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Towards classifying Finite Point-Set Configurations

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Introduction and Extended Abstract

Our objective is to study the moduli space $Y = Y^{(m,n)}$ of n-point configurations in the $m-1$ dimensional projective space $\mathbf{P}^{m-1}$. (We're working over a fixed field of characteristic zero, which shall remain unmentioned almost always, except when it is essential to refer to it). This means, Y is the geometric quotient of the n^{th} cartesian power of $\mathbf{P}^{m-1}$ by the (diagonal action of the) automorphism group $\mathbf{PGL}_m$ of the projective space; since the problem is 'trivial' when $n \leq m$, we assume throughout that $n > m$. This is a special case of an ancient problem (cf. Coble [Co]) wherein one further attaches to each of the n points some given multiplicities (lately such 'multiple points' have been called "fat points" in literature, – cf. Geramita [Ge] and Harbourne [Ha]). However, we shall see that the sort of most 'elementary' information that we seek (in our "particular case") along *group-theoretic*, or rather *invariant-theoretic* lines, already involves difficulties that seem almost insurmountable in the generality that is inherent in our problem, and very little existing information in contemporary literature seems to be helpful; a notable exception is (some initial sections in) a modern update of Coble's work by Dolgachev and Ortland [DO] (that we shall return to in course of the paper). These difficulties are basically of a combinatorial nature, and we describe below our partial success in overcoming those for the $m = 2$ case. In the course of our paper the reader shall encounter the sort of complications we've just hinted, which may account for the neglect this problem has received (along the natural directions of interest to us).

The algebraic description of the space Y is in terms of a certain ring Q of invariants, inside the polynomial ring S in the set of nm variables X_{ij}; this set of (commuting) variables is identified with the entries of the generic matrix $X = [X_{ij}]$ having n rows and m columns, so that the set of row-vectors in X provide us with the "generic n-point configuration"). To define Q we first consider the graded subring R of S generated by the maximal minors (i.e. by the determinants of the $m \times m$ submatrices) of X: To fix notations (that matter a lot in this problem) once and for all, for an m-subset

$$I = \langle i_1, i_2, \cdots, i_m \rangle$$

of the 'row-index set' $[1, n]$, let us denote by ξ_I the corresponding maximal minor, where we invariably view I as a column-vector (*this by itself is perhaps the most important innovation of this author, that is fundamental to our approach to the so-called "Young tableaux" theory one needs to crucially rely upon*); we shall treat these generators

$$\{ \ \xi_I \ \mid \ \text{terms of } I \text{ in strictly increasing order} \ \}$$

as elements of the first degree in the grading of R, so that the d^{th} degree homogeneous component R_d of R is spanned by the d-fold products $\xi_{\mathbf{I}} := \xi_{I_1}.\xi_{I_2}.\cdots.\xi_{I_d}$ for $\mathbf{I}$ running over d-tuples $< I_1, I_2, \cdots, I_d >$. Leaving apart a discussion of the internal structure and significance of R (to which we do return soon), we may define now the graded ring Q as consisting of all invariants in R under the 'maximal torus' $\mathbf{Diag}_n \cap \mathbf{SL}_n$ of $\mathbf{SL}_n$ (for the natural action of $\mathbf{GL}_n$ on R). Thus $Q \cap R_d$ is precisely the zero-weightspace of the $\mathbf{GL}_n$-module R_d (the latter is known to be irreducible, obviously of highest weight d times the m^{th} fundamental weight). [A direct and more elementary description of Q is the span of those 'monomials' $\xi_{\mathbf{I}}$ (for $\mathbf{I}$ as above) for which each row-index between 1 to n occurs equally often (amidst the multiset of all indices in the various columns of $\mathbf{I}$).] In order that this intersection be nonzero md should be precisely a (nonnegative) multiple of n, which is the same as d being a multiple of n' where $n' = \frac{n}{\gcd(m,n)}$. For this reason it is good to set the k^{th} homogeneous component Q_k of Q as the above intersection $Q \cap R_d$ for $d = k.n'$; however, let us hasten to warn the reader that in our bid to present our results for the $m = 2$ case, we shall be modifying this convention – at formula (1) onwards (by replacing the index-set for the homogeneous (nonzero) components of Q by the set of (non-negative) half-integers – instead of integers).

