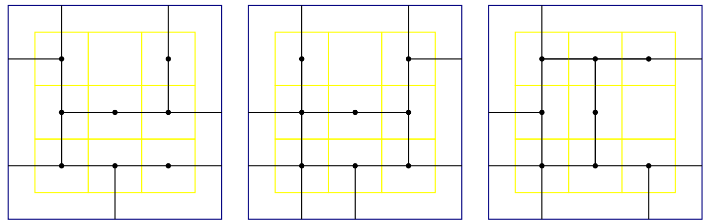


On dividing rectangles into rectangles

Joshua Smith[*] and Helena Verrill

Department of Mathematics, Louisiana State University,
Baton Rouge, LA 70803-4918
verrill@math.lsu.edu

Abstract

We consider the problem of counting the number of ways of dividing a rectangle of integer side lengths n, m, into rectangles of integer side lengths. Similar problems have been considered by many authors, though most authors put more restrictions on the problem. E.g., see [1]. We give a recursive method of finding the sequences of solutions. The sequence with $n = 2$ is sequence A034999 in Sloan's database [2].

1 Result and Examples

Let $a_{m,n}$ be the number of ways of dividing an $n \times m$ rectangle into a disjoint union of rectangles with integer length sides. We prove the following:

Theorem 1.

$$a_{m,n} = \mathbf{1} \cdot (M_m)^{n-1} \cdot \mathbf{1}^t, \tag{1}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbf{Z}^{2^{m-1}}$, *and* M_m *is a* $2^{m-1} \times 2^{m-1}$ *matrix defined recursively as follows:*

$$M_1 = (2),\ B_1 = (1), \quad M_{m+1} = \begin{pmatrix} M_m & B_m \\ B_m & 2M_m \end{pmatrix},\ B_{m+1} = \begin{pmatrix} B_m & B_m \\ B_m & M_m \end{pmatrix}.$$

Remark 1. *We can convert these expressions to give recurrence relations, with coefficients as in the characteristic polynomial of* M_m. *E.g.,* $a_{3,1}, \ldots a_{3,4}$ *are given in Tables 1, and for* $n \geq 4$,

$$a_{3,n} = 18a_{3,n-1} - 100a_{3,n-2} + 216a_{3,n-3} - 153a_{3,n-4}.$$

[*] Whilst working on this article, first year undergraduate Joshua Smith held a Louisiana State University "Chancellor's Future Leaders in Research" scholarship.

	1	2	3	4	5	6
1	1	2	4	8	16	32
2	2	8	34	148	650	2864
3	4	34	322	3164	31484	314662
4	8	148	3164	70878	1613060	36911922
5	16	650	31484	1613060	84231996	4427635270
6	32	2864	314662	36911922	4427635270	535236230270
7	64	12634	3149674	846280548	233276449488	64878517290010

Table 1: Numbers of ways of dividing an $n \times m$ rectangle, $a_{m,n}$.

Example 2. *Table 1 shows some data computed using the formula (1), Figure 1 gives an example, and the matrices M_2, M_3, M_4 are as follows:*

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 4 & 2 \\ 1 & 2 & 2 & 8 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 4 & 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 8 & 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 1 & 4 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 & 2 & 8 & 2 & 4 \\ 1 & 1 & 2 & 1 & 2 & 2 & 8 & 4 \\ 1 & 2 & 1 & 4 & 2 & 4 & 4 & 16 \end{pmatrix}.$$

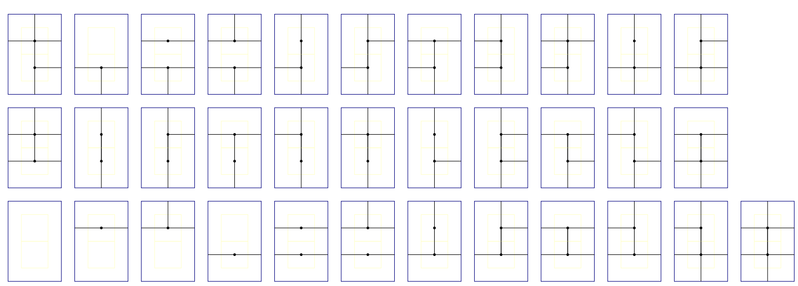


Figure 1: Ways of dividing a 3×2 rectangle into a union of rectangles.

