Belgian Numbers

(formerly *Eric Numbers*)

176 is an "Belgian-**0** number" because, starting from **0**, one can build a sequence containing 176 in this way:

0 1 8 14 15 22 28 29 36 42 43 50 ... 155 162 168 169 176 ...
 1 7 6 1 7 6 1 7 6 1 7 ...   7   6   1   7

The "first differences" *building rule* is easy to understand. The above example shows that one doesn't have to add the full digit-pattern [1+7+6] to produce the according *Belgian number*: 176 already appears when 7 is added to the previous sum – *not* after 6 is added.

Here are the first **Belgian-0** numbers:

*Be**0*** = **0** 1 2 3 4 5 6 7 8 9 10 11 12 13 17 18 20 21 22 24 26 27 30 31 33 35 36 39 40 42 44 45 48 50 53 54 55 60 62 63 66 70 71 72 77 80 81 84 88 90 93 99 100 101 102 106 108 110 111 112 114 117 120 ...

Here is another example in order to explain how the above sequence works. Take its integer 17 for instance; 17 is a *Belgian-**0** number* because 17 belongs to this infinite sequence:

0 1 8 9 16 17 24 25 32 ...
 1 7 1 7 1 7 1 7

Now, we have started from 0 (zero) but we could have started from any other "seed", ranging from 0 to 9 (in the *Belgian number's world*, seeds cannot be greater than 9 – this will be explained later).

Belgian-1 numbers (seed in bold):

*Be**1*** = **1** 10 11 13 16 17 21 23 41 43 56 58 74 81 91 97 100 101 106 110 111 113 115 121 122 130 131 137 142 155 157 161 170 171 172 178 179 181 184 188 193 201 ...

179, for instance, is a *Belgian-**1** number* because (seed in bold):
1 2 9 18 19 26 35 36 43 52 53 ... 155 162 171 172 179.
 1 7 9 1 7 9 1 7 9 1 ...   7   9   1   7

Belgian-2 numbers:

*Be**2*** = **2** 10 11 12 15 16 20 22 25 26 32 38 41 42 46 67 72 82 86 91 95 100 101 102 103 105 107 110 111 112 113 115 116 120 121 122 123 124 125 130 131 132 134 136 138 142 143 ...

138, for instance, is a *Belgian-**2** number* (seed in bold):
2 3 6 14 15 18 26 27 30 38 39 ... 122 123 126 134 135 138.
1 3 8 1 3 8 1 3 8 1 ... 1 3 8 1 3

Belgian-3 numbers:

*Be**3*** = **3** 10 11 12 14 15 21 23 30 31 33 34 35 39 47 51 52 59 63 69 73 75 78 94 100 101 102 103 104 105 107 110 111 112 113 115 116 120 123 133 141 146 147 151 153 154 158 159 163 164 166 168 183 185 191 196 ...

159, for instance, is a *Belgian-**3** number* (seed in bold):
3 4 9 18 19 24 33 34 39 48 49 ... 139 144 153 154 159.
1 5 9 1 5 9 1 5 9 1 ... 5 9 1 5

Belgian-4 numbers:

*Be**4*** = **4** 10 11 13 14 20 21 22 24 25 31 32 37 40 43 44 51 54 57 64 65 76 82 84 87 89 92 98 100 101 104 110 111 112 114 116 121 122 124 125 127 128 137 140 141 142 144 145 148 149 151 154 158 172 177 191 196 ...

149, for instance, is a *Belgian-**4** number* (seed in bold):
4 5 9 18 19 23 32 33 37 46 47 ... 131 135 144 145 149.
1 4 9 1 4 9 1 4 9 1 ... 4 9 1 4

Belgian-5 numbers:

*Be**5*** = **5** 10 11 12 13 29 38 45 50 52 53 55 61 100 101 102 110 111 114 120 121 124 125 130 131 132 134 135 136 137 138 139 140 145 148 150 151 160 174 175 182 186 191 195 211 ...

