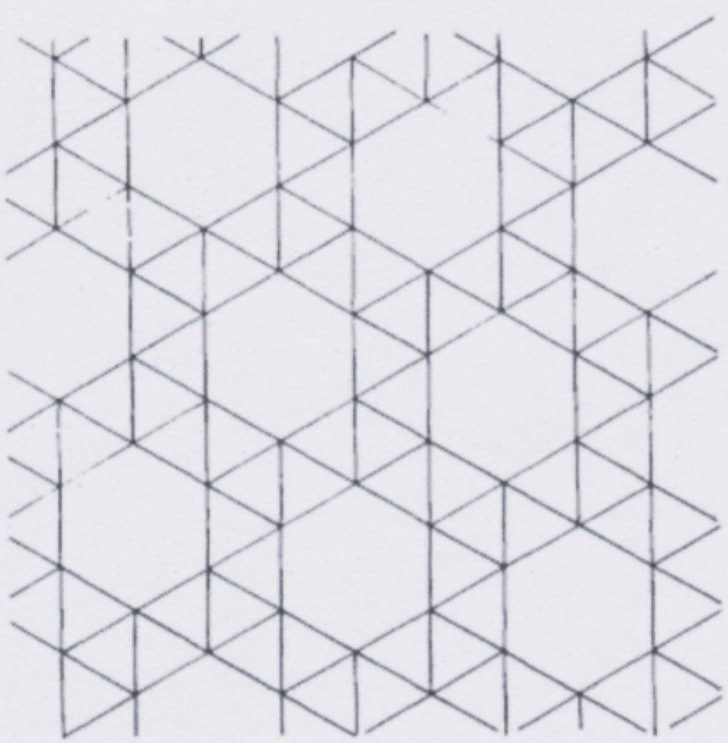
3^6 (3^6) A8458



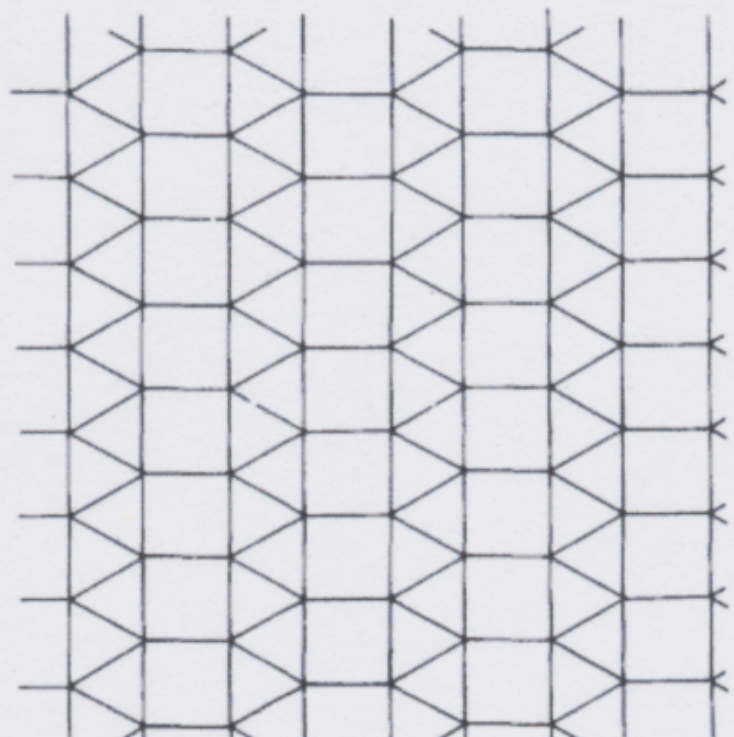
(4^4) A8574



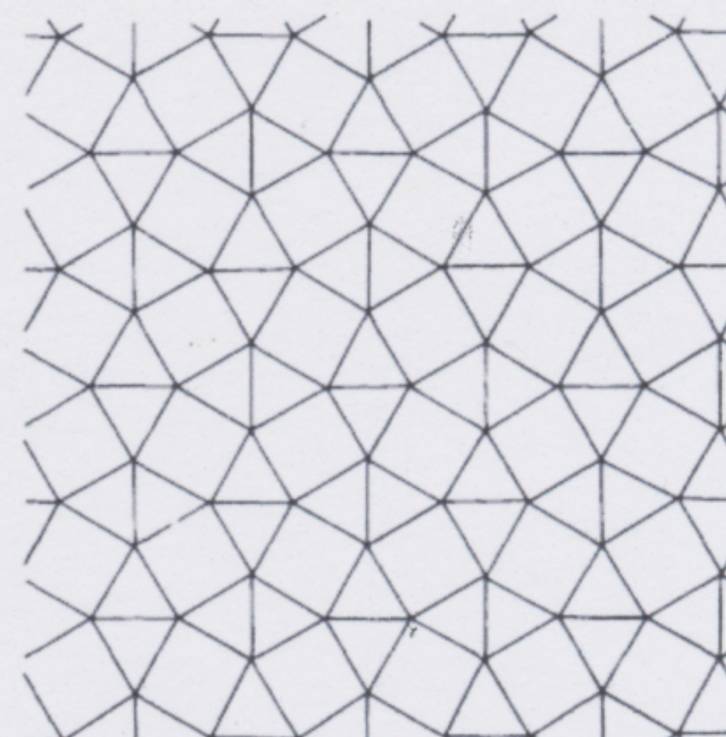
(6^3) A8486



$(3^4.6)$ A250120

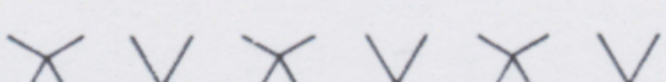


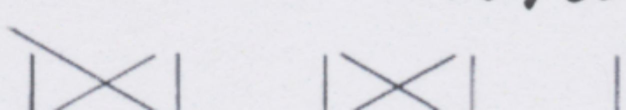
$(3^3.4^2)$ A8706

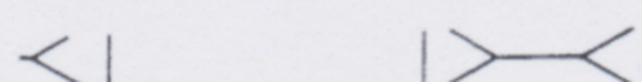


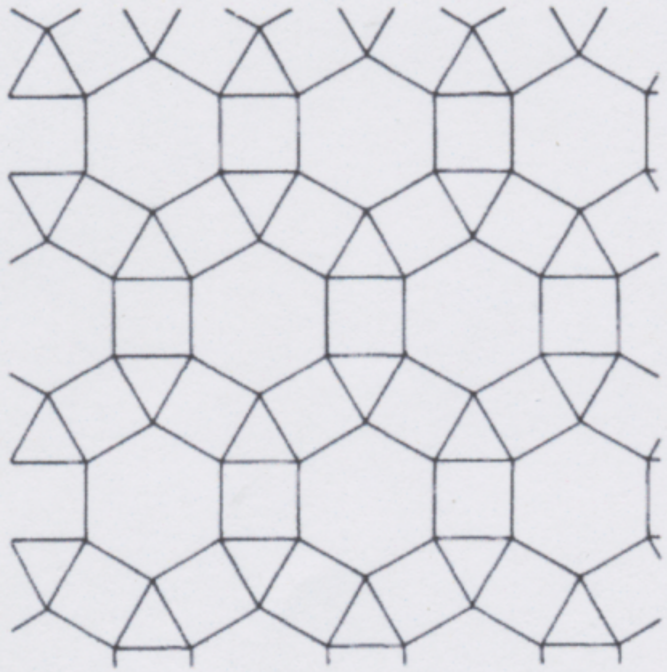
$(3^2.4.3.4)$ A219529

Dual to Cairo

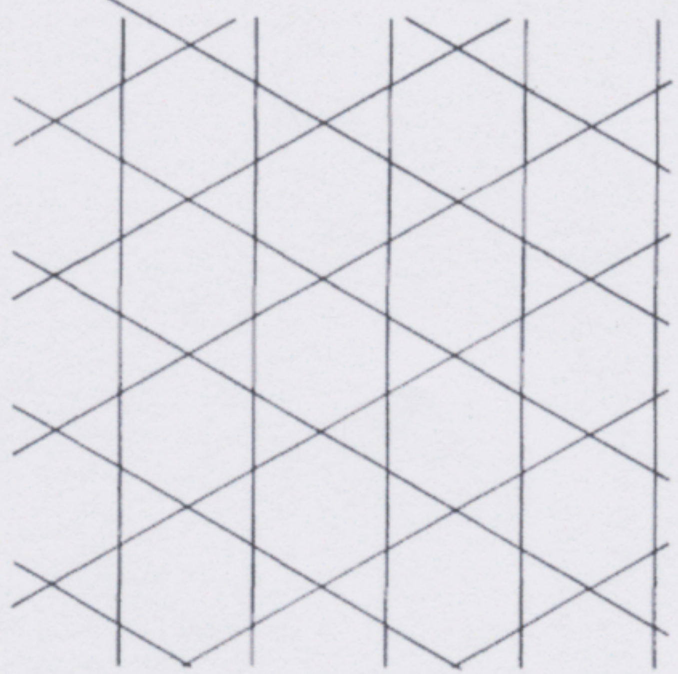