2 Dividing rectangles into strips

In order to count divisions of rectangles, we cut the large rectangle into strips, as in the example in Figure 2. Two strips can only occur next to each other if the “fingers” sticking out to the right of one agree with those on

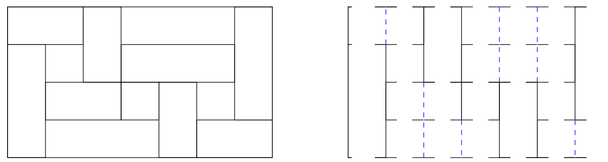


Figure 2: Slicing up a 4×7 rectangle (divided into rectangles) into 6 strips (plus 2 end edges on far left and right).

the left of the other. So counting divisions of rectangles becomes a problem about sequence of strips. We formally define the strips as follows:

Definition 1. *A* **strip** *of length m is a triple (F_l, U, F_r), where the* **left** *of the strip is a sequence $F_l = (f_{l,1}, f_{l,1}, \ldots, f_{l,m-1})$, the* **right** *is $F_r = (f_{r,1}, f_{r,1}, \ldots, f_{r,m-1})$, with $f_{l,i}, f_{r,i} \in \{0, 1\}$ for $1 \leq i < m$. The* **middle** *is a sequence $U = (u_1, \ldots, u_m)$ satisfying a kind of"gluing" condition:*

$$\begin{aligned} f_{li} \neq f_{ri} \quad &\Rightarrow \quad u_i = u_{i+1} = 1 \\ f_{li} = f_{ri} = 0 \quad &\Rightarrow \quad u_i = u_{i+1}. \end{aligned}$$

We define $\mathcal{S}_m$ to be the set of all possible strips of length m.

Example 3. *Figure 3 shows the geometrical interpretation of all elements of $\mathcal{S}_2$, given below, where commas in sequences of 0s and 1s are omitted:*

$$\mathcal{S}_2 = \left\{ \begin{array}{cccc} (0, 00, 0), & (0, 11, 0), & (0, 11, 1), & (1, 11, 0), \\ (1, 00, 1), & (1, 01, 1), & (1, 10, 1), & (1, 11, 1), \end{array} \right\}$$

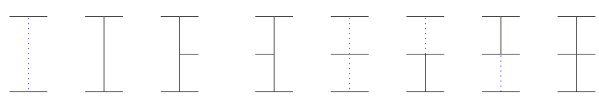


Figure 3: Possible strips in a $2 \times n$ divided rectangle.

Definition 2. *We define a directed graph $\mathcal{G}_m$. The vertices of $\mathcal{G}_m$ are given by $\mathcal{S}_n$. There is a directed edge from $S_a = (F_{al}, U_a, F_{ar})$ to $S_b = (F_{bl}, U_b, F_{br})$ if and only if $F_{ar} = F_{bl}$.*

Remark 2. *Note that the $\mathcal{G}_m$ has $a_{2,m}$ elements, the number of ways of dividing a $2 \times m$ rectangle, since this is essentially what the strips are.*

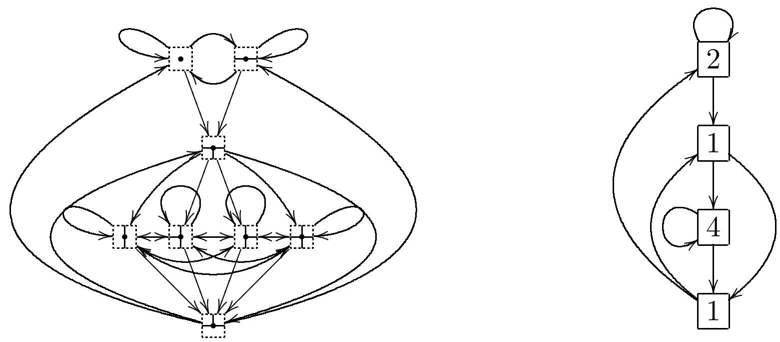


Figure 4: Paths in $\mathcal{G}_2$, on the left, correspond to divisions of an $2\times n$ rectangle into rectangles. The weighted graph $\mathcal{H}_2$, on the right, is a projection of $\mathcal{G}_2$. The weights of the vertices are the number of vertices of $\mathcal{G}_2$ which map to a vertex of $\mathcal{H}_2$. The strips in this picture are rotated through 90°.

