International Mathematical Olympiad 2001
Hong Kong Preliminary Selection Contest
(Sponsored by the Quality Education Fund)
Solutions

1. (1 mark) Find the sum of all real x satisfying $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$.
Solution: Let $a = 2^x - 4$, and $b = 4^x - 2$, then the equation becomes
$a^3 + b^3 = (a + b)^3$, giving $3ab(a + b) = 0$
Either a=0, or b = 0, or a + b = 0
i.e., $2^x - 4 = 0$, or $4^x - 2 = 0$, or $4^x + 2^x - 6 = (2^x + 3)(2^x - 2) = 0$
Get x = 2, 1/2 or 1. The sum is 7/2 .

2. (1 mark) In how many ways can 30! be expressed as the product of two integers p and q such that $0 < \frac{p}{q} < 1$ and p and q are relatively prime.
Solution: There are 10 prime factors of 30!, namely 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.
Actually, $30! = 2^\alpha \times 3^\beta \times 5^7 \times 7^4 \times 11^2 \times 13 \times 17 \times 19 \times 23 \times 29$.
Each of these (e.g. 2^α and 3^β), can only appear in one piece in either p or q, this gives 1024 choices. Half of them (so that p < q) will be 512 choices.
Considering negative integers as well, there are 1024 choices.

3. (1 mark) Find the coefficient of x^{17} in the expansion of $(1 + x^5 + x^7)^{20}$.
Solution: x^{17} can only be obtained by multiplying two x^5s and one x^7. There are 20 ways to get x^7 and ${}^{19}C_2 = 171$ ways to get two x^5s in the remaining 19 factors. So the answer is 20 X 171 = 3420.

4. (1 mark) If [x] represents the greatest integer less than or equal to x, find the sum of
$$\left[\frac{1\times1999}{2001}\right]+\left[\frac{2\times1999}{2001}\right]+\left[\frac{3\times1999}{2001}\right]+\cdots+\left[\frac{2000\times1999}{2001}\right].$$
Solution: Note that if n is an integer and x is not an integer, then
[n + x] = n + [x] and [-x] = -1 – [x]
$$\text{Hence } \left[\frac{2000 \times 1999}{2001}\right] = \left[1999 - \frac{1999}{2001}\right] = 1999 - 1 - \left[\frac{1999}{2001}\right]$$

$$\text{Therefore } \left[\frac{1 \times 1999}{2001}\right] + \left[\frac{2000 \times 1999}{2001}\right] = 1999 - 1 = 1998$$

$$\text{Similarly } \left[\frac{2 \times 1999}{2001}\right] + \left[\frac{1999 \times 1999}{2001}\right] = 1999 - 1 = 1998$$

$$\left[\frac{3 \times 1999}{2001}\right] + \left[\frac{1998 \times 1999}{2001}\right] = 1999 - 1 = 1998$$

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$$\left[\frac{1000 \times 1999}{2001}\right] + \left[\frac{1001 \times 1999}{2001}\right] = 1999 - 1 = 1998$$

Summing up

$$\left[\frac{1 \times 1999}{2001}\right] + \left[\frac{2 \times 1999}{2001}\right] + \left[\frac{3 \times 1999}{2001}\right] + \ldots + \left[\frac{2000 \times 1999}{2001}\right] = 1{,}998{,}000$$

5. (1 mark) If x, y are nonzero numbers satisfying $x^2 + xy + y^2 = 0$. Find the value of

$$\left(\frac{x}{x+y}\right)^{2001} + \left(\frac{y}{x+y}\right)^{2001}.$$

Solution: Let $t = x/(x + y)$.

Note that $[x/(x + y)]\,[y/(x + y)] = xy/(x^2 + 2xy + y^2) = xy/xy = 1$

Therefore $y/(x + y) = 1/t$, and $t + 1/t = 1$.

$t^2 - t + 1 = 0$, and $(t^3 + 1) = 0$.

Hence $t^3 = -1$

$$\left(\frac{x}{x+y}\right)^{2001} + \left(\frac{y}{x+y}\right)^{2001}. = (t^3)^{667} + (1/t^3)^{667} = -2 .$$

6. (1 mark) For how many real numbers a do the quadratic equations $x^2 + ax + 8a = 0$ have only integral roots?

Solution: Let m, n be the integral roots of the equation, with $m \leq n$.

