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PASCAL'S TRIANGLE: TOP GUN OR JUST ONE OF THE GANG?

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INTRODUCTION

Pascal's triangle can appear as a member of classes of triangular arrays where presumably no class member should be ranked in importance over any other. Two such cases which came to mind were the multinomial triangles [6] and the Hoggatt triangles [2]. No doubt there are others. We selected the multinomial triangles. Was Pascal's triangle only a binomial triangle in a sea of trinomial, quadrinomial, pentanomial, etc., triangles, or might it exhibit a significant influence on the makeup of the other multinomial triangles? We admit a certain prejudice in our choice. Computer experimentation with partition counting, large multinomial expansions, and generating functions using computer algebra systems (muMath, Derive, Mathematica) hinted at a definite Pascal influence. A few years ago, such experimentation would have been virtually impossible.

In this note, we take advantage of features of multinomial triangles to search for influence of Pascal rows and diagonals on their counterparts in the other multinomial triangles. After having seen our results, we leave it to the reader to answer the title question.

SOME INTERESTING AND USEFUL OBSERVATIONS ON MULTINOMIAL TRIANGLES

If the coefficients of the expansions of $(1+x+x^2+x^3+\cdots+x^t)^m$ for fixed t are symmetrically arranged by rows for $m=0, 1, 2, \ldots$, the results are triangular arrays of integers popularly known as *multinomial triangles*. The degenerate triangle consisting of a single vertical

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line of 1's is of order $t=0$. The bi- or 2-nomial triangle of order $t=1$ is Pascal's triangle. We take the liberty of discarding the Latin or Greek prefix where convenient and use instead a numerical term, $(t+1)$-nomial. As with the Pascal triangle, all $(t+1)$-nomial triangles exhibit symmetry about a vertical centerline. The symbol $\left\langle{m \atop p}\right\rangle_t$ serves to identify uniquely the integer in position p of row m of triangle t, i.e., the $(t+1)$-nomial triangle. All m, p, and t may take on non-negative integer values 0, 1, 2, 3, ... An example row m is shown in (1).

$$\left\langle{m \atop 0}\right\rangle_t \left\langle{m \atop 1}\right\rangle_t \left\langle{m \atop 2}\right\rangle_t \cdots \left\langle{m \atop p}\right\rangle_t \cdots \left\langle{m \atop mt-2}\right\rangle_t \left\langle{m \atop mt-1}\right\rangle_t \left\langle{m \atop mt}\right\rangle_t \tag{1}$$

There are many straight paths through $(t+1)$-nomial triangles which have interesting numerical properties and which might qualify as diagonals of some type. However, there seems to be one *defacto* "diagonal" sequence designation. The defacto diagonal sequences appear as columns in the left-justified versions of multinomial triangles (See Hoggatt and Bicknell [6]). In the nomenclature of this note, diagonal d of triangle t is the series of integer coefficients

$$\left\langle{\lceil\frac{d}{t}\rceil \atop d}\right\rangle_t, \left\langle{\lceil\frac{d}{t}\rceil+1 \atop d}\right\rangle_t, \left\langle{\lceil\frac{d}{t}\rceil+2 \atop d}\right\rangle_t, \left\langle{\lceil\frac{d}{t}\rceil+3 \atop d}\right\rangle_t, \left\langle{\lceil\frac{d}{t}\rceil+4 \atop d}\right\rangle_t, \ldots . \tag{2}$$

Although the functionally equivalent left-justified version of the $(t+1)$-nomial triangle was very useful to Hoggatt and Bicknell [6] in the study of diagonal generating functions and generalized Fibonacci sequences and to Greenbury [5] for the direct display of generalized Fibonacci sequences, we use the isosceles form where it suits our purposes. Several $(t+1)$-nomial triangles through $m=5$ are shown in Figure 1. Diagonals can be traced on the triangles of Figure 1 by starting at the first integer of (2) and moving "left $t/2$, down one." Half column widths apply for odd t.

m	$t=0$	$t=1$	$t=2$
0	1	1	1
1	1	1 1	1 1 1
2	1	1 2 1	1 2 3 2 1
3	1	1 3 3 1	1 3 6 7 6 3 1
4	1	1 4 6 4 1	1 4 10 16 19 16 10 4 1
5	1	1 5 10 10 5 1	1 5 15 30 45 51 45 30 15 5 1