Example 4. *Figure 4 shows $\mathcal{G}_2$ (turned on its side).*

The graph $\mathcal{G}_m$ has been constructed so that:

Proposition 5. *The number of ways of dividing a $m \times n$ rectangle into a union of rectangles with integer side lenghts is given by the number of paths of length $n-1$ in the graph $\mathcal{G}_m$.*

Corollary 6. *The number of ways of dividing a $m\times n$ rectangle into integer length sided rectangles is given by $\mathbf{1}\cdot A_m^{n-1}\cdot \mathbf{1}^t$ where $\mathbf{1} = (1,1,\ldots,1) \in \mathbf{Z}^{a_{2,m}}$, and A_m is the adjacency matrix of $\mathcal{G}_m$.*

3 The matrix M_m

We will now replace the matrix A_m with a matrix M_m, which corresponds to projecting from $\mathcal{G}_m$ to a certain weighted graph $\mathcal{H}_m$. Vertices of $\mathcal{H}_m$ are given by pairs (F_a, F_b) of possible left and right parts of of strips, and the weight is the number of ways to insert a U to get a valid strip (F_a, U, F_b). M_m will be the weighted adjacency matrix for $\mathcal{H}_m$. It is not hard to see that a path $v_1, \ldots, v_k$ in $\mathcal{H}_m$, with the vertex v_i having weight w_i, lifts to $w_1 w_2 \cdots w_k$ paths in $\mathcal{G}_m$. To define M_m, first we index the F_a as follows.

Definition 3. *For an integer x, $1 \le x \le 2^{m-1}$, with $x - 1 = \sum_{j=0}^{m-2} e_j 2^j$, where $e_j \in \{0,1\}$ for $0 \le j < m$, we set $F_x = (e_0, e_1, \ldots, e_{m-2})$.*

Now we define the matrix M_m so that the entry $M_m(a,b)$ gives the number of strips with left side F_a and right side F_b:

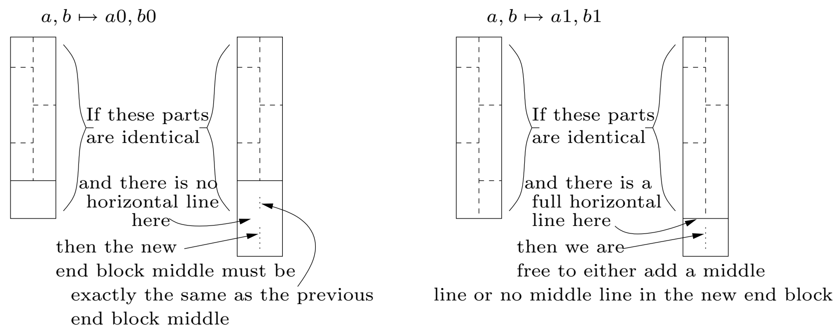


Figure 5: Extending a length m strip to a length $m+1$ strip

Definition 4. *For a positive integer m, we define the matrix M_m to be a 2^{m-1} by 2^{m-2} matrix, with entry $M_m(a,b)$ given by*

$$M_m(a,b) = \#\{x \in \mathcal{S}_m \ : \ x = (F_a, U, F_b) \text{ for some } U\}.$$

Since there are $M_n(i,j)$ strips with left side F_i and right side F_j, and since a strip with left side F_i and right side F_j can join to a strip with left side F_k and right side F_l if and only if $j = k$, we have the following result:

Proposition 7. *The number of ways of dividing an $n \times m$ rectangle into integer sided rectangles is given by*

$$\mathbf{1} \cdot (M_m)^{n-1} \cdot \mathbf{1}^t,$$

where M_m is as in Defintion 4.

4 Inductive construction of M_m

Because of the correspondence between sequeneces in $\{0,1\}$ and integers, we will abuse notation and write a to mean either an integer $1 \le a \le 2^{m-1}$, or a sequence of 2^{m-1} 0 and 1s. Then for $\epsilon \in \{0,1\}$, the notation $a\epsilon$ means the sequence formed by appending ϵ to the end of the sequence a, which also corresponds to the integer $a + 2^m$. Throughout this section $k = M_m(a,b)$.