148, for instance, is a *Belgian-**5** number* (seed in bold):
5 6 10 18 19 23 31 32 36 44 45 ... 127 135 136 140 148.
1 4 8 1 4 8 1 4 8 1 ... 8 1 4 8

Belgian-6 numbers:

*Be**6*** = **6** 10 11 12 20 21 22 23 24 28 30 33 34 36 41 42 46 49 58 60 61 62 66 68 73 83 92 96 100 101 102 103 110 111 112 113 114 118 120 121 122 123 126 127 128 129 130 131 132 133 134 136 138 143 150 155 156 ...

138, for instance, is a *Belgian-**6** number* (seed in bold):
6 7 10 18 19 22 30 31 34 42 43 ... 118 126 127 130 138.
1 3 8 1 3 8 1 3 8 1 ... 8 1 3 8

Belgian-7 numbers:

*Be***7** = **7** 10 11 21 27 29 31 32 37 41 56 70 71 77 85 94 100 101 103 106 110 111 112 113 117 118 119 122 127 128 131 133 143 152 173 176 201 205 ...

128, for instance, is a *Belgian-**7** number* (seed in bold):
7 8 10 18 19 21 29 30 32 40 41 ... 109 117 118 120 128.
1 2 8 1 2 8 1 2 8 1 ... 8 1 2 8

Belgian-8 numbers:

*Be**8*** = **8** 10 11 12 13 14 15 16 17 18 19 20 22 23 26 28 31 35 40 42 43 44 48 53 62 64 71 74 75 79 80 86 88 97 100 101 102 104 105 106 108 109 110 111 112 113 115 117 118 119 120 121 123 126 129 132 135 139 141 142 144 149 152 153 154 157 159 161 ...

119, for instance, is a *Belgian-**8** number* (seed in bold):
8 9 10 19 20 21 30 31 32 41 42 ... 107 108 109 118 119.
1 1 9 1 1 9 1 1 9 1 ... 1 1 9 1

Belgian-9 numbers:

*Be**9*** = **9** 10 11 12 13 14 15 16 17 18 19 21 25 27 30 32 33 36 45 51 54 57 63 67 69 72 81 83 90 93 99 100 101 102 104 105 108 109 110 111 115 117 119 120 121 122 123 124 126 129 130 135 139 140 141 142 144 146 149 153 159 161 162 164 165 166 169 ...

149, for instance, is a *Belgian-**9** number* (seed in bold):
9 10 14 23 24 28 37 38 42 51 52 ... 126 135 136 140 149.
1 4 9 1 4 9 1 4 9 1 ... 9 1 4 9

None of those sequences are yet in the **OEIS**. They will be submitted soon. (They are now)

****

Two types of **Self-Belgian Numbers** (SBN) could be also defined – if you are not asleep yet!

The first type (SBN_1) would only consist in *Belgian numbers* whose building sequence begins with the same seed as their leftmost digit.

179 is an example of *Self-Belgian Number* of type_1. The "seed" is **1** because 1 is the leftmost digit of **1**79. Here is the complete sequence leading to 179:

1 2 9 18 19 26 35 36 43 52 53 60 69 70 77 86 87 94 103 104 111 120 121 128 137 138 145 154 155 162 171 172 179....
1 7 9 1 7 9 1 7 9 1 7 9 1 7 9 1 7 9 1 7 9 1 7 9 1 7 9 1 7 9 1 7...

And here are the first **Self-Belgian Numbers** of type_1 (SBN_1):

SBN_1 = 0 1 2 3 4 5 6 7 8 9 10 11 13 16 17 20 22 25 26 30 31 33 34 35 39 40 43 44 50 52 53 55 60 61 62 66 68 70 71 77 80 86 88 90 93 99 100 101 106 110 111 113 115 121 122 130 131 137 142 155 157 161 170 171 172 178 179 181 184 188 193 200 ...