$t=3$

m	
0	1
1	1 1 1 1
2	1 2 3 4 3 2 1
3	1 3 6 10 12 12 10 6 3 1
4	1 4 10 20 31 40 44 40 31 20 10 4 1
5	1 5 15 35 65 101 135 155 155 135 101 65 35 15 5 1

$t=4$

m	
0	1
1	1 1 1 1 1
2	1 2 3 4 5 4 3 2 1
3	1 3 6 10 15 18 19 18 15 10 6 3 1
4	1 4 10 20 35 52 68 80 85 80 68 52 35 20 10 4 1
5	1 5 15 35 70 121 185 255 320 365 381 365 320 255 185 121 70 35 15 5 1

Figure 1. Some partial $(t+1)$-nomial or Multinomial Triangles.

One of the features all $(t+1)$-triangles share in common is, except for the single 1 in row zero, that every element is the sum of the $(t+1)$ consecutive elements centered on and placed directly above it. It is also obvious that there are $mt+1$ integers per row and that a row sum equals $(t+1)^m$. For the Pascal triangle these are often a student's first discoveries. To satisfy the construction for all $mt+1$ row elements, it may be necessary to visualize some blank positions outside the triangle filled with 0's. The construction idea is important in later manipulation of the general $(t+1)$-nomial triangle. In addition, the concept can be the basis of spectacular digital computer spreadsheet displays of multinomial triangles.

It can be observed that the class of isosceles form triangles is partitioned into two subclasses according to column structure. The triangles for even t have an odd number of elements in upper row sums. This leads to triangles with unbroken columns. For odd t, the result is row-staggered columns. The column difference is not obvious in left-justified triangles.

Two crucial features of multinomial triangles are:

(a) The first t coefficients, $\left\langle {m \atop 0} \right\rangle_{t-1}, \left\langle {m \atop 1} \right\rangle_{t-1}, \left\langle {m \atop 2} \right\rangle_{t-1}, \cdots \left\langle {m \atop t-1} \right\rangle_{t-1}$, of row m of triangle t-1 equal term-by-term the first t coefficients, $\left\langle {m \atop 0} \right\rangle_{t}, \left\langle {m \atop 1} \right\rangle_{t}, \left\langle {m \atop 2} \right\rangle_{t}, \cdots \left\langle {m \atop t-1} \right\rangle_{t}$, of row m of triangle t.

and

(b) The first $t+1$ coefficients, $\left\langle {m \atop 0} \right\rangle_{t}, \left\langle {m \atop 1} \right\rangle_{t}, \left\langle {m \atop 2} \right\rangle_{t}, \cdots \left\langle {m \atop t} \right\rangle_{t}$, of row m of triangle t equal term-by-term the first $t+1$ coefficients,

$$\left\langle {\lceil \frac{d}{t} \rceil \atop d} \right\rangle, \left\langle {\lceil \frac{d}{t} \rceil + 1 \atop d} \right\rangle, \left\langle {\lceil \frac{d}{t} \rceil + 2 \atop d} \right\rangle, \ldots, \left\langle {\lceil \frac{d}{t} \rceil + t \atop d} \right\rangle$$

of diagonal $d=m-1$ of Pascal's triangle. It can be seen that coefficient $\left\langle {m \atop t+1} \right\rangle_t$ and all beyond are always less than their diagonal counterparts from Pascal's triangle.

Observation of $(t+1)$-nomial triangles reinforces the central rôle so many times accorded Pascal's triangle. Aside from being a member of other sets of triangles [5], it is a member of the class of all $(t+1)$-nomial triangles as well as a member of the subclass with row-staggered columns. Another observation indicated that sets of diagonals from Pascal's triangle are included in every $(t+1)$-nomial triangle. For $t \geq 2$ the diagonals 0, 1, 2, . . . , t of a $(t+1)$-nomial triangle are identically the same numbered diagonals of Pascal's triangle. The equality ends with diagonal t. This fact can be directly associated with the abrupt change in the form of diagonal generating functions experienced by Hoggatt and Bicknell. (See [6], bottom of p. 341 and eq. (1), p. 342.) It was the last observation, coupled with Hoggatt and Bicknells' analytic results, which challenged us to search for a layered mathematical structure for $(t+1)$-triangles where outer layers could be repeatedly "peeled" off to reveal other layers functionally related to parts of Pascal's triangle. Since this analysis of multinomial triangles appeared new, we concentrated our computer experimental efforts on it.