Then $m + n = -a$ and $mn = 8a$.

Hence $8(m + n) = -mn$, and $mn + 8m + 8n = 0$, $(m+8)(n+8)= 64$.

$64 = 1 \times 64 = 2 \times 32 = 4 \times 16 = 8 \times 8 = -64 \times -1 = -32 \times -2 = -16 \times -4 = -8 \times -8$

giving the solutions (-7, 56), (-6, 24), (-4, 8), (0, 0), (-72, -9), (-40, -10), (-24, -12) and (-16, -16).

Hence a can have 8 different values, namely –49, -18, -4, 0, 81, 50, 36 and 32.

7. (1 mark) Suppose tan α and tan β are the roots of $x^2 + \pi x + \sqrt{2} = 0$. Evaluate

$\sin^2(\alpha + \beta) + \pi \sin(\alpha + \beta)\cos(\alpha + \beta) + \sqrt{2}\,\cos^2(\alpha + \beta)$.

Solution: Using the formulae for sum of rots and product of roots,

$\tan\alpha + \tan\beta = -\pi$, $(\tan\alpha)(\tan\beta) = \sqrt{2}$

Therefore $\tan(\alpha + \beta) = -\pi/(1 - \sqrt{2})$

Now $\sin^2(\alpha + \beta) + \pi \sin(\alpha + \beta)\cos(\alpha + \beta) + \sqrt{2}\,\cos^2(\alpha + \beta)$.

$= \cos^2(\alpha + \beta)[\tan^2(\alpha + \beta) + \pi \tan(\alpha + \beta) + \sqrt{2}\,]$

$= [\tan^2(\alpha + \beta) + \pi \tan(\alpha + \beta) + \sqrt{2}\,] / [1 + \tan^2(\alpha + \beta)]$

$$= \frac{(1-\sqrt{2})^2}{\pi^2+(1-\sqrt{2})^2} \times \left(\frac{\pi^2}{(1-\sqrt{2})^2} - \frac{\pi^2}{(1-\sqrt{2})^2} + \sqrt{2} \right)$$
$$= \sqrt{2}$$

8. (1 mark) 2000 lamps are controlled by 2000 switches, numbered 1, 2, 3, …, 2000. A click on each switch will either turn the lamp on or off. In the beginning, all the lamps are off. On the first day, all the switches are clicked once. On the second day, all the switches numbered 2 or a multiple of 2 are clicked once. Similarly on the n^{th} day, all the switches numbered n or a multiple of n are clicked once, and so on. How many lamps will be on after the operation on the 2000^{th} day?

Solution: After the 2000^{th} operation, only those lamps with numbers which have an odd number of integral factors will be left open.
This is equal to the number of perfect squares less than 2000.
Since $44^2 = 1936$, and $45^2 = 2025$,
Therefore the number of lamps which are on = 44.

9. (1 mark) Point B is in the exterior of the regular n-sided polygon $A_1A_2 \ldots A_n$ and A_1A_2B is an equilateral triangle. Find the largest value of n such that A_n, A_1 and B are consecutive vertices of a regular polygon.

Solution: Let m be the number of sides of regular polygon with An, A1 and B as consecutive vertices.
The degree measure of the interior angles of the three polygons are
$180 - 360/n$, 60 and $180 - 360/m$.
Hence $180 - 360/n + 60 + 180 - 360/m = 360$
Giving $n = 6 + 36/(m-6)$
n is largest when $m = 7$, and $n = 42$.

10. (1 mark) There are three parallel lines L_1, L_2 and L_3 on the plane, with L_2 in between. The distance between L_1 and L_2 is 4, and the distance between L_2 and L_3 is 3. A, B and C are points on L_1, L_2 and L_3 respectively, such that ΔABC is an equilateral triangle. Find the area of the triangle.

Solution: Let the circumcircle of ΔABC cut L_2 at the point P.
Note that $\angle APT = \angle BPT = 60°$
$AP = 4/\sin 60° = 8/\sqrt{3}$, $BP = 3/\sin 60° = 6/\sqrt{3}$
$AB = \sqrt{[64/3 + 36/3 - 2(8/\sqrt{3})(6/\sqrt{3}) \cos 120°]} = 148/3$
Area of $\Delta ABC = (\sqrt{3}/4)(148/3) = 37\sqrt{3}/3$.

