Notes on logarithmic differentiation, the binomial transform and series reversion

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We consider a modified logarithmic differentiation operator $\mathcal{L} : A(x) \to 1 + x\frac{A^{'}(x)}{A(x)}$ acting on power series $A(x)$. We show that the operator $\mathcal{L}$ commutes with the binomial transform and give some examples involving sequences from the OEIS. We show that $\mathcal{L}$ relates the Bell subgroup of the Riordan group to the Hitting time subgroup. $\mathcal{L}$ also links the sequence of Legendre polynomials $P_n(t)$ with the sequences of Chebyshev polynomials of the first kind $T_n(t)$.

1. The modified logarithmic differentiation operator $\mathcal{L}$.

Define a functional $\mathcal{L}$ by

$$\mathcal{L}(A(x)) \quad = \quad 1 + x\frac{A^{'}(x)}{A(x)}. \tag{1}$$

where the prime indicates differentiation with respect to x. For our purposes, the function $A(x)$ will be an ordinary generating function of a sequence $a(n)$ in the OEIS

$$A(x) \quad = \quad \sum_{n\geq 0} a(n)x^n,$$

where either $a(n)$ is an integer sequence or $a(n) = a(n,t)$ is the sequence of integral row polynomials of some lower triangular array - in which case $A(x) \equiv A(x,t)$ will be a bivariate generating function. We will often abuse notation and write $\mathcal{L}(a(n))$ to mean the sequence (or array) whose generating function is $\mathcal{L}(A(x))$. We restrict attention throughout these notes to sequences $a(n)$ with initial term $a(0) = 1$. Then $\mathcal{L}$ becomes a bijection with inverse $\mathcal{L}^{-1}$, where if

$$\mathcal{L}^{-1}\left(a(n)\right) \quad = \quad b(n),$$

the sequence $b(n)$ is determined by

$$\exp\left(\sum_{n\geq 1} a(n)\frac{x^n}{n}\right) \quad = \quad \sum_{n\geq 0} b(n)x^n. \tag{2}$$

If $a(n)$ is an integer sequence then $\mathcal{L}(a(n))$ will also be an integer sequence and we can construct an infinite chain of integer sequences

$$a(n) \to \mathcal{L}(a(n)) \to \mathcal{L}^2(a(n)) \to \cdots$$

In general, the operator $\mathcal{L}^{-1}$ does not preserve the integrality of a sequence. However, there are some well-known sequences $a(n)$ for which $\mathcal{L}^{-1}(a(n))$ appears to be an integer sequence including A000364 - the sequence of Euler numbers, A005258 and A005259 - the two sequences of Apéry numbers arising in Apéry's proof of the irrationality of $\zeta(3)$, and the sequene of Domb numbers A002895. Paul D. Hanna has contributed many other examples to the database. If we examine Hanna's contributions we are lead to a number of conjectures (C1 - C8 below) about the action of the inverse operator $\mathcal{L}^{-1}$ on sequences and arrays.

Suppose sequences $a(n)$ and $b(n)$ are such that $\mathcal{L}^{-1}(a(n))$ and $\mathcal{L}^{-1}(b(n))$ are integer sequences. It is clear from (2) that $\mathcal{L}^{-1}(a(n)+b(n))$ will also be an integer sequence. Are the following sequences also integral?

C1. $\mathcal{L}^{-1}(a(n)b(n))$

C2. $\mathcal{L}^{-1}(a(kn))$ for $k = 2, 3, ...$

C3. $\mathcal{L}^{-1}(a(n^k))$ for $k = 2, 3, ...$

Similar questions may be asked about the action of the inverse operator $\mathcal{L}^{-1}$ on arrays. Let $A = (u_{i,j}), B = (v_{i,j})$ with $u_{0,0} = v_{0,0} = 1$ be lower triangular arrays with integer entries. Suppose both $\mathcal{L}^{-1}(A)$ and $\mathcal{L}^{-1}(B)$ are integral arrays. It is clear that $\mathcal{L}^{-1}(A + B)$ will be integral. Are the following arrays guaranteed to be integral?

C4. $\mathcal{L}^{-1}(AB)$

C5. $\mathcal{L}^{-1}(A \odot B)$, where $A \odot B = (u_{i,j}v_{i,j})$ denotes the Hadamard product of A and B.