Suppose we have computed M_m. I.e., we have computed the number of strips with left side F_a and right side F_b, for $1 \le a, b \le 2^{m-1}$. There are two ways to extend F_a and F_b to the $n+1$ case—either add 0 or 1 to a and b.

Extending from $M_m(a,b)$ to $M_{m+1}(a,b)$

Suppose there are k strips with left side F_a and right side F_b, for a,b some sequences of $m-1$ 0s and 1s. Then F_{a0}, F_{b0} are also joined in exactly k ways to form a strip, since the mid-line of the last section must be continued exactly as the immediately above section. An example is shown in Figure 5. Thus for $1 \leq a,b \leq 2^{m-1}$ we have, $M_{m+1}(a,b) = M_m(a,b)$.

Extending from $M_m(a,b)$ to $M_{m+1}(a+2^m, b+2^m)$

The strips F_{a1} and F_{b1} can be joined in $2k$ ways, because there is now a box at the bottom, which is free to either contain or not contain a vertical line. An example is shown in Figure 5. Thus $M_{m+1}(a+2^m, b+2^m) = 2M_m(a,b)$.

Considerations so far show that for some B_m, we have

$$M_{m+1} = \begin{pmatrix} M_m & B_m \\ B_m & 2M_m \end{pmatrix}. \tag{2}$$

Extending from $M_m(a,b)$ to $M_{m+1}(a+2^m, b)$ or $M_{m+1}(a, b+2^m)$

The strips corresponding to $a0$ and $b1$ join in either k or $k/2$ ways, since by Definition 1, in (F_{a0}, U, F_{b1}), we must have $u_m = 1$ and $u_{m-1} = 1$, whereas for (F_a, U, F_b) possibly no constraint on u_m. To determine which case we are in, we consider the last elements of the sequences a and b.

The case $a = a'1$, $b = b'1$

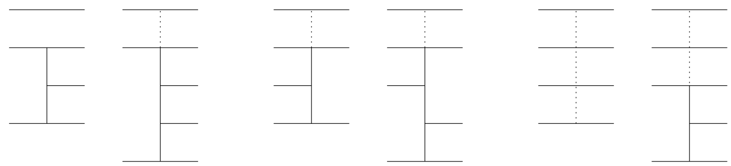
If we have $a = a'1$ and $b = b'1$, then strips corresponding to $a = a'10$ and $b = b'11$ can join in $k/2$ ways, since half the possible strips will not contain a vertical line in the nth place, but in the extension they must contain this line. An example is shown in the right picture in Figure 6.

We have seen that the part of M_{m-1} corresponding to pairs of the form $a'1, b'1$ is given by $2M_{m-2}$. Similarly, values for pairs of the form $a'10, b'11$ are given by M_{m-2}, so B_m has the form

$$B_m = \begin{pmatrix} ? & ? \\ ? & M_{m-1} \end{pmatrix}.$$

The cases $a = a'0$, $b = b'1$ and $a = a'1$, $b = b'0$

Strips corresponding to $a'00$ and $b'11$ can join in k ways, since we will be able to continue exactly in the same manner as for the middle of the mth



$a'0,\ b'1 \mapsto a'00, b'11$ $\quad$ $a'1,\ b'0 \mapsto a'00, b'11$ $\quad$ $a'1,\ b'1 \mapsto a'00, b'11$

Figure 6: Example of extending strips.

block, since the m and $m+1$ situations will be the same. An example is shown in Figure 6.

Similarly, strips corresponding to $a'10$ and $b'01$ can join in k ways, since there must be a vertical line at the mth position, which continues to the $m+1$th row. So B_m has the form

$$B_m = \begin{pmatrix} C_m & B_{m-1} \\ B_{m-1} & M_{m-1} \end{pmatrix}, \tag{3}$$

where C_m is still to be determined.