GENERAL DISCUSSION OF COEFFICIENT CALCULATIONS

One obvious way to find the members of row m of a $(t+1)$-nomial triangle is to expand $(1+x+x^2+\cdots+x^t)^m$ and collect the (integer) coefficients of powers of x in ascending order from 0 through tm. This is an effortless task even for tremendous t and m values if muMath, Derive, Mathematica, or other computer algebra systems perform the computations. The method, however, is certainly not new.

Our approach to obtaining the coefficients of expanded $(1+x+x^2+\cdots+x^t)^m$ is semi-heuristic. We found two different ways and, correspondingly, two different formulations.

In the first, we establish a tabular procedure in which column sums are the coefficients of the expansion, and the column designators are the corresponding powers of x. Through a very selective choice of row values for the examples, a pattern emerges. The pattern evokes a very safe conjecture that the coefficients of row m of triangle t are functions of row m of Pascal's triangle and rows m down through 0 of triangle $t-1$. By concentrating on how single columns are formed, we can predict general formulas for single row members, the $\left\langle {m \atop p} \right\rangle_t$'s.

In the second, we capitalized on the fact that parts of certain diagonals of Pascal's triangle are also parts of rows of $(t+1)$-nomial triangles. In experimental examples of widely varying size, we forced diagonals of Pascal's triangles to equal, term-by-term, the row coefficients of rows of a $(t+1)$-nomial triangles. As will be detailed later, through careful (crafty?) choice of forcing terms it is shown that coefficients of $(t+1)$-triangles can be found from rows and diagonals of Pascal's triangle alone!

MULTINOMIAL TRIANGLE COEFFICIENTS. FIRST METHOD

After rephrasing, material from Parzan (see [8], page 40, formula 118) states

$$(x^0+x^1+x^2+\cdots+x^t)^m = \sum_{k_0=0}^{m}\sum_{k_1=0}^{m}\cdots\sum_{k_t=0}^{m}\binom{m}{k_0k_1\cdots k_t}(x^0)^{k_0}(x^1)^{k_1}\cdots(x^t)^{k_t}$$

$$k_0+k_1+k_t\cdots k_t = m, \tag{3}$$

where the integers

$$\binom{m}{k_0k_1\cdots k_t} = \left(\frac{m!}{k_0!k_1!\cdots k_t!}\right) \tag{4}$$

are universally called *multinomial coefficients.* From a combinatorics point of view, they are the numbers of permutations, or arrangements, of m objects, k_0 of one kind, k_1 of another kind, $\ldots k_t$ of the last kind. However, the coefficients found in $(d+1)$-nomial, i. e., multinomial, triangles do not, in general, equal multinomial coefficients. The reason is simple. The coefficients of multinomial triangles are coefficients of powers of x in $(x^0+x^1+x^2+\cdots+x^t)^m$. In (3) there may be several different multinomial coefficients having the same total power of x. This establishes multinomial triangle coefficients as being permutations or sums of permutations. We use restricted partitions of the $(tm+1)$ powers of x in the expansion $(x^0+x^1+x^2+\cdots+x^t)^m$ to guide us to the permutations which eventually sum to the $(d+1)$-nomial triangle coefficients.

As a first example, consider row 4 of the 3-nomial triangle. The corresponding expansion is $(x^0+x^1+x^2)^4$ for $t=2$ and $m=4$. The $mt+1 = 9$ powers of x in the expansion are 0, 1, 2, 3, 4, 5, 6, 7, 8. These are the column headings for the subsequent layers of rows. Because a power of x in the expansion is always the sum of powers of x from each of the four $(x^0+x^1+x^2)$'s, the nine powers can be partitioned into four-part partitions with 0 considered as a possible member of a partition. Before tabulating the set of four-part partitions, however, consider the same partitions with all 0's excluded except for the inevitable, single 0 for the zero power of x. Including the single zero partition, we have now the restricted partitions of 0, 1, 2, 3, 4, 5, 6, 7, 8 whose member size cannot exceed t=2 and whose number of members cannot exceed m=4.