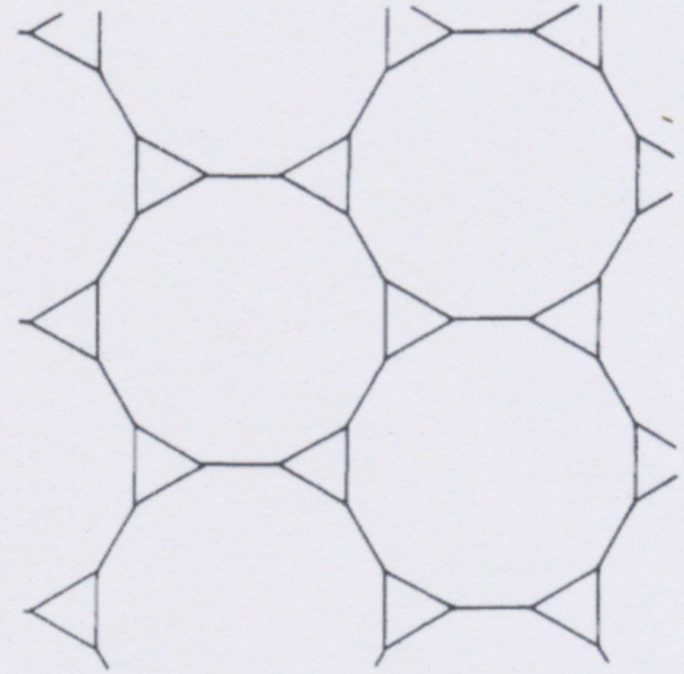




(3.4.6.4) A 8574 again



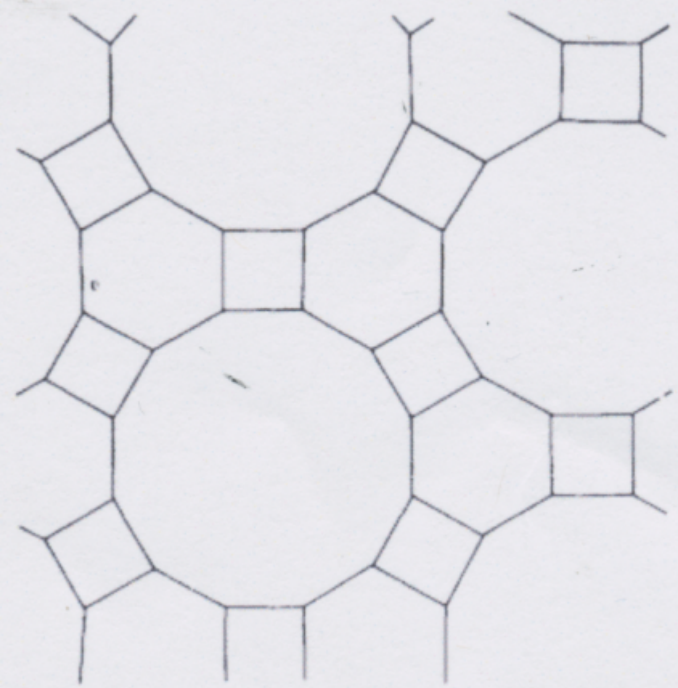
(3.6.3.6) A8579



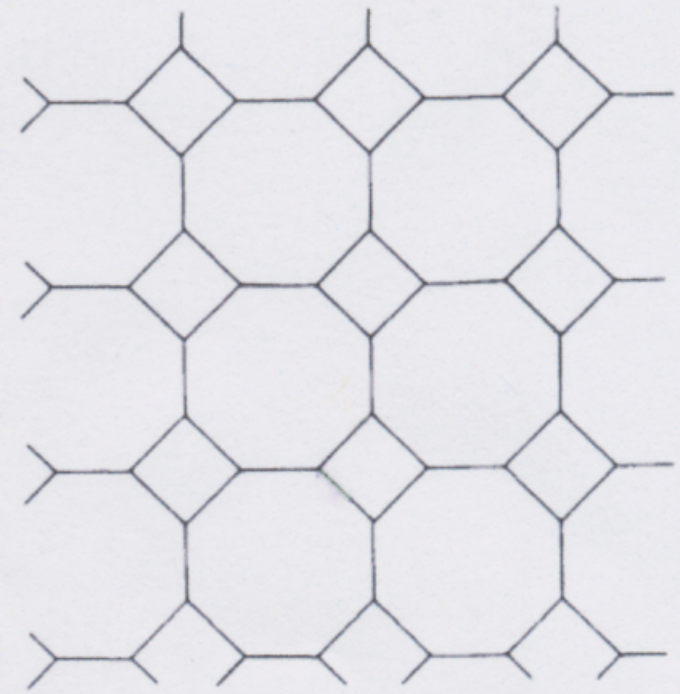
(3.12²) A 250122

The 11 distinct types of Archimedean tilings of the plane. The tiling of type (3⁴.6) exists in two mirror-symmetric (enantiomorphic) forms.

FIGURE 7



(4.6.12) A72154



(4.8²) A8576

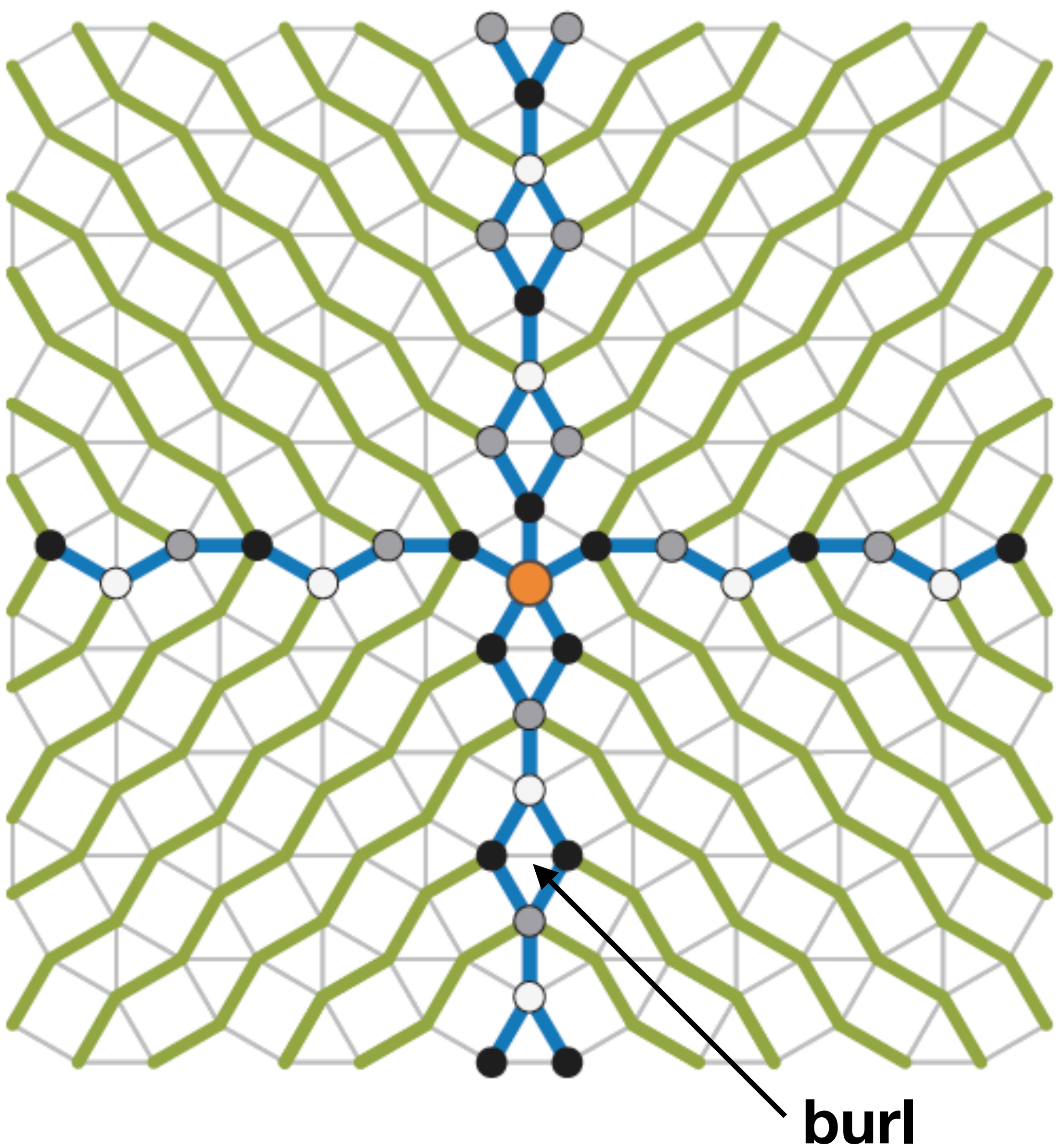
The 11 uniform or Archimedean tilings (part 2)