C6. $\mathcal{L}^{-1}((u_{ki,kj}))$ for $k = 2, 3, ...$

C7. $\mathcal{L}^{-1}((u_{i^2,ij}))$

C8. $\mathcal{L}^{-1}((u_{ij,j^2}))$

2. $\mathcal{L}$ and the binomial transform.

Let Bin denote the binomial transform of a sequence

$$\text{Bin } (a(n)) \quad = \quad \sum_{k=0}^{n} \binom{n}{k} a(k).$$

At the generating function level, if $A(x) = \sum_{n\geq 0} a(n)x^n$ is the ordinary generating function for the sequence $a(n)$ then

$$\mathrm{Bin}\,(A(x)) \quad = \quad \frac{1}{1-x}A\left(\frac{x}{1-x}\right). \tag{3}$$

One advantage of using the operator $\mathcal{L}$ instead of the the usual logarithmic differentiation operator $A'(x)/A(x)$ is that $\mathcal{L}$ (and hence also $\mathcal{L}^{-1}$) commutes with the binomial transformation

$$\mathcal{L}\text{ o Bin} \quad = \quad \text{Bin o }\mathcal{L} \tag{4}$$

$$\mathcal{L}^{-1}\text{ o Bin} = \text{Bin o }\mathcal{L}^{-1}. \tag{5}$$

These follow easily from (3) and the definition of $\mathcal{L}$ in (1). A simple induction argument then shows that for $m \in \mathbb{Z}$

$$\mathcal{L}\text{ o Bin}^m \quad = \quad \text{Bin}^m\text{ o }\mathcal{L} \tag{6}$$

$$\mathcal{L}^{-1}\text{ o Bin}^m = \text{Bin}^m\text{ o }\mathcal{L}^{-1}. \tag{7}$$

Consequently, if $\mathcal{L}^{-1}(a_n)$ is an integer sequence (or sequence of polynomials with integer coefficients) then for $m \in \mathbb{Z}$,

$$b(n) := \mathcal{L}^{-1}\left(\text{Bin}^m(a(n))\right) = \text{Bin}^m(\mathcal{L}^{-1}(a(n)))$$

will also be an integer sequence (or sequence of polynomials with integer coefficients).

Example 1. Starting with A199572 = [1,0,4,0,16,0,64,...], the powers of 4 aerated with zeros, we produce the following commutative diagram of OEIS sequences. The diagram links several classic sequences: A002426, the central trinomial coefficients, A000984, the central binomial coefficients (A126869 is an aerated version), A001006, the Motzkin numbers, and A000108, the Catalan numbers (A126120 is an aerated version).

$$\begin{tikzcd}
A199572 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A046717 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A081294 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A034478 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A081335 \arrow[d, "\mathcal{L}^{-1}"'] \\
A126869 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A002426 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A000984 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A026375 \arrow[r, "Bin"] \arrow[d, "\mathcal{L}^{-1}"'] & A081671 \arrow[d, "\mathcal{L}^{-1}"'] \\
A126120 \arrow[r, "Bin"] & A001006 \arrow[r, "Bin"] & A000108^{*} \arrow[r, "Bin"] & A002212^{*} \arrow[r, "Bin"] & A005572
\end{tikzcd}$$

Fig. 1 (* indicates the initial term of the sequence is omitted)

The central trinomial coefficients A002426 occurring on the middle row of Fig. 1 are defined as $[x^n]\,(1+x+x^2)^n$ where $[x^n]$ is the coefficient extraction operator. We can find similar expressions for the terms of the other sequences in the middle row of Fig. 1 using the following result.

Proposition 1. *Let $F(x)$ and $G(x)$ be formal power series. If $a(n) = [x^n]\,F(x)G(x)^n$ then for $m \in \mathbb{Z}$,*

$$\mathrm{Bin}^m(a(n)) = [x^n]\,F(x)(mx+G(x))^n.$$

Proof.

$$\begin{aligned}
[x^n]F(x)(mx+G(x))^n &= [x^n]\left(F(x)\sum_k \binom{n}{k} m^{n-k}x^{n-k}G(x)^k\right)\\
&= \sum_k \binom{n}{k} m^{n-k}[x^k]\left(F(x)G(x)^k\right)\\
&= \sum_k \binom{n}{k} m^{n-k}a(k)\\
&= \mathrm{Bin}^m(a(n)).
\end{aligned}$$

□

Example 1 continued. Starting with $A002426(n) = [x^n]\,(1+x+x^2)^n$ we apply Proposition 1 to the middle row of Fig. 1 to complete Table 1.