11. (1 mark) A circle is inscribed in ΔABC. D, E are points on AB and AC respectively, such that DE is parallel to BC and is tangent to the circle. If the perimeter of ΔABC is p, find the maximum length of DE.

Solution: Let BC = a, and DE = x
Using tangent property, the perimeter of $\Delta ADE = p - 2a$
Using property of similar triangles, $x/a = (p-2a)/p$
$x = (ap - 2a^2)/p = 2[(p/4)^2 - (a - p/4)^2]/p$
x is maximum when $a = p/4$, and $x_{max} = p/8$.

12. (1 mark) In ΔABC, $BC = 5$, $AC = 12$, $AB = 13$. D, E are points on AB and AC respectively such that DE divides ΔABC into two parts of equal area. Find the minimum length of DE.

Solution: Area of $\Delta ABC = (5)(12)/2 = 30$, and $\sin A = 5/13$.

Let $AD = x$, $AE = y$, then area of $\Delta ADE = (xy \sin A) / 2 = 15$

Therefore $xy = 78$.

By Cosine Law, $DE^2 = x^2 + y^2 - 2xy \cos A$

$$= (x-y)^2 + 2xy(1 - \cos A)$$
$$= (x-y)^2 + 2(78)(1 - 12/13)$$
$$= (x-y)^2 + 12$$

Minimum length of DE = $\sqrt{12}$.

13. (1 mark) D is a point inside ΔABC. PDS, QDT and RDU are lines parallel to BA, CA and CB respectively such that P, Q lie on BC, R, S lie on CA, and T, U lie on AB. If the areas of ΔTUD, ΔPQD and ΔRSD are respectively 8, 128 and 32, find the area of ΔABC.

Solution: Let the Area of $\triangle ABC$ be S.

Note that $\triangle TUD \sim \triangle ABC$, and their areas are in the ratio $UD^2 : BC^2$

Therefore $\sqrt{8} / \sqrt{S} = UD / BC$

Similarly $\sqrt{128} / \sqrt{S} = PQ / BC$

Therefore $\sqrt{32} / \sqrt{S} = DR / BC$

Furthermore, $BC = BP + PQ + QC = UD + PQ + DR$

Therefore $(\sqrt{8} + \sqrt{128} + \sqrt{32}) / \sqrt{S} = (UD + PQ + DR) / BC = 1$

$$\sqrt{S} = 14\sqrt{2}$$
$$S = 392$$

14. (2 marks) The numbers $x_1, x_2, \ldots, x_{2000}$ are such that $|x_1 - x_2| + |x_2 - x_3| + \ldots + |x_{1999} - x_{2000}| = 2000$. Find the largest value of

$|y_1 - y_2| + |y_2 - y_3| + \ldots + |y_{1999} - y_{2000}|$,

where $y_k = \dfrac{x_1 + x_2 + \cdots + x_k}{k}$, for $k = 1, 2, \ldots, 2000$.

Solution: $|y_k - y_{k+1}| = |(x_1 + x_2 + \ldots + x_k)/k - (x_1 + x_2 + \ldots + x_{k+1})/(k+1)|$

$$= |(x_1 + x_2 + \ldots + x_k - k\, x_{k+1})/k(k+1)|$$
$$\leq (|x_1 - x_2| + 2|x_2 - x_3| + \ldots + k|x_k - x_{k+1}|) / k(k+1)$$

Hence $|y_1 - y_2| + |y_2 - y_3| + \ldots + |y_{1999} - y_{2000}|$

$$\leq |x_1 - x_2| (1/1.2 + 1/2.3 + \ldots + 1/1999.2000)$$
$$+ 2|x_2 - x_3| (1/2.3 + 1/3.4 + \ldots + 1/1999.2000)$$
$$+ \ldots + 1999|x_k - x_{k+1}| (1/1999.2000)$$
$$= |x_1 - x_2| (1 - 1/2000) + |x_2 - x_3| (1 - 2/2000) + \ldots$$
$$+ |x_{1999} - x_{2000}| (1 - 1999/2000)$$
$$\leq 2000(1 - 1/2000)$$
$$= 1999$$

Note that $x_1 = 2000$ and $x_2 = x_3 = \ldots = x_{2000} = 0$ gives the extreme case.

