REVIEWS

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How Does One Cut A Triangle? By Alexander Soifer, Center for Excellence in Mathematical Education, 885 Red Mesa Drive, Colorado Springs, CO 80906, 1990, xiii + 138 pp.

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It is some thirty years since Dieudonné's Royaumont Seminar Address [1] in which he proclaimed "Euclid Must Go" and that it should be replaced by two-dimensional linear algebra. Dieudonné's quarrel was not with the purpose of geometry, which he considered important, but with the *method* of teaching geometry. Dieudonné's reasons were that Euclid as taught was a "Process Fantastically Laborious" as well as having "the affine and metric properties hopelessly mixed up." While there are many who would agree with the latter reasons, there are many who do not agree with the linear algebra replacement as proposed. For a good late survey of this, see Burn [2] and references therein. Burn notes that Freudenthal [3] in his chapter on "The Case of Geometry" offers a profound critique of Dieudonné and recommends a piecemeal school geometry program with strong intuitive spatial content. As to "What is Geometry?," see Freudenthal [3], Chern [4, 5], Yaglom [6] and Kelly [7].

Although Dieudonné's message was not heeded in the U.S. and Canada, it now appears that in our secondary schools Euclid and other geometry is essentially gone and has not been replaced. This treatment of geometry in our schools should be contrasted with that in the U.S.S.R., Hungary and Romania, for example, in which geometry is covered repeatedly in the different grades. They have many good books with challenging problems, e.g., [8–17] (fortunately, these have been translated into English) unlike most of the sterile ones used in the U.S. and Canada which take great pains to prove very obvious theorems but not unobvious attractive theorems and insist that proofs be done in "two columns."†

Some time ago Poincaré addressed the downgrading of geometry in his classic book *The Foundation of Science* [18] which I strongly believe should be required reading for all mathematical educators and teachers. I quote:

> It looks as if geometry could contain nothing which is not already included in algebra or analysis; that geometrical facts are only algebraic or analytic facts expressed in another language. It might then be thought that after our review there would remain nothing more for us to say relating specially to geometry. This would be to fail to recognize the importance of well-constructed language, not to comprehend what is added to the things themselves by the method of expressing these things and consequently of grouping them.
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> First the geometric considerations lead us to set ourselves new problems; these may be, if you choose, analytic problems, but such as we never would have set ourselves in connection with

† Apparently in the 1990 geometry syllabus in Alberta, all proofs have been removed.

analysis. Analysis profits by them however, as it profits by those it has to solve and satisfy the needs of physics.

A great advantage of geometry lies in the fact that in it the senses can come to the aid of thought, and help find the path to follow, and many minds prefer to put the problems of analysis into geometric form. Unhappily our senses can not carry us very far, and they desert us when we wish to soar beyond the classical three dimensions. Does this mean that, beyond the restricted domain wherein they seem to wish to imprison us, we should rely only on pure analysis and that all geometry of more than three dimensions is vain and object-less? The greatest masters of a preceding generation would have answered 'yes'; today we are so familiarized with this notion that we can speak of it, even in a university course, without arousing too much astonishment.

But what good is it? That is easy to see: First it gives us a very convenient terminology, which expresses concisely what the ordinary analytic language would say in prolix phrases. Moreover, this language makes us call like things by the same name and emphasize analogies it will never again let us forget. It enables us therefore still to find our way in this space which is too big for us and which we can not see, always recalling visible space, which is only an imperfect image of it doubtless, but which is nevertheless an image. Here, again, as in all the preceding examples, it is analogy with the simple which enables us to comprehend the complex.

The downgrading of geometry also affects many courses down the line. In the last few years, there has been lots of activity about improving our calculus courses **[19–20]**. No doubt there are improvements to be made in our textbooks and teaching methods but another improvement should be in the students' secondary school preparation. In particular, I and a number of my colleagues have found that wherever there is any geometry involved in the calculus, and there is quite a bit, students have all kinds of difficulties. In this regard, I quote from the beginning of the late N. Steenrod's first lecture at a geometry conference **[21]** which addresses this point and which is highly in tune with Pioncaré (above).