Table 1.

$a(n) = [x^n]\,G(x)^n$
$\mathrm{A126869}(n) = [x^n]\,(1+x^2)^n$
$\mathrm{A002426}(n) = [x^n]\,(1+x+x^2)^n$
$\mathrm{A000984}(n) = [x^n]\,(1+2x+x^2)^n$
$\mathrm{A026375}(n) = [x^n]\,(1+3x+x^2)^n$
$\mathrm{A081671}(n) = [x^n]\,(1+4x+x^2)^n$

The next result relates sequences belonging to the same column in diagrams such as Fig. 1.

Proposition 2. *Let $G(x) = 1 + g_1x + g_2x^2 + \cdots$ be a formal power series and define the sequence $a(n) = [x^n]\,G(x)^n$ with generating function $A(x) = \sum_{n\geq 0} a(n)x^n$. Then*

(i)

$$\mathcal{L}\,(a(n)) = [x^n]\left(\frac{x}{\mathrm{Rev}(xA(x))}\right)^n$$

where Rev denotes the series reversion with respect to x.

(ii) The sequence $\mathcal{L}^{-1}(a(n))$ has the ordinary generating function

$$\mathcal{L}^{-1}(A(x)) \quad = \quad \frac{1}{x} \mathit{Rev} \left(\frac{x}{G(x)} \right).$$

(iii) If $G(x)$ has integer coefficients then the sequence $\mathcal{L}^{-1}(a(n))$ is an integer sequence.

Proof.

(i) By [1, Proposition 2] (stated there for integer sequences but this restriction is inessential) applied to the sequence $\mathcal{L}(a(n))$ there is a power series $H(x)$ such that

$$\mathcal{L}\left(a(n)\right) \quad = \quad [x^n]\, H(x)^n\,,$$

where $H(x)$ is given by

$$\begin{aligned} H(x) \quad &= \quad \frac{x}{\operatorname{Rev}\left(x \exp\left(\sum_{n \geq 1} \mathcal{L}(a(n)) \frac{x^n}{n} \right)\right)}. && (8) \\ &= \quad \frac{x}{\operatorname{Rev}\left(x \mathcal{L}^{-1}\left(\mathcal{L}(A(x)) \right)\right)} && (9) \\ &= \quad \frac{x}{\operatorname{Rev}(xA(x))}\,. && (10) \end{aligned}$$

(ii) By [1, Proposition 2] applied to the sequence $a(n)$, the power series $G(x)$ satisfies

$$G(x) \quad = \quad \frac{x}{\operatorname{Rev}\left(x \exp\left(\sum_{n \geq 1} a(n) \frac{x^n}{n} \right)\right)}. \qquad (11)$$

Thus

$$\begin{aligned} \mathcal{L}^{-1}\left(A(x)\right) \quad &= \quad \exp\left(\sum_{n \geq 1} a(n) \frac{x^n}{n} \right) \text{ by (2)} \\ &= \quad \frac{1}{x} \operatorname{Rev}\left(\frac{x}{G(x)} \right) \text{ by (11).} \end{aligned}$$

(iii) By assumption, the constant term of $G(x)$ equals 1. Hence the series expansion of $x/G(x)$ and thus also of $\operatorname{Rev}(x/G(x))$ will have integer coefficients. Thus by part (ii), $\mathcal{L}^{-1}(A(x))$ has integer coefficients. $\Box$

Example 1 continued. Applying Proposition 2 to the results in Table 1 gives the following information about the sequences in Fig. 1.

Table 2.