The case $a = a'0$, $b = b'0$

If we have $a = a'0$ and $b = b'0$, and $M(a'0, b'0) = k$, then to determine $M(a'00, b'01)$, we must consider the last terms of a' and b'. By similar considerations to the previous 3 cases, we find that

$$C_m = \begin{pmatrix} C_{m-1} & B_{m-2} \\ B_{m-2} & M_{m-2} \end{pmatrix}, \tag{4}$$

which tells us that $C_m = B_{m-1}$. From (3) and (4), we obtain

$$B_m = \begin{pmatrix} B_{m-1} & B_{m-1} \\ B_{m-1} & M_{m-1} \end{pmatrix}. \tag{5}$$

Proof of Theorem 1. Theorem 1 now follows from (5), (2), and the determination of B_1 and M_1, which by considering 1×1 and 2×2 rectangles are easily seen to be the 1×1 matrices (1) and (2) respectively. $\square$

Remark 3. *Instead gluing together strips, one can consider gluing together small squares, which come in eight types, as in Figure 4. This was the basis of the computer program written to draw the diagrams in this article.*

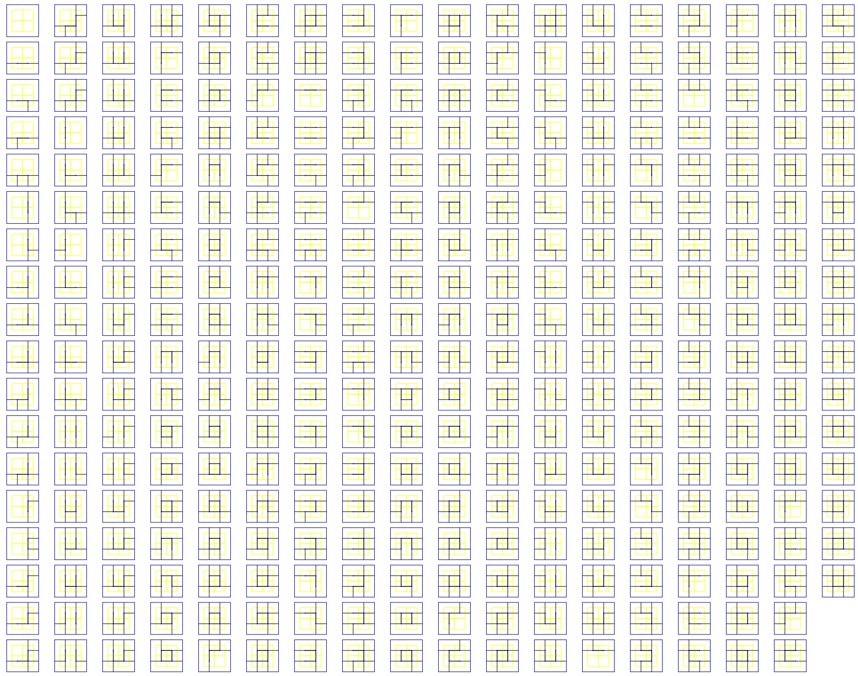


Figure 7: Ways of dividing a 3×3 rectangle into a union of rectangles.

Remark 4. *Since there is a symmetry between F_a and $F_{\overline{a}}$, where $\overline{\epsilon_1 \dots \epsilon_n} = \epsilon_n \dots \epsilon_1$, the matrix M_n can be replaced by a matrix N_n with $2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$ rows and columns, indexed by the sets $[a] = \{a, \overline{a}\}$ for $0 \leq a < 2^n$. We have $a_{m,n} = \mathbf{1} \cdot N_m^{n-1} \cdot \mathbf{v}$, where $\mathbf{v}$ has entries $v_{[a]} = |\{a, \overline{a}\}|$, and $\mathbf{1} = (1, \dots, 1)$. For example, for $m = 4$, with ordering of $[a]$ given by* $[000], [001], [010], [011], [101], [111]$, *we have* $\mathbf{v} = (1, 2, 1, 2, 1, 1)$, *and*

$$N_4 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 5 & 2 & 3 & 4 & 4 \\ 1 & 1 & 4 & 2 & 1 & 1 \\ 2 & 3 & 4 & 9 & 4 & 8 \\ 1 & 2 & 1 & 2 & 8 & 4 \\ 1 & 2 & 1 & 4 & 4 & 16 \end{pmatrix}.$$

References

[1] Ronald Read, *The dimer problem for narrow rectangular arrays: a unified method of solution, and some extensions.* Aequationes Math. 24 (1982), no. 1, 47–65.

[2] Neil Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/∼njas/sequences

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