15. (2 marks) There are n distinct points on a plane. Eight different circles $C_1, C_2, \ldots, C_8$ are drawn such that C_1 passes through one of the points, C_2 passes through two of the points, C_3 passes through three of the points, and so on. Find the minimum value of n.

Solution: Consider drawing the circles in the order $C_8, C_7, C_6, \ldots C_1$, and then locating the points on the circle. Draw C_8, and 8 points will be marked on the circle. Draw C_7 so that it passes through 2 of the 8 points already existing, and 5 new points will have to be created. Now draw C_6 so that it passes through 2 existing points on C_8 and 2 existing points on C_7, and this leaves 2 more points to be created.

Now the situation is we have drawn C_8, C_7 and C_6, with 6 points fixed, and 9 points (4 on C_8 only, 3 on C_7 only and 2 on C_6 only) that can be fixed at a later stage.

Now attempt to draw C_5 and C_4 by suitably selecting the positions of these 9 points.

It may be observed that there will be no difficulty to draw C_3 and C_2 by selecting three or two existing point not the same circle. Finally C_1 may be drawn to pass any one existing point.

Hence the minimum number of points is $8 + 5 + 2 = 15$.

16. (2 marks) Let S denotes a finite sequence of the letters a and b, and f denotes a function defined by

f(S) = the new sequence formed by changing all 'a's to 'a, b' and all 'b's to 'b, a'.

For example, f(b, a, a, b) = (b, a, a, b, a, b, b, a), and the number of pairs of consecutive 'b's in f(b, a, a, b) is 1. If $f^{(n)}(S)$ denotes f(f(...(S)...) (n times), find the number of pairs of consecutive 'b's in $f^{(n)}(a)$.

Solution: Denote the number of 'b,b' pairs in $f^n(a)$ by P_n, and the number of 'a,b' pairs by Q_n. The number of 'b,b' pairs in $f^n(a)$ is equal to the number of 'a,b' pairs in $f^{n-1}(a)$. The number of 'a,b' pairs in $f^{n-1}(a)$ is equal to the number of 'a's plus the number of 'b,b' pairs in $f^{n-2}(a)$. Furthermore, $f^{n-2}(a)$ consists of 2^{n-2} letters, and half of them are 'a's.

Hence $P_n = Q_{n-1} = 2^{n-3} + P_{n-2}$

Iteratively, $P_n = 2^{n-3} + 2^{n-5} + \ldots + \begin{cases} 2^0 + P_1 & \text{if n is odd} \\ 2^1 + P_2 & \text{if n is even.} \end{cases}$

Now $P_1 = 0$, and $P_2 = 1$.

Hence if n is odd, $P_n = 2^{n-3} + 2^{n-5} + \ldots + 1 = (2^{n-1} - 1)/3$

If n is even, $P_n = 2^{n-3} + 2^{n-5} + \ldots + 2 + 1 = 2(2^{n-2} - 1)/3 + 1 = (2^{n-1} + 1)/3$.

Combining the results, $P_n = [2^{n-1} + (-1)^n]/3$.

17. (2 marks) The circumcircle of the isosceles triangle ΔABC has AB as a diameter. There is a circle Γ tangent to BC at its midpoint E and tangent to the minor arc BC at F. If AB = 4, find the length of the tangent from A to Γ.

Solution: Let T be a point on Γ such that AT is a tangent to Γ.

Since $\triangle ACE \sim \triangle EKF$, so EK/AC = EF/AE,

$EK = AC.EF/AE = 2\sqrt{2}(2 - \sqrt{2})/\sqrt{10} = (4 - 2\sqrt{2})/\sqrt{5}$.

$AK = AE + EK = (4 + 3\sqrt{2})/\sqrt{5}$.

$AT = \sqrt{(AE.AK)} = \sqrt{(6 + 4\sqrt{2})} = 2 + \sqrt{2}$.

18. (3 marks) In ΔABC, BC, CA and AB are divided by P, Q, and R respectively in the same ratio. AP intersects BQ at X, BQ intersects CR at Y, and CR intersects AP at Z. Each of the areas of ΔARZ, ΔBPX, ΔCQY and ΔXYZ equals 1 cm^2. Find the area of the quadrilateral PCYX.

Solution: Let AR:RB = BP:PC = CQ:QA = 1 : k .