Branko Grünbaum and G. C. Shephard, **Tilings** and Patterns.

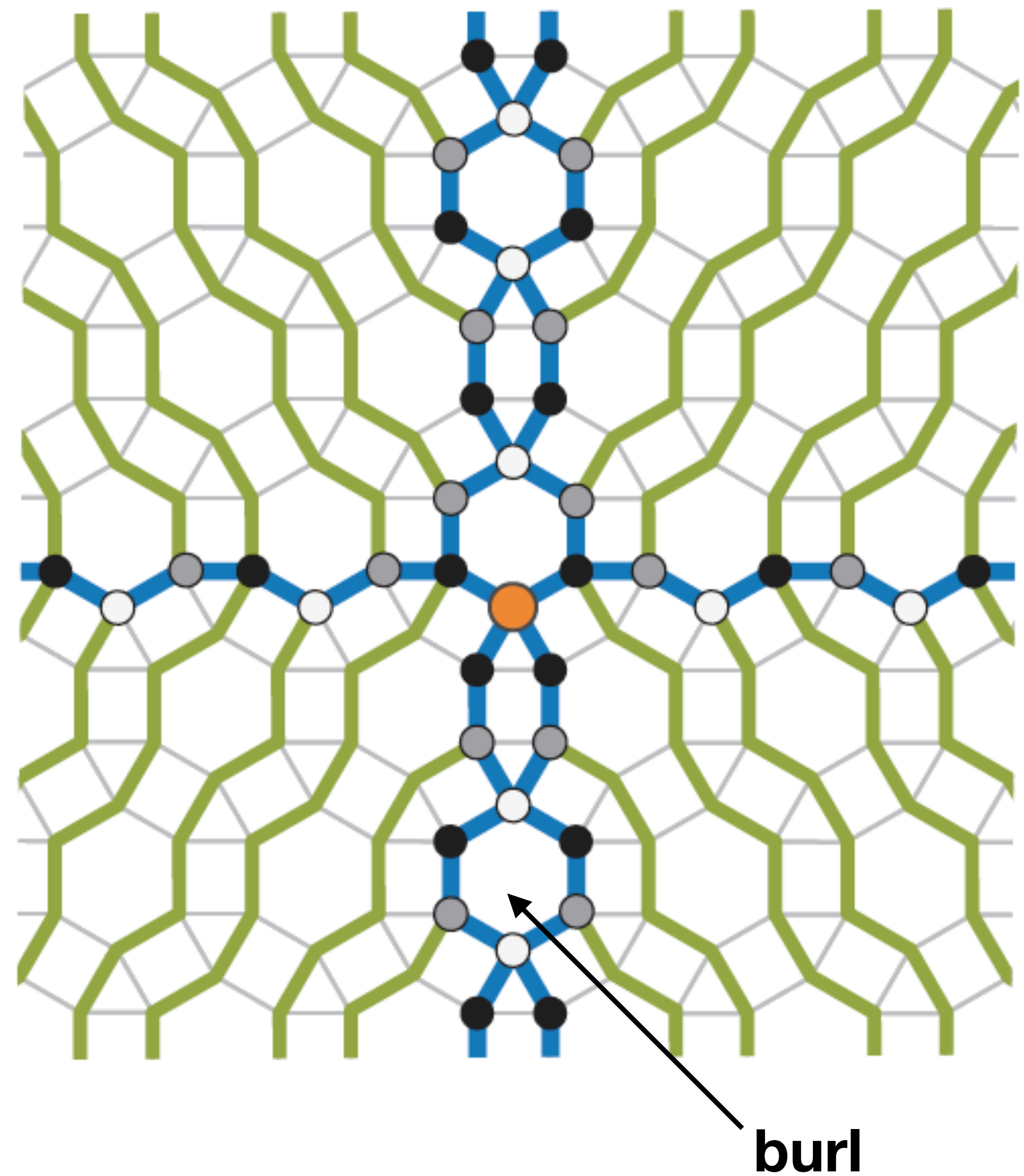
burl

From Wikipedia

Trunks and Branches for 2 of the 11 Uniform Tilings



3.3.4.3.4 (dual to Cairo), A219529



3.4.6.4, A8574 again!

The k-uniform tilings of the plane

(Tiles are regular polygons, group has k orbits on nodes.)

Brian Galebach, 2002, A68599:

k:	1	2	3	4	5	6
#:	11	20	61	151	332	673

No. of coord. seqs. = 6536, all in OEIS

Stages in studying coord. seqs.:

- Compute initial terms
- Look up in OEIS
- Guess generating function
- Prove g.f. is correct (done for k=1, partly for k=2)

Duals done only for k=1, 2?

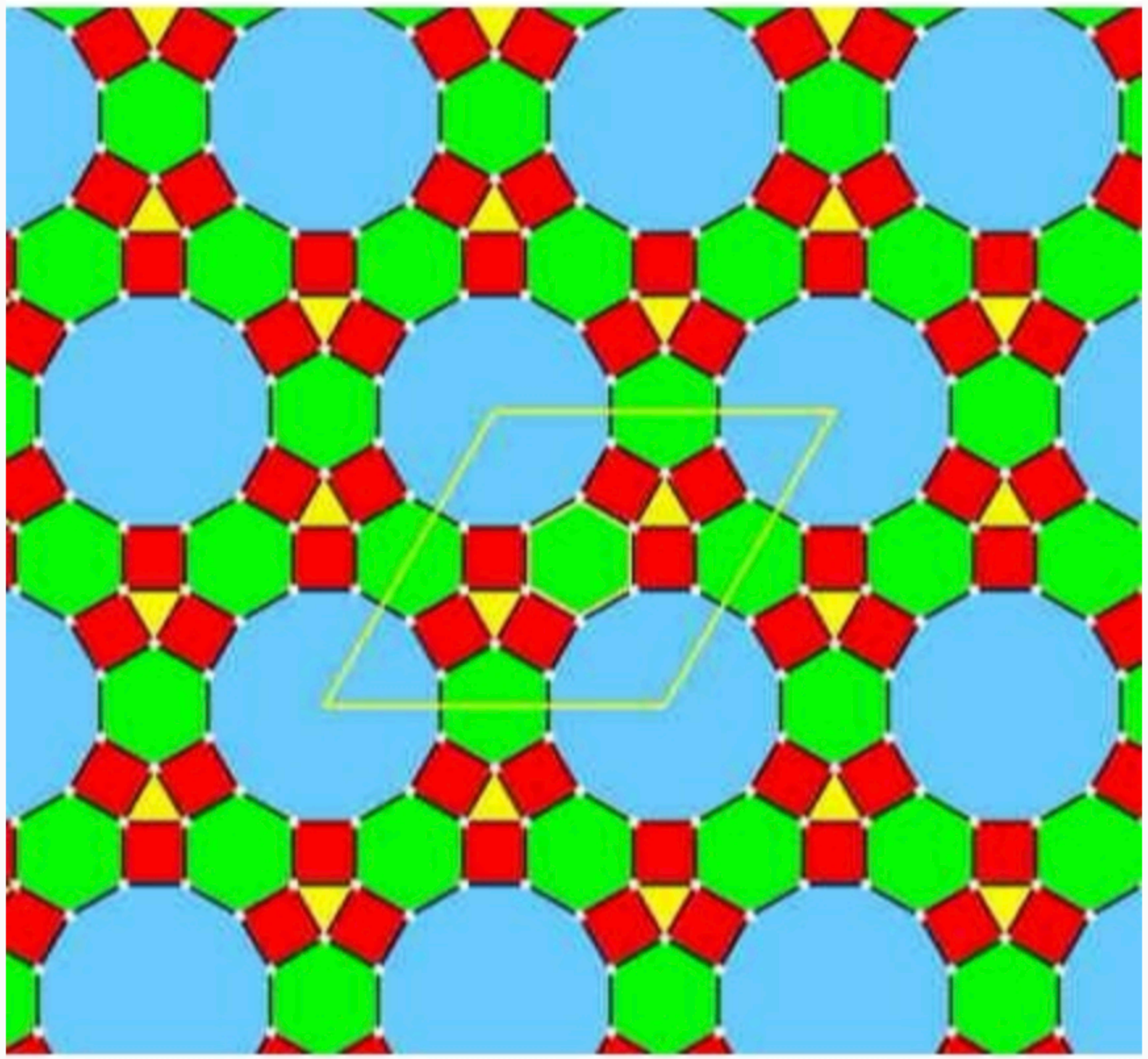
The “coloring book” approach is a “method”, not yet an “algorithm”
It would be nice to automate it.