$a(n)$	$\mathcal{L}(a(n))$	G.f. $\mathcal{L}^{-1}(a(n))$
A126869(n)	A199572(n) $= [x^n]\left(\sqrt{1+4x^2}\right)^n$	G.f. A126120 $= \frac{1}{x}\mathrm{Rev}\left(\frac{x}{1+x^2}\right)$
A002426(n)	A046717(n) $= [x^n]\left(x+\sqrt{1+4x^2}\right)^n$	G.f. A001006 $= \frac{1}{x}\mathrm{Rev}\left(\frac{x}{1+x+x^2}\right)$
A000984(n)	A081294(n) $= [x^n]\left(2x+\sqrt{1+4x^2}\right)^n$	G.f. A000108* $= \frac{1}{x}\mathrm{Rev}\left(\frac{x}{1+2x+x^2}\right)$
A026375(n)	A034478(n) $= [x^n]\left(3x+\sqrt{1+4x^2}\right)^n$	G.f. A002212* $= \frac{1}{x}\mathrm{Rev}\left(\frac{x}{1+3x+x^2}\right)$
A081671(n)	A081335(n) $= [x^n]\left(4x+\sqrt{1+4x^2}\right)^n$	G.f. A005572 $= \frac{1}{x}\mathrm{Rev}\left(\frac{x}{1+4x+x^2}\right)$

* indicates the initial term of the sequence is omitted

The following result reveals a surprising connection between the binomial transform Bin and the series reversion operator Rev.

Proposition 3. *Let* $F(x) = 1 + f_1x + f_2x^2 + \cdots$ *be a power series. Then for integer* m *we have*

$$\mathrm{Bin}^m(F(x)) = \frac{1}{x}\mathrm{Rev}\left(\frac{\mathrm{Rev}(xF(x))}{1 + m\mathrm{Rev}(xF(x))}\right).$$

Proof. Define the power series $G(x)$ by

$$G(x) = \frac{x}{\mathrm{Rev}(xF(x))} \tag{12}$$

so that

$$F(x) = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{G(x)}\right). \tag{13}$$

Define the sequence $a(n) = [x^n]\,G(x)^n$. By Proposition 1, the m-th binomial transform of $a(n)$ is the sequence $[x^n]\,(mx + G(x))^n$. By (7) and Proposition 2 (ii), we have the commutative diagram

$$\begin{array}{ccc} a(n) = [x^n]G(x)^n & \xrightarrow{Bin^m} & [x^n](mx+G(x))^n \\ \big\downarrow \mathcal{L}^{-1} & & \mathcal{L}^{-1} \big\downarrow \\ \text{g.f.} = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{G(x)}\right) & \xrightarrow{Bin^m} & \text{g.f.} = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{mx+G(x)}\right) \end{array}$$

The bottom row of the diagram reads

$$\mathrm{Bin}^m\left(\frac{1}{x}\mathrm{Rev}\left(\frac{x}{G(x)}\right)\right) \quad = \quad \frac{1}{x}\mathrm{Rev}\left(\frac{x}{mx+G(x)}\right),$$

which we can rewrite in terms of $F(x)$ using (12) and (13) to give

$$\mathrm{Bin}^m(F(x)) \quad = \quad \frac{1}{x}\mathrm{Rev}\left(\frac{\mathrm{Rev}(xF(x))}{1+m\mathrm{Rev}(xF(x))}\right).$$

□

3. The $\mathcal{T}$ transformation.

We define a transformation $\mathcal{T}$ of lower triangular arrays and show it commutes with the operator $\mathcal{L}$.

Let B (for binomial) denote Pascal's triangle A007318 and let M be an arbitrary lower triangular array with row generating polynomials $R(n,t)$ with $R(0,t) = 1$. The ordinary generating function $G(x,t)$ for the array M is thus

$$G(x,t) \quad = \quad \sum_{n\geq 0} R(n,t)x^n.$$

The binomial transform of M is obtained by pre-multiplying M by B. Let now $\mathcal{T}$ denote the transformation of lower triangular arrays that acts by post-multiplying by B:

$$\mathcal{T} : M \quad \to \quad MB.$$

To see how the operator $\mathcal{T}$ acts on generating functions we calculate

$$\begin{aligned}\mathcal{T}(M)\begin{bmatrix}1\\t\\t^2\\\vdots\end{bmatrix} &= MB\begin{bmatrix}1\\t\\t^2\\\vdots\end{bmatrix}\\ &= M\begin{bmatrix}1\\1+t\\(1+t)^2\\\vdots\end{bmatrix}\\ &= \begin{bmatrix}1\\R(1,1+t)\\R(2,1+t)\\\vdots\end{bmatrix}.\end{aligned}$$

Hence the transformed array $\mathcal{T}(M)$ has the generating function

$$G(x,t+1) = \sum_{n\geq 0} R(n,t+1)x^n.$$

Thus $\mathcal{T}$ acts as translation operator in the variable t. It then follows easily from the definition (1) of $\mathcal{L}$ that the operators $\mathcal{L}$ and $\mathcal{T}$ commute.