Again, this sequence should be red like this: **6**8 (for instance) is a *Belgian-6 number*; **7**0 is a *Belgian-7 number*; (and so are also 71 and 77); **8**0 is a *Belgian-8 number*, etc. All the above SBN_1 integers use their leftmost digit as seed for their building sequence.

____

The second types of *Self Belgian Numbers* (my favorites, SBN_2) are numbers who fully show all their digits (in the same order) at the beginning of their building sequence – and not only their leftmost one. **61** is the first such integer with more than one digit:

6 12 13 19 20 26 27 33 34 40 41 47 48 54 55 61.
6 1 6 1 6 1 6 1 6 1 6 1 6 1 6

As one can see, the seed remains **6** -> and *not* 61. If we allow seeds to have more than one digit, then all integers would be SEN_2, right from the beginning of their building sequence! This is why seeds cannot be greater than 9.

The beginning of the SBN_2 sequence looks like this (more terms in OEIS's A107070, here):

SBN_2 = 1, 2, 3, 4, 5, 6, 7, 8, 9, 61 71 918 3612 5101 8161 ... (a huge file by **Robert G. Wilson** is there)

Again, this last integer belongs to the SBN_2 family because its building sequence shows at the very beginning all it's digits (in the same order), see here:

```
    8 16 17 23 24 32 33 39 40 ... 8145 8151 8152 8160 8161
     8  1  6  1  8  1  6  1  ...      6    1    8    1
```

Hans Havermann has computed the prime terms of SBN_2, here.

____

The comment on this from **Eugene McDonnell**, here.
The wonderful page from **Jean-Paul Davalan**, with lots of applets to compute Belgian numbers, there.
And many, many thanks to **Robert G. Wilson** for his work and remarks on A107070.

____

May 5th update:

[**Mauro Fiorentini**]:

(...)
I suggest adding some (fairly obvious) notices to [your] page.

— All numbers that are multiple of the sum of their digits (like 48) are *0-Belgian*.

— The number of *Belgian-k numbers* is infinite for every *k*, as integers using only the digits 0 and 1 belong to all the classes.

— There is no *non-Belgian number*: every integer belongs to at least one class.

> If S(n) is the sum of the digits, take the sequence starting with 0; if it contains *n mod S(n)*, it will contain *n* as well, after adding S(n) several times; otherwise let *m* be the largest integer of the sequence not exceeding *n mod S(n)*, then *n* is *Belgian-(n mod S(n) - m)*, the difference being a single digit.
>
> For example, for n = 1949, S(n) = 23 and n mod S(n) = 17. The sequence goes 0, 1, 10, 14, 23, ... and the largest integer not exceeding 17 is 14, therefore 1949 is *Belgian-3*.

— In a similar way it can be proved that any *Belgian-0 number* is also *Belgian-k* for at least another *k*.

— *Belgian-k* numbers have a positive density for every *k*.

> Given a number *n* ending in 0, suppose it is not *Belgian-k*; then somewhere the sequence "skips over" *n*, because adding a digit *m*, the sum becomes too large. Then reduce that digit, incrementing the final 0 of the same amount (to preserve the sum of digits), and you'll get a number that is *Belgian-k*. So at least a number out of 100 is *Belgian-k*.

— The number of SBN_1 is infinite, as numbers using only the digits 0 and 1 belong to this class.

And now my (hard) question: do you have any idea about proving that SNS_2 is infinite? The best approach seems to build an infinite sequence of numbers belonging to this class, but I did not succeed.

____

Thanks, **Mauro**! Unfortunately I cannot ask your (hard) question!