A 2-uniform tiling with only conjectured g.f.'s

Type (3.4.6.4, 4.6.12), name = krt net

RCSR

krt



vertex	cs_1	cs_2	cs_3	cs_4	cs_5	cs_6	cs_7	cs_8	cs_9	cs_{10}	cum_{10}	vertex symbol
V1	4	6	7	10	14	20	24	24	23	26	159	3.4.6.4
V2	3	6	9	11	14	17	21	25	28	30	165	4.6.12

Reticulated Chemistry Structure Resource

Have 1000 terms of coord. seqs.

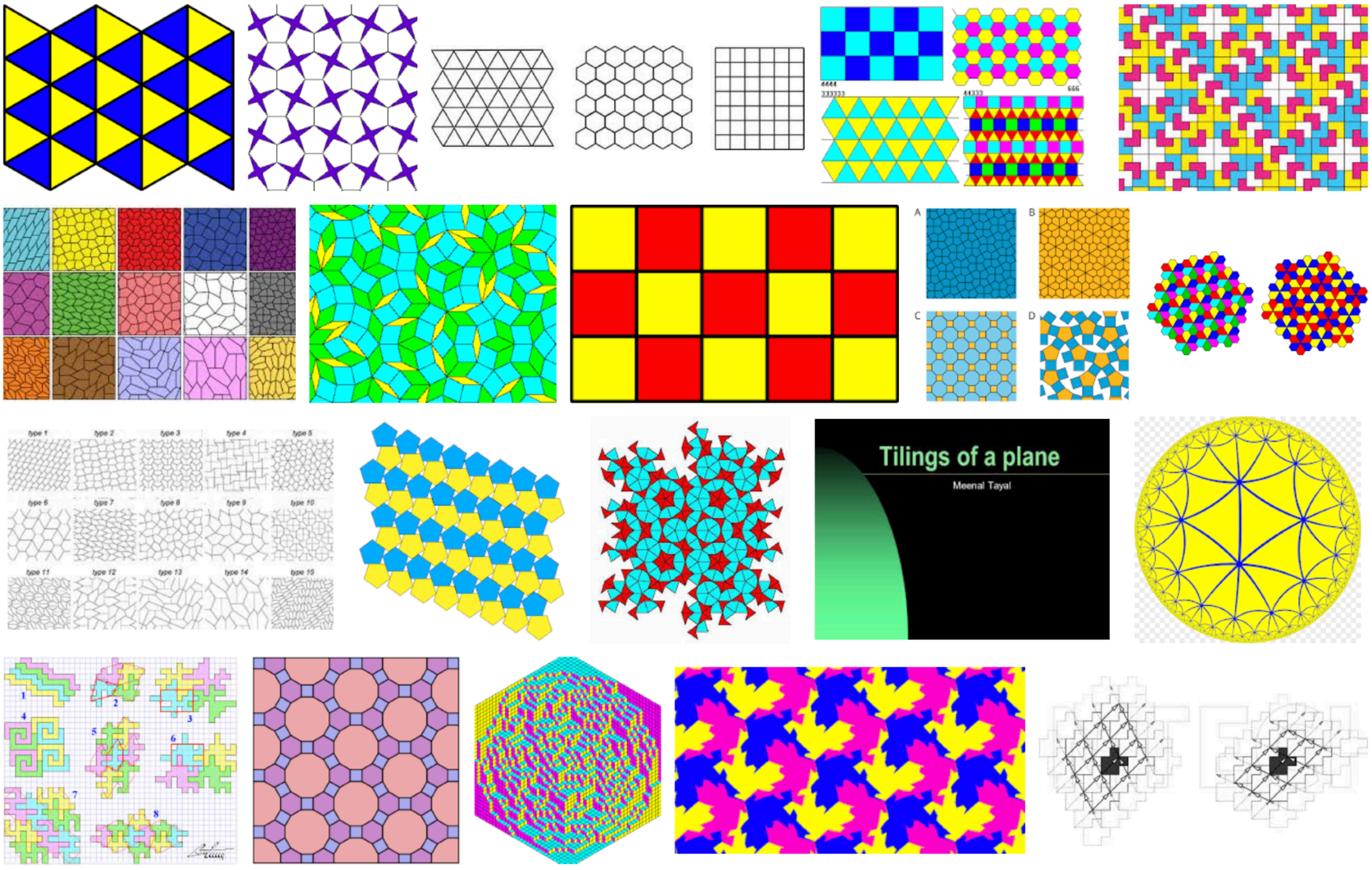
(Joseph Myers)

For 4.6.12 node, g.f. appears to be

$$\frac{1 + x^2 + 2x^5 - 2x^6 + 2x^7 - x^8}{(1-x)^2(1-x+x^2)}$$

A265035

There are a LOT of tilings!



And there are a LOT of articles about coord. seqs, many web sites, …

Our “Coloring Book” paper has extensive bibliography

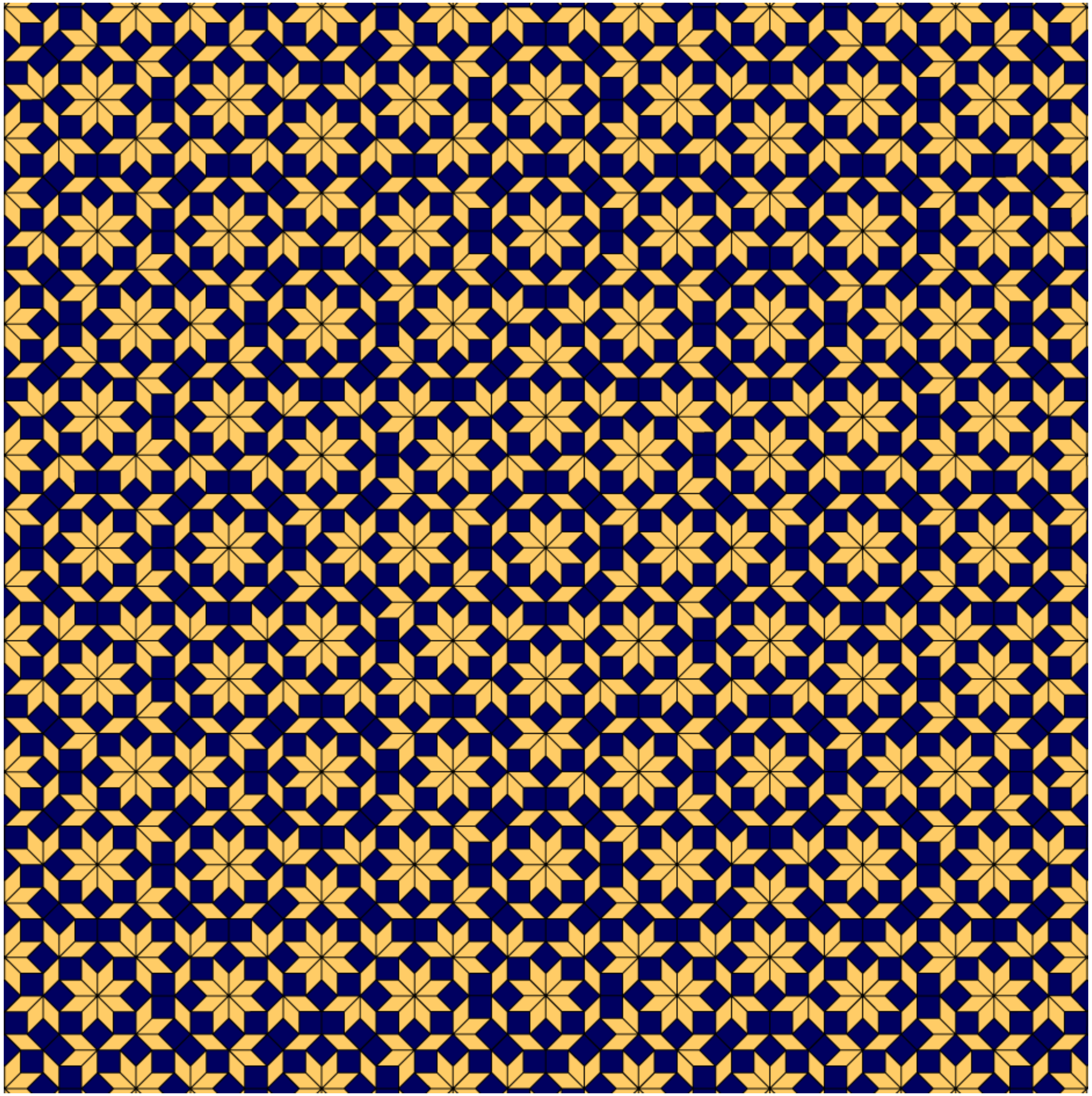
See especially the RCSR (Reticular Chemistry Structure Resource) of O’Keeffe et al.) and ToposPro (Blatov et al.) web sites