Example 2. We have the following commutative diagram of triangular arrays taken from the OEIS - n.l. indicates that the array is not currently listed in the OEIS.

Fig. 2

$$\begin{tikzcd}
A065600 \arrow[r, "\mathcal{T}"] \arrow[d, "\mathcal{L}"'] & A033184 \arrow[r, "\mathcal{T}"] \arrow[d, "\mathcal{L}"'] & A039598 \arrow[r, "\mathcal{T}"] \arrow[d, "\mathcal{L}"'] & A035324 \arrow[d, "\mathcal{L}"'] \\
n.l \arrow[r, "\mathcal{T}"'] & A100100 \arrow[r, "\mathcal{T}"'] & A094527 \arrow[r, "\mathcal{T}"'] & A113955
\end{tikzcd}$$

All the arrays in Fig. 2 are Riordan arrays. The arrays in the top row of the diagram belong to the Bell subgroup of the Riordan group while the arrays in the bottom row of the diagram belong to the Hitting time subgroup of the Riordan group. In the next section we shall show how the modified logarithmic differentiation operator $\mathcal{L}$ relates the Bell subgroup and the Hitting time subgroup of the Riordan group.

4. The Riordan group.

A ***proper Riordan array*** [3] is a lower unit triangular array such that the k-th column of the array has the ordinary generating function

$$f(x)F(x)^k$$

for $k = 0, 1, 2, 3, \ldots$, where $f(x)$ and $F(x)$ are formal power series in the indeterminate x of the form

$$f(x) = 1 + a_1x + a_2x^2 + \cdots \tag{14}$$
$$F(x) = x + b_2x^2 + b_3x^3 + \cdots . \tag{15}$$

The coefficients a_i and b_i could be real or complex but for examples drawn from the OEIS they will be integers. We denote the proper Riordan array associated with the powers series $f(x)$ and $F(x)$ by $(f(x), F(x))$. The bivariate generating function of the Riordan array $(f(x), F(x))$ equals $f(x)/(1 - tF(x))$.

The set of proper Riordan arrays forms a group called the ***Riordan group*** under the operation of matrix multiplication. The multiplication law is

$$(f(x), F(x))\,(g(x), G(x)) = (f(x)g(F(x)), G(F(x)))\,. \tag{16}$$

The proper Riordan array $(1, x)$ is the infinite identity matrix. The inverse of the proper Riordan array $(f(x), F(x))$ is given by

$$(f(x), F(x))^{-1} \quad = \quad \left(\frac{1}{f(\overline{F}(x))}, \overline{F}(x)\right), \tag{17}$$

where $\overline{F}$ denotes the compositional inverse of F, that is,

$$F(\overline{F}(x)) = \overline{F}(F(x)) = x.$$

For a pair of numbers r, s we define a subset $G_{r,s}$ of the Riordan group by

$$G_{r,s} \quad = \quad \left\{\left(\left(\frac{F(x)}{x}\right)^{r} \left(F^{'}(x)\right)^{s}, F(x)\right) : F(x) = x + b_2x^2 + b_3x^3 + \cdots\right\}$$

Using (16) and (17) it is straightforward to check that the set $G_{r,s}$ is closed under multiplication and taking inverses and hence $G_{r,s}$ is a subgroup of the Riordan group. The subgroups $G_{r,s}$ are isomorphic to each other for different choices of r and s. The mapping $\phi \equiv \phi(r, s, r^{'}, s^{'})$ defined by

$$\left(\left(\frac{F(x)}{x}\right)^{r} \left(F^{'}(x)\right)^{s}, F(x)\right) \quad \xrightarrow{\phi} \quad \left(\left(\frac{F(x)}{x}\right)^{r^{'}} \left(F^{'}(x)\right)^{s^{'}}, F(x)\right) \tag{18}$$

is easily shown to give an isomorphism between the groups $G_{r,s}$ and $G_{r^{'},s^{'}}$.

Some of the subgroups $G_{r,s}$ have been named and studied in the literature.