Netto (see [7], page 122) suggests a very simple way of generating unrestricted partitions of an integer n. An algorithm based on Netto's work and adapted for computer use appears in Fielder [1] and Fielder and Alford [3]. To demonstrate the restricted partitions described in the previous paragraph, columns of unrestricted partitions of 0, 1, 2, 3, 4, 5, 6, 7, 8 are recorded as generated by the algorithm. Those partitions which cannot pass the restriction test, either because of member size or number of members or both, are crossed out. The survivors are retained in the order of their generation on their same relative rows. Table 1 verifies the choice of restricted partitions needed in our example development.

Powers of x

0	1	2	3	4	5	6	7	8
0	1	11	111	1111	~~11111~~	~~111111~~	~~1111111~~	~~11111111~~
		2	12	112	1112	~~11112~~	~~111112~~	~~1111112~~
			~~3~~	~~13~~	~~113~~	~~1113~~	~~11113~~	~~111~~[illegible]
				22	122	1122	~~11122~~	~~11122~~
				~~4~~	~~14~~	~~114~~	~~1114~~	~~11114~~
					~~23~~	~~123~~	~~1123~~	~~11123~~
					~~5~~	222	1222	~~11222~~
						~~15~~	~~115~~	~~1115~~
						~~24~~	~~124~~	~~1124~~
						~~33~~	~~133~~	~~1133~~
						~~6~~	~~223~~	~~1223~~
							~~16~~	2222
							~~25~~	~~116~~
							~~34~~	~~125~~
							~~7~~	~~134~~
								~~224~~
								~~233~~
								~~17~~
								~~26~~
								~~35~~
								~~44~~
								~~8~~

Table 1. Table of Unrestricted Partitions of Powers of x in Order Generated by Special Algorithm. Desired Restricted Partitions Shown Not Cancelled

In anticipation of what is to follow, the successful partitions from Table 1, with the zero members restored, are tabulated as shown in Table 2 below.

Powers of x

0	1	2	3	4	5	6	7	8
0000	0001	0011	0111	1111				
		0002	0012	0112	1112			
				0022	1122	1122		
						0222	1222	
								2222

Table 2. Intentionally Arranged Set of m=4-Part Restricted Partitions of x With No Member Less Than 0 or Greater Than t=2

By considering the partition members as "objects", derangements of the partitions yield the permutation counts which equal the desired coefficients. This is shown in Table 3 where the "layers" are accentuated by boundary lines.

Powers of x

0	1	2	3	4	5	6	7	8
4!/4! 0!	4!/3! 1!	4!/2! 2!	4!/3! 1!	4!/4! 0!				
		4!/3! 1!	4!/2!1!1!	4!/2!1!1!	4!/3! 1!			
				4!/2!2!	4!/2!1!1!	4!/2! 2!		
						4!/3! 1!	4!/3!1!	
								4!/4!0!
1	4	10	16	19	16	10	4	1

Table 3. Permutation Counts Derived From Set of m=4-Part Restricted Partitions of Table 2.

The column totals are the coefficients shown at the bottom of Table 3. Now suppose that binary coefficient $\binom{4}{0}$ is factored out of each permutation of layer 0, $\binom{4}{1}$ is factored out of each coefficient of layer 1, $\binom{4}{2}$ is factored out of layer 2, $\binom{4}{3}$ is factored out of layer 3, and $\binom{4}{4}$ is factored out of layer 4.

The new tabular arrangement with the factored binomial coefficients in the left column is shown in Table 4.

CONCLUDING REMARKS

By approaching the construction of multinomial triangles through partition and derangement techniques, we were able, with judicious forcing of component positions, to observe successive patterns of Pascal triangle components in the completed multinomial triangles. As a result, we were able to find finite, closed summation formulas for elements of multinomial triangles in terms of rows of the next lower order multinomial triangles and one row only of Pascal's triangle. This first formula applied to trinomial triangles uses rows of Pascal's triangle. Further work produced general formulas for elements of multiomial triangles of any size (not just trinomial) using one row and one diagonal from Pascal's triangle.

We showed that any diagonal of any multinomial triangle can be completely expressed using at most one row and one diagonal of Pascal's triangle. The diagonal properties documented were homogeneous difference equations, generating functions, and general sequence terms.

Again, we highly recommend the experimental opportunities which modern computer algebra systems offer.

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