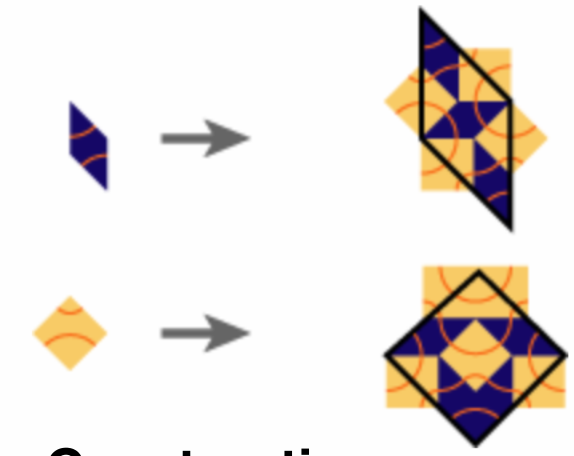
Conjecture: The coord. seq. of a periodic tiling of d-dimensional Euclidean space by polytopes always has a rational generating function.

What about aperiodic tilings?

There is recent work by Anton Shutov and Andrey Maleev, and Rémy Sigrist

An example of an Ammann-Beenker tiling with a unique vertex with global 8-fold symmetry





Construction

Rémy Sigrist, A303981:

1, 8, 16, 32, 32, 40, 48, ...

(900 terms, no g.f. known)

Coordination Sequences (cont.)

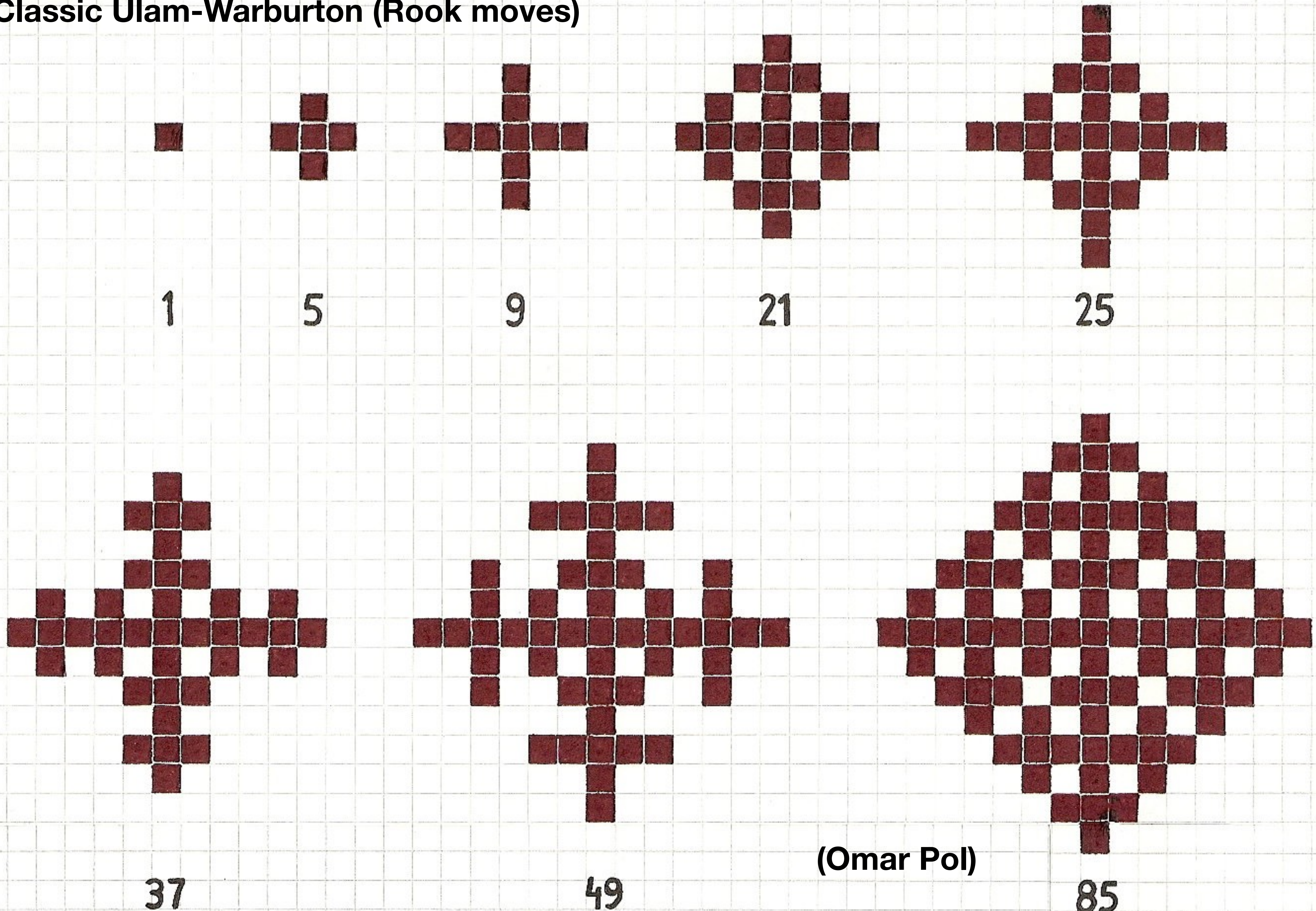
Limit of contour lines

There is work on the limiting shape of the contour lines in a tiling by Vladimir Zhuravlev and independently by Shigeki Akiyama (arXiv:1707.02373)

Interesting topic for future work!