Table 3.

r, s	$G_{r,s}$	Riordan arrays
$r = 0, s = 0$	Associated subgroup	$\{(1, F(x))\}$
$r = 1, s = 0$	Bell subgroup	$\{(\frac{F(x)}{x}, F(x))\}$
$r = 0, s = 1$	Derivative subgroup	$\left\{\left(F^{'}(x), F(x)\right)\right\}$
$r = -1, s = 1$	Hitting-time subgroup	$\left\{\left(x\frac{F^{'}(x)}{F(x)}, F(x)\right)\right\}$

Proposition 4. *The isomorphism $\phi : G_{1,0} \to G_{-1,1}$ between the Bell subgroup and the Hitting time subgroup of the Riordan group is given by the modified logarithmic differentiation operator $\mathcal{L}$.*

Proof. Let $(F(x)/x, F(x))$ be a Riordan array in the Bell group. The bivariate generating function for the array is

$$\frac{F(x)}{x}\frac{1}{1-tF(x)}.$$

A short calculation gives

$$\begin{aligned}\mathcal{L}\left(\frac{F(x)}{x}\frac{1}{1-tF(x)}\right) &= 1+x\frac{d}{dx}\left(\log\left(\frac{F(x)}{x}\frac{1}{1-tF(x)}\right)\right)\\ &= x\frac{F^{'}(x)}{F(x)}\frac{1}{1-tF(x)},\end{aligned}$$

which is the generating function for the Hitting time array $\left(xF^{'}(x)/F(x), F(x)\right)$. $\square$

5. Legendre and Chebyshev Polynomials

The operator $\mathcal{L}$ links the sequence of Legendre polynomials $P_n(t)$ with the sequences of Chebyshev polynomials of the first kind $T_n(t)$. The Legendre polynomials $P_n(t)$ have the generating function

$$\sum_{n\geq 0} P_n(t)x^n = \frac{1}{\sqrt{1-2tx+x^2}}. \tag{19}$$

The Chebyshev polynomials of the first kind $T_n(t)$ have the generating function

$$\sum_{n\geq 0} T_n(t)x^n = \frac{1-tx}{1-2tx+x^2}. \tag{20}$$

Applying the operator $\mathcal{L}$ to (19) gives

$$\begin{aligned}\mathcal{L}\left(\sum_{n\geq 0} P_n(t)x^n\right) &= \mathcal{L}\left(\frac{1}{\sqrt{1-2tx+x^2}}\right)\\ &= 1+x\frac{d}{dx}\log\left(\frac{1}{\sqrt{1-2tx+x^2}}\right)\\ &= \frac{1-tx}{1-2tx+x^2}\\ &= \sum_{n\geq 0} T_n(t)x^n \text{ by (20).}\end{aligned}$$

APPENDIX

We give two more examples of commutative diagrams of OEIS sequences. A * indicates the sequence is aerated with zeros. For example, A001018* is [1,0,8,0,64,0,512,0,...]. A ** indicates the sequence has its initial term omitted.

Example 3.

$$\begin{array}{ccccccccc}
A001018^{*} & \xrightarrow{Bin} & A084058 & \xrightarrow{Bin} & A084128 & \xrightarrow{Bin} & A001541 & \xrightarrow{Bin} & A084130 \\
\downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} \\
A059304^{*} & \xrightarrow{Bin} & A084601 & \xrightarrow{Bin} & A006139 & \xrightarrow{Bin} & A001850 & \xrightarrow{Bin} & A080609 \\
\downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} \\
A151374^{*} & \xrightarrow{Bin} & A025235 & \xrightarrow{Bin} & A071356 & \xrightarrow{Bin} & A001003^{**} & \xrightarrow{Bin} & A068764^{**}
\end{array}$$

Fig. 3

Example 4.

$$\begin{array}{ccccccccc}
A001021^{*} & \xrightarrow{Bin} & A090042 & \xrightarrow{Bin} & \text{not listed} & \xrightarrow{Bin} & A141041 & \xrightarrow{Bin} & A090965 \\
\downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} \\
A098658^{*} & \xrightarrow{Bin} & A084603 & \xrightarrow{Bin} & A084609 & \xrightarrow{Bin} & A122868 & \xrightarrow{Bin} & A069835 \\
\downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} & & \downarrow{\scriptstyle \mathcal{L}^{-1}} \\
A005159^{*} & \xrightarrow{Bin} & A025237 & \xrightarrow{Bin} & A122871 & \xrightarrow{Bin} & A107264 & \xrightarrow{Bin} & A007564^{**}
\end{array}$$

Fig. 4

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