In this series of lectures I am going to discuss the geometric content of the freshman and sophomore mathematics courses. I shall criticize what we as teachers are now doing and suggest what we might do. Let me begin with what I believe to be the chief criticisms.

(1) Although geometry pervades all of mathematics and is present at every stage of a development, too often do we fail to point this out to our students. We rely on analytical formulations since we realize that they are complete and we are in a hurry to get on to other material. We do not take time to look at geometric formulations.

(2) We are too greatly impressed by the rigor of analysis. We seem to feel that geometry is not rigorous, or at least that the background needed for rigor is not avaialble. We feel that it is better not to do anything that is not rigorous. I think we are buffaloed too much by this.

(3) When we do present geometry, it is too often the instructor who does the geometry while the student is merely a passive recipient. We present the geometry to him in order to explain the analysis, but then we require him to do only the analysis—no geometry.

(4) We tend to avoid geometric formulations of questions in examinations. Questions are difficult to formulate geometrically. Almost any time you try such a question, you find that a large group of students misinterpret it. Such questions are hard to grade because the answers are so varied. The absence of geometric questions on final exams tends to degrade the geometric content of the course, and leads to its neglect.

Now that I have listed the main criticisms, let me take them up one at a time and fill in some details. The pervasiveness of geometry is an idea that goes back to Descartes, for a coordinate system in the plane or in space sets up an equivalence between geometry and algebra-analysis. Every geometric proposition can be translated into its algebraic-analytic analog and vice versa. I am not proposing that we lead the student through the details of the formal isomorphism between these two systems, but I am trying to remind you that the geometry is always there and to keep in mind that the geometric language for the conversion is always at hand. For example, A5792, A74764

here are a few geometric terms and their algebraic-analytic counterparts:

Geometric language	*Algebraic-Analytic language*
point, vector	number triple (x, y, z)
projection	coordinate, variable
surface	equation
plane	linear equation
region	system of inequalities
mapping, transformation	function
neighborhood	ε, δ
limit (using deleted neighborhoods)	limit (using ε, δ)
tangent	derivative

One might ask, in view of this equivalence, why bother with the geometry at all? The answer is clear to all of us. The first main reason is that many applications of calculus are to problems of conics, oscillating systems, the 2-body problem. A complete solution of such a problem has three main steps: The reformulation in analytic terms, the derivation of an analytic solution, and the interpretation of the solution in geometric terms.

	Geometry		Analysis
Problem	.	$\longrightarrow$	.
Solution	.	$\longleftarrow$	.

A second reason is psychological. Two views of the same thing reinforce one another. Most of us are able to remember the multitudinous formulas of analysis mainly because we attach to each a geometric picture that keeps us from going astray. Even better than that, the geometric view of a problem helps us to focus on the invariants and to weed out the irrelevant details. For example, a poor choice of a coordinate system may lead to a horrible mess in the analytic formulation, but with some geometric insight, we may be able to choose a much better coordinate system.

Since students are not getting enough geometry in school, one of the avenues left is self-study. Any new attractive books in this field are indeed very welcome. *How does one cut a triangle*? is one of these and is mostly combinatorial geometry that is accessible to bright high school and college students even without too much prior knowledge. Also, there is quite a number of unsolved problems which will challenge professional mathematicians and other scientists. There are various monetary awards for the first received acceptable solutions of some of these latter problems. Like the challenge of climbing mountains because they are, there is a mental challenge of solving the problems not only because they are here (in this book), but also because they are simple to comprehend but still challenging.

A5792 → A74764 →

Two of the opening problems are "**Grand Problem I. Find all positive integers *n*, such that every triangle can be cut into *n* triangles congruent to each other,**" and "**Grand Problem II. Find all positive integers *n*, such that every triangle can be cut into *n* triangles similar to each other.**" Even deciding which of these two problems is harder is a problem. Although Dieudonné is strongly opposed to "triangles," he may be attracted to the solutions of these problems since one of them requires quite a bit of linear algebra. However, I think Soifer's excursion into the linear algebra necessary in the solution is perhaps too brief.

Another set of problems deals with the problem of given a certain number of points within or on a triangle to determine the maximum of the minimum area formed by three of the points. In particular, three proofs are given to show that for any five points in a triangle of area 1, there are three points that form a triangle of