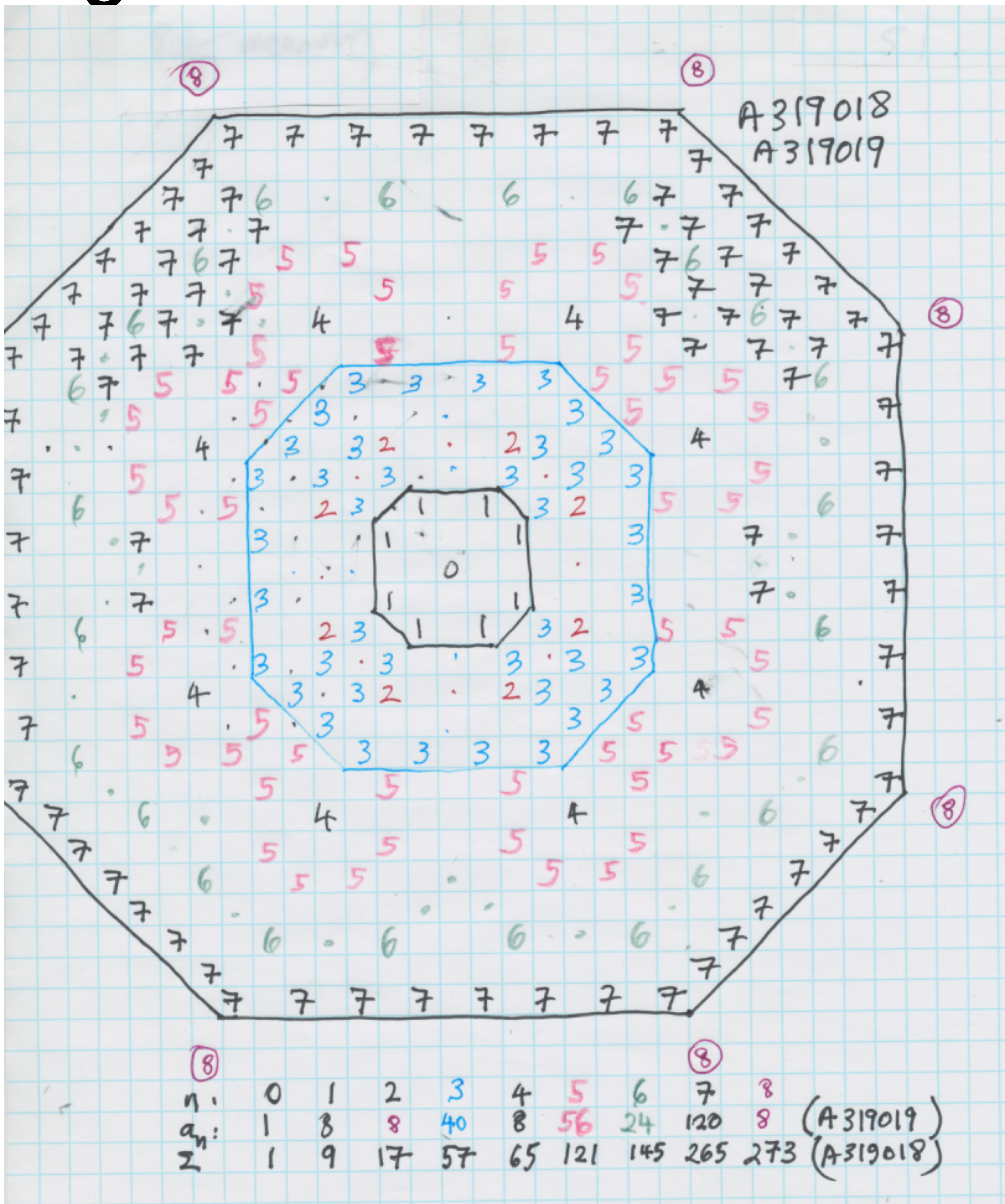
Knight’s Move
Ulam-Warburton
Cellular Automaton

Classic Ulam-Warburton (Rook moves)



(Omar Pol)