But **Hans Havermann** made this remark (on May 7th, 2011):

> > a (hard) question by **Mauro Fiorentini**
>
> There are many questions in mathematics where it is hard to prove something, even where it is (empirically) obvious that it is (probably) true. A more productive question in this instance might be:
> Is there any reason to doubt that there are an infinite number of SNS_2 solutions?
>
> In order to answer this, I have put up the first 97550 terms of A107070 (a 2.2 MB file, in an economically structured and visually pleasing format):
>
> http://chesswanks.com/num/Type2Belgians.html
>
> Each blue digit marks the terminus of a solution (< 10^48969). A plot of the accumulating number of solutions as a function of digit-length is pretty much a straight line showing no sign of abating and averaging ~2 solutions per power-of-ten.
>
> I stopped at 48969 digits so that it would be easy for one to find an 8-solution result (by scrolling to the far right, it's missing the start-with-9). Other 8-solution results in the region are for 21955 digits (missing the start-with-4), 40727 digits (missing the start-with-2), and 48504 digits (missing the start-with-8). My hard question is: What is the first number-of-digits after 1 that has another full complement of 9 solutions?

And **Hans** sent me this, 2 weeks later:

> On 7 May 2011, I asked:
>
> > What is the first number-of-digits after 1 that has another full complement of 9 solutions?
>
> Answer: 1899283
> Here are the nine solutions (16.3 MB file):
> http://chesswanks.com/num/NineSolutionType2Belgians.html

Hans wrote me again around mi-August 2011:

> I spent a couple of months working out *all* solutions up to length 1899283, at which length all nine numbers (terms 3594728-3594736, I believe) are again solutions.

I wanted to replace the 16 MB < http://chesswanks.com/num/NineSolutionType2Belgians.html > (which colours just the first and last digits) with it, but the file ended up at a large 81 MB (with the solution-termini html-coloured as I did for the much smaller 48969-digit < http://chesswanks.com/num/Type2Belgians.html >) and it wouldn't display properly in *any* browser that I tried: the far-right digits did not align, even though all nine numbers have the same number of digits and they are rendered in a monospace font. Today, Firefox released version 6 of its browser and it renders my file correctly! So, if you can use that application and are willing to wait for the 81 MB to download, here it is:
http://chesswanks.com/num/bookends.html

Finally, a note about the 3594736 solutions up to and including length 1899283: wWhen I first counted the number of solutions, I thought that perhaps my program had miscalculated somehow, because I was expecting 2*1899283 or ~3.8 million solutions. The expectation was based on the assumption that an equidistribution of the ten base-ten digits within our nine templates predicts a long-term average of *two* solutions per digit-length. I think that the shortfall is because the distribution of digits within our nine templates, at that particular length, is *not* (approximately) equal. To show this, I counted the digits:

digit =>	1	2	3	4	5	6	7	8	9	0
template 1:	279021	196582	175420	172770	181449	186511	181568	176272	175570	174120
template 2:	279020	196582	175420	172770	181449	186511	181569	176272	175570	174120
template 3:	277935	197181	175429	173402	179434	185921	182257	177036	175605	175083
template 4:	279020	196581	175420	172771	181449	186511	181569	176272	175570	174120
template 5:	275807	199026	176656	174054	177416	185361	184155	177409	174746	174653
template 6:	277935	197182	175428	173402	179434	185921	182257	177036	175605	175083
template 7:	276749	198535	176024	173113	178675	186638	182192	177043	175360	174954
template 8:	279020	196581	175420	172770	181449	186511	181569	176273	175570	174120
template 9:	277932	197070	175464	173595	179346	186953	183096	176303	175219	174305

So, in fact, there are many more ones, and slightly more twos, in our nine templates (up to this length) and, hopefully, that explains the shortfall.

____

Many thanks to everyone for those nice results,
The concept of *Belgian numbers* came to the lousy author after his

discovery of the *Keith Numbers* (or Repfigits), there.
More terms (remarks and corrections) are always welcome (here).
Best,
É.

____

Les *nombres belges* firent l'objet d'une question des *Olympiades académiques de mathématiques*, le 23 mars 2011, comme on le verra page 89, ici.

____

[First draft: June 7th, 2005.]
Back to the main page (in French)