Then $S(\triangle ABP) / S(\triangle ACP) = 1 / k$

$S(\triangle ABZ) / S(\triangle ACZ) = 1 / k$ ---------------------------------------(1)

$S(\triangle ARZ) / S(\triangle ABZ) = 1 / (1+k)$ ---------------------------------(2)

(1).(2) $S(\triangle ARZ) / S(\triangle ACZ) = 1 / k(1+k)$ ------------------------------(3)

$\therefore$ $S(\triangle ARZ) / S(\triangle ACR) = 1 / (k^2 + k + 1)$ ----------------------------(4)

$S(\triangle ARZ) / S(\triangle ABC) = 1 / (k + 1)(k^2 + k + 1)$ ------------------- (5)

(4)/(3) $S(\triangle ACZ) / S(\triangle ACR) = (k^2 + k) / (k^2 + k + 1)$

$S(\triangle ACZ) / S(\triangle ABC) = k / (k^2 + k + 1)$

Similarly $S(\triangle BAX) / S(\triangle ABC) = k / (k^2 + k + 1)$

$S(\triangle CBY) / S(\triangle ABC) = k / (k^2 + k + 1)$

$\therefore S(\triangle XYZ) / S(\triangle ABC) = 1 - 3k / (k^2 + k + 1) = (k^2 - 2k + 1)/ (k^2 + k + 1)$ ---- (6)

Comparing (5) and (6),

$$1 / (k + 1)(k^2 + k + 1) = (k^2 - 2k + 1)/ (k^2 + k + 1)$$

$$(k + 1) (k^2 - 2k + 1) = 1$$

$$k^2 - k - 1 = 0$$

$$k = (1 + \sqrt{5}) / 2$$

Now Area of $\triangle ARZ = 1$ cm^2,

Area of $\triangle ABC = (k + 1)(k^2 + k + 1)$ cm^2,

Therefore, area of PCYX = $[(k + 1)(k^2 + k + 1)/(k + 1) - 2]$ cm^2

$= (k^2 + k - 1)$ cm^2

$= (1 + \sqrt{5})$ cm^2.

19. (3 marks) B is a point on the line segment AC such that AB = 1 and BC = 3. Semicircles Γ_1, Γ_2, Γ_3 are drawn with diameters AC, AB, BC respectively, and all are on the same side of AC. Let E be on Γ_1 such that $EB \perp AC$. Let U on Γ_2, V on Γ_3 be such that UV is a common tangent to Γ_2 and Γ_3. Find the ratio of the area of ΔEUV over the area of ΔEAC.

Solution: Suppose AE and CE intersect Γ_2 and Γ_3 at U' and V' respectively.

Using angles in semicircle, it is easy to show that EU'BV' is a rectangle.

Since $\angle V'U'B = \angle EBU' = \angle EAB$, U'V' is tangent to Γ_2 .

Similarly it is tangent to Γ_3.

Thus U = U' and V = V', and $S(\Delta EUV)/S(\Delta EAC)$

$= (UV/AC)^2$

$= (EB/AC)^2$

$= (AB.BC/AC^2$

$= 3/16$

20. (3 marks) In each of 12 photographs, there are 3 women; the woman in the middle is the mother of the person on her left and is a sister of the person on her right. The women in the middle of the 12 photographs are all different persons. Determine the smallest number of different persons in the photographs.

Solution: Draw family tree diagrams for the persons in the photographs.

The 0th row denotes the persons whose mothers do not appear in the photographs.

Let r_k be the number of persons appearing in the middle of some pictures and are place in the kth row, t_k be the number of other persons in the kth row, and s_k be the total number of mothers of the persons in the kth row.

Now $s_{k+1} \leqq r_{k+1}/2 + t_{k+1}$,(every middle woman has a sister in the picture)

and $r_k \leqq s_{k+1}$,(every middle woman has a daughter in the picture),

Thus $r_k \leqq r_{k+1}/2 + t_{k+1}$, for $k = 0, 1, 2\ldots$

And $1 \leqq r_0/2 + t_0$

Summing up, we have

$(r_0 + r_1 + \ldots) + 1 \leqq (r_0 + r_1 + \ldots)/2 + (t_0 + t_1 + \ldots)$

giving $(r_0 + r_1 + \ldots) + (t_0 + t_1 + \ldots) \geqq 3(r_0 + r_1 + \ldots)/2 + 1$

$= 3(12)/2 + 1$

$= 19$