Knight's-move Ulam-Warburton cellular automaton

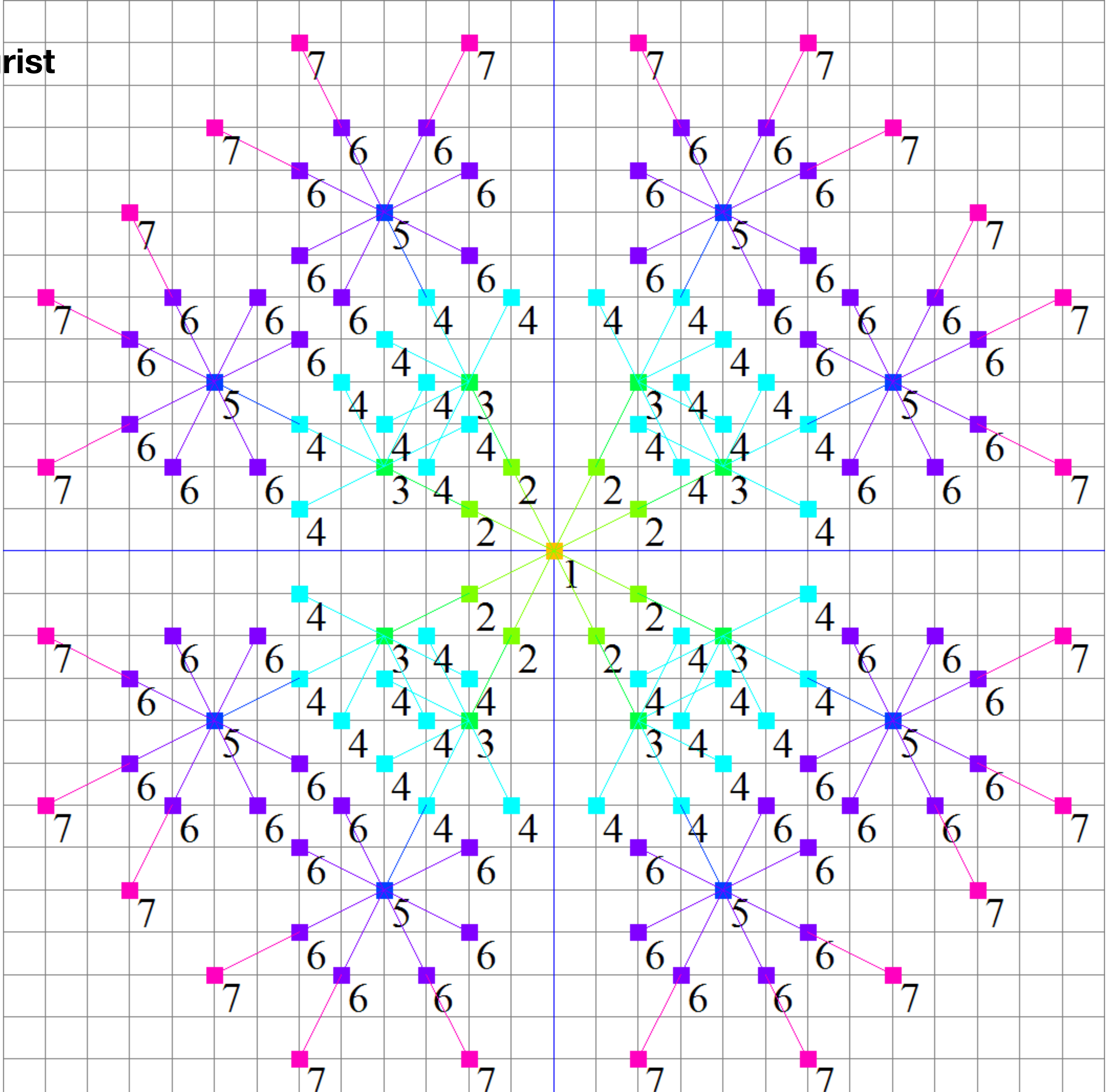


Neighbor means the 8 knight-move neighbors

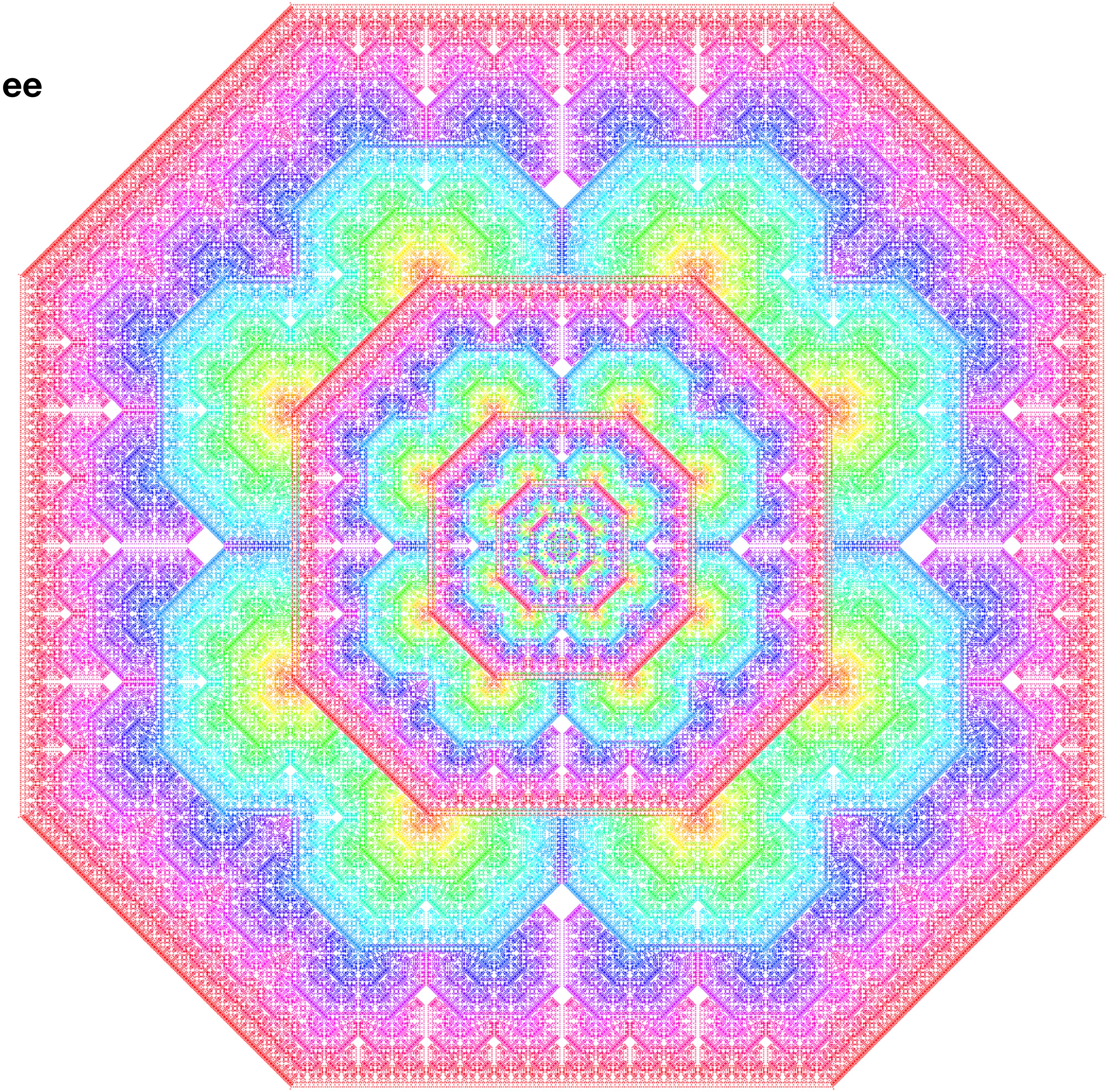
You turn ON iff exactly one of your 8 nbrs is ON

A319018

Remy Sigrist



Brad Klee



Some Recent Sequences and Unsolved Problems

For example, any recent submission by Eric Angelini or Rémy Sigrist is worth studying

Typical questions to ask:

- **is the sequence infinite?**
- **does every number appear?**
- **is there a formula, recurrence, g.f.?**
- **how fast does it grow?**

Eric Angelini’s remove-repeated-digits operation

Drop any digit from n that appears more than once

1231, 1123, 123111, 11023 all become 23

Write 0 if nothing left.

A320486 says what happens to n:
1, 2, 3,…,10, 0, 12, 13,…, 21, 0, 23,…

Get 0 with probability 1, so easy to analyze!

“Factorials” 1, 2, 6, 24, 120, 720, (5040) 54, 432, (3888) 3, 30, (330) 0

A321008

Start with n, and repeatedly square-and-delete:
Conjecture (Lars Blomberg) : Reach one of 5 fixed points:
0, 1, 1465, 4376, 89476. (A321010)
or one of two nontrivial loops

(1465 is a fixed point: 1465^2 = 2146225 -> 1465)

1 2 4 8 16 32 64 128 256 512

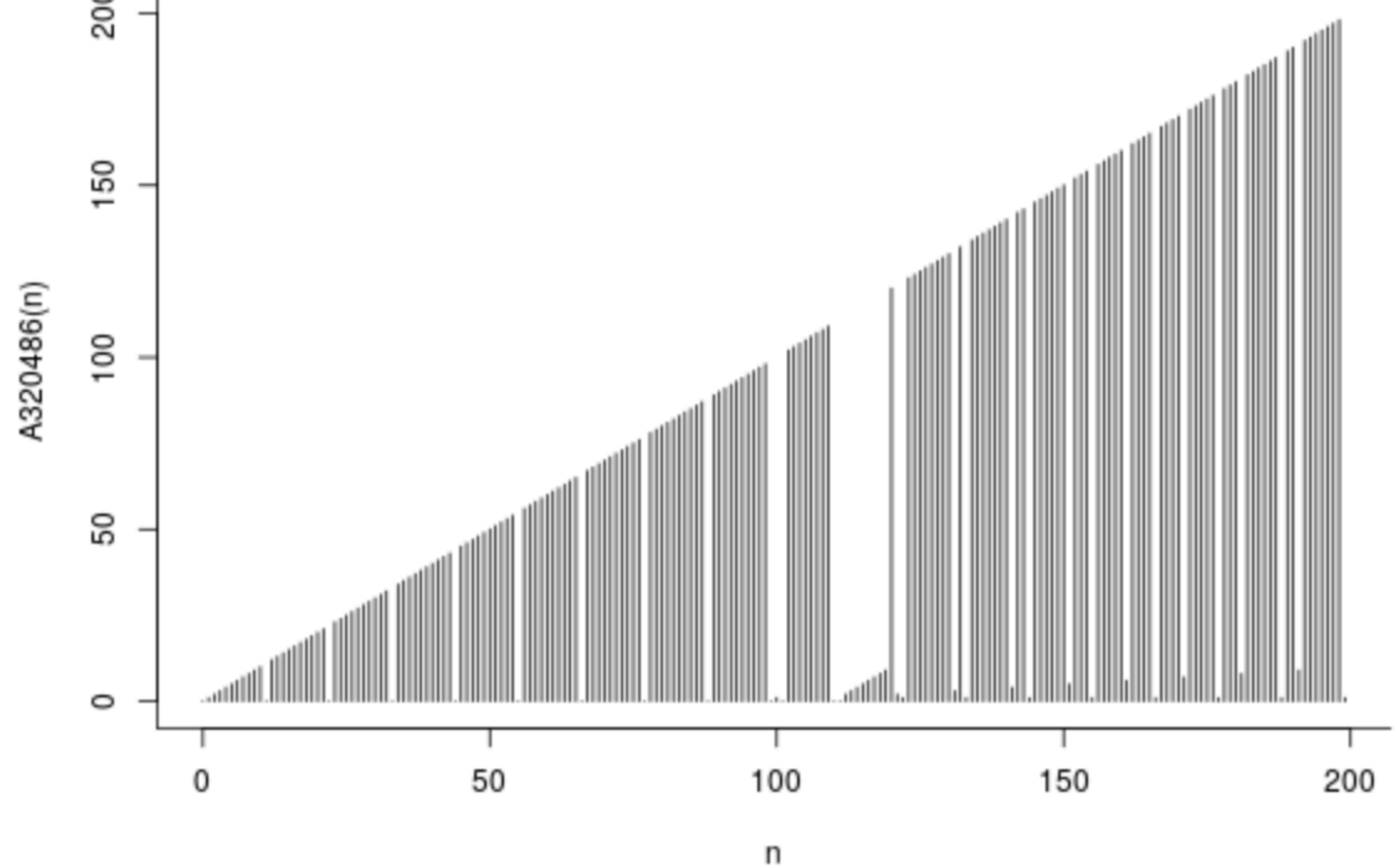
1024 2048 4096 8192 16384

32768 3 6 12 24 48 96 192

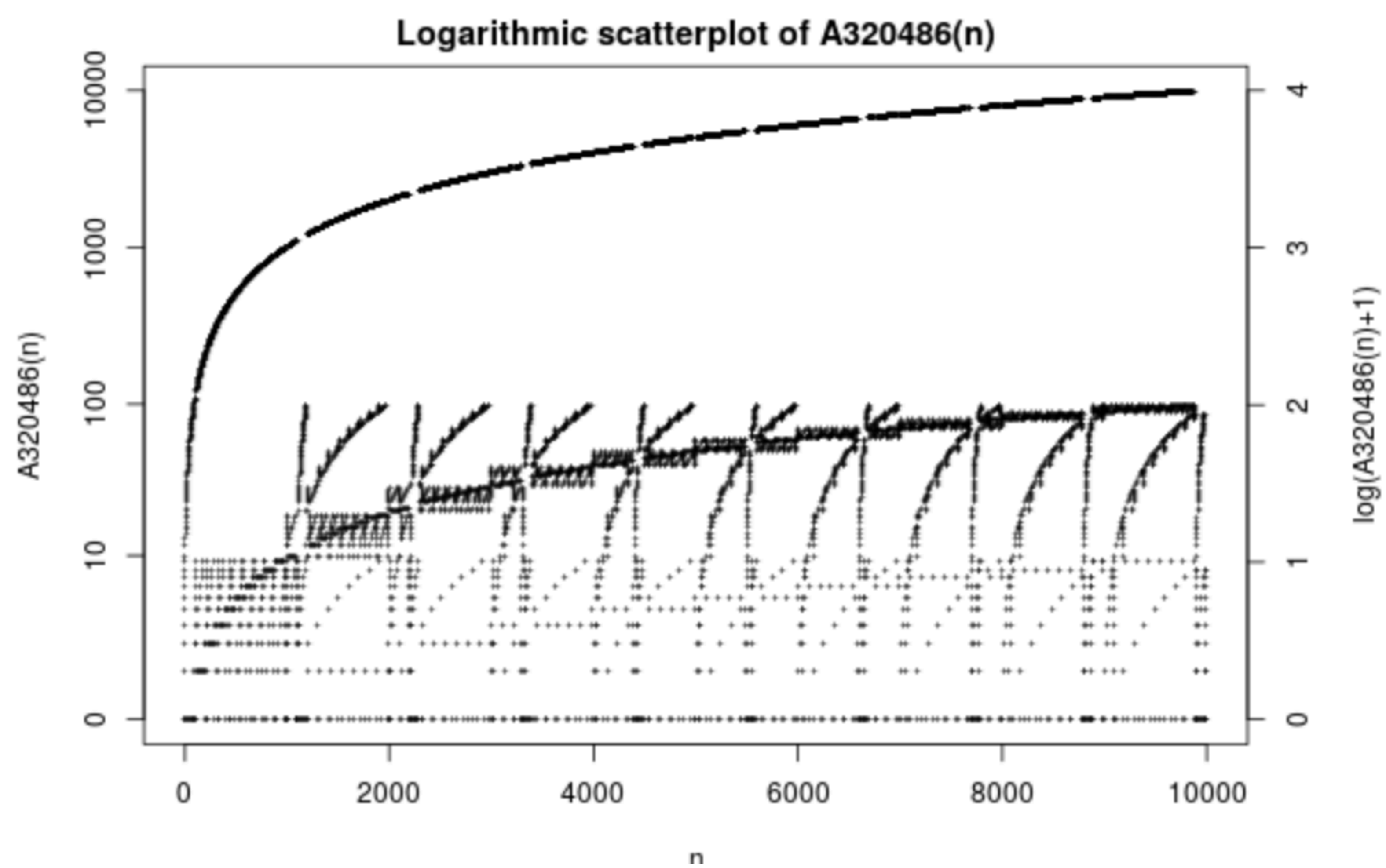
384 768 1536 3072 61 1 2 4 8

Periodic, easy - explain!

Two plots of A320486, Angelini's Remove repeated digits from n



200 terms



Log plot of 10K terms

Georg Fischer has been searching for duplicates, Many unsolved and solved problems!

1

A045318 Primes p such that x^8 = 3 has no solution mod p.
A301916 Primes which divide numbers of the form 3^k + 1.
are almost the same, the terms
in the latter but not in the former being A320481

2, 769, 1297, 6529, 7057, 8017, 8737, 12097, 12289. ...

The question is, what are these primes?
Solved by Don Reble, Oct 25 2018 and Richard Bumby, Nov 12 2018

2

Are A027595 and A007212 the same?

A027595 satisfies T^2(a)=a: given a1<=a2<=..., let b(n) = number of ways of partitioning n into parts from a1, a2, ... such that parts == 0 mod 5 do not occur more than once.

A007212 has similar definition, but w/o the mod 5 condition.

Either there is a mistake, or there is a theorem here!

Many conjectured formulas from R. H. Hardin

Typical recent example (A250352):
How many lists x of length 3 with x(i) in [i, i+1, …, i+n]
and no term appearing more than twice in a list?

Examples: a(6) includes 2,4,6; 0,4,4; 1,7,7; …

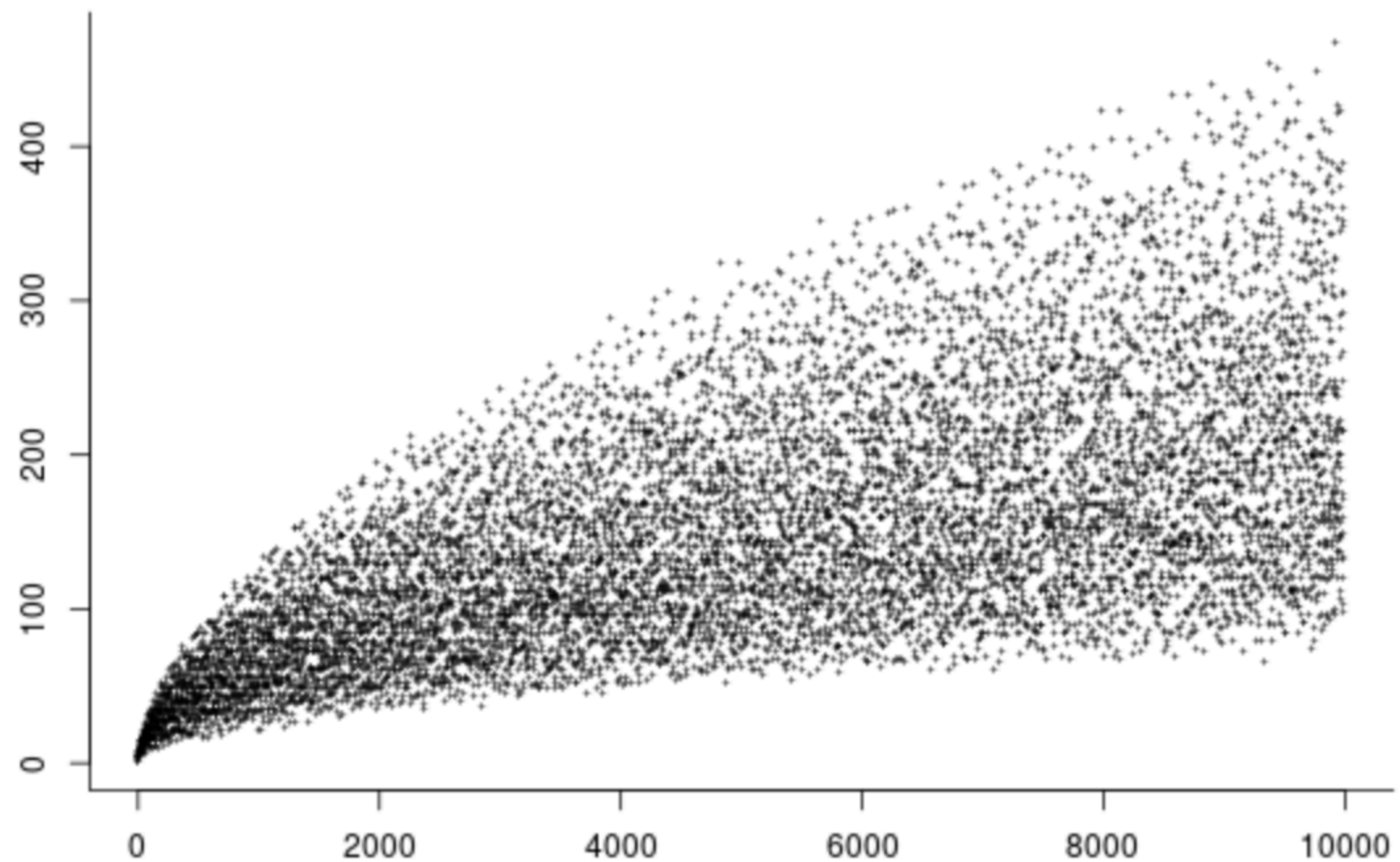
Empirical: $a(n) = n^3 + 3n^2 + 2n + 2$

Search for R. H. Hardin AND empirical

Allan Wechsler
No. of partitions into parts that are consecutive, all parts singletons except the largest

A321440, Nov 9 2018

a(9)=7: 1^9, 12222, 1233, 234, 333, 45, 9



WHY?

(Hint: Partitions into consec. parts = no. of odd divisors)

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