It is not hard to identify R with the homogeneous coordinate ring of the grassmannian $\mathbf{Grass}_m(n)$ of all m-dimensional subspaces of an n-dimensional vectorspace V, for its canonical embedding in the projective space associated to the m^{th} exterior power of V; this rests on the observation that R is just the subring of invariants in S for the natural action of $\mathbf{SL}_m$ (coming from the canonical action of $\mathbf{GL}_m$ on the span of the entries of X, viz. on the degree 1 component of S). Then one is left with the natural action of $\mathbf{GL}_n$ on (each homogeneous component of) R, and (as said above) each of these component $\mathbf{GL}_n$-modules is irreducible. One knows that R is generated by its degree 1 component R_1, subject only to a well-known set of quadratic relations, originally due to Plücker from the later decades of the last century; these identities amount to a family of determinantal identities (discovered by Sylvester) that are fundamental to this work (as well as in the general 'tableaux theory' fleetingly mentioned below in this Abstract). One also has some neat formulae for the dimensions of the various components R_d; apart from the formula of Weyl (available for all irreducible representations), one has a lesser known formula of Philip Hall (for the $\mathbf{GL}_n$-irreps, which has sometimes been attributed in literature to R. Stanley) from which the dependence of dimR_d as a function of the 3 parameters n, m, d becomes more transparent; besides these there also exists a recasting by this author (in an unpublished privately announced work from 1993) which expresses the requisite dimension function as a 'higher beta function' (a function of 3 variables), which readily impies a seemingly miraculous symmetry for the dimension function when recast as a function of 3 arguments $m, n-m, d$ (asserting full symmetry for these 3 arguments). At any rate, it is safe to say that the structure of the graded ring R is well-understood. By contrast, relatively very little is known about the structure of the **zero-weight ring** Q (the sum of zero-weghtspaces from R). The first notable departure is the fact that (unlike R) Q is not always generated by the its lowest positive degree component; our success in certain computations that we report below indicates that this 'pathology' may never arise for m=2, but certainly (m,n) = (3,6) provides a low case for this phenomenon). This (and many other rather ill-understood) ring-theoretic features of Q account for the complications in the geometry of Y and its mysterious singularities; one should note that thus the natural geometric embedding of Y that arises from Q is only in a 'weighted projective space' which already acquires singularities from the natural cyclic group actions.

There is another twist to viewing the components Q_k of the ring Q, related to what is sometimes called 'noncommutative Invariant Theory' (say as in the paper of Almkvist, Dicks and Formanek [ADF]). It is captured by the following:

PROPOSITION: The space Q_k is identifiable with the space of $\mathbf{SL}_m$-invariants in the $\mathbf{GL}_m$-module $\bigotimes^n(\mathrm{Sym}^k(\mathrm{nat}_m))$, where nat_m)) stands for the m-dimensional 'birth-certificate' representation of $\mathbf{GL}_m$, and Sym^k (resp., $\bigotimes^n$) stands for the process of taking the k^{th} symmetric power (resp., that of taking the n^{th} tensor power) of any given vectorspace. Further, this identification respects the canonical action of the symmetric group $\mathbf{S}_n$ available in both cases.

Note that the said $\mathbf{SL}_m$ invariants in the tensor-space above, are nonzero if and only if m divides nk, which is precisely the earlier condition we had for Q_k to be nonzero (emanating from $md = kn$ before).

This result 'explains' the nature of difficulties in our problem, viz. as coming from the tough challenge offerred by the demands of the 'noncommutative Invariant Theory' – which one may "define" (in a limited sense) as an investigation of $\mathbf{SL}_m$-invariants in the tensor algebra of the space of homogeneous forms (of given degree) in m (commuting) variables (in analogy with the Classical 19^{th}-century Invariant Theory). For the tensors in $m = 2$ variables (i.e. for forms of degree 1) this was done in an expository and exploratory work of this author [Ve] in 1982, which implies that dim Q_1 (which is nonzero just when n is even) equals the s^{th} Catalan number $\frac{1}{s+1}\binom{2s}{s}$ for $n = 2s$; and the analogous numerology for the next case is given in [ADF] that gives the value of dim Q'_k for ($m = 2$ and) $k' = \gcd(2, n)$ as given by the socalled 'Motzkin sums' (cf. [DS], which we discovered via entry # M2587 in [SP]):

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
dim Q_k	1	0	1	1	3	6	15	36	91	232	603	1585	4213	11298	30537	...

However, it is quite hard to proceed further along these lines; but a different approach works below (for the $m = 2$ case).

Our view in this study is mostly on understanding the combinatorics of the graded ring Q, for which even an explicit clean and closed formula for the dimension N_k of Q_k is lacking.

From general considerations it is clear that the N_k's form a family of Kostka numbers; more precisely, if λ (resp., μ) denotes the rectangular partition (i.e. a partition all whose (nonzero) 'parts' are equal) having m rows (resp., n rows) such that they have both equal and minimal sizes (which forces λ to have n' columns, and μ to have m' columns, with n' as defined earlier and m' defined similarly to mean $\frac{m}{\gcd(m,n)}$), then N_k counts the number of (what we call) 'standard semi-tableaux' (SST's in short, that have often been called semi-standard tableaux in literature) of shape $k.\lambda$ and content $k.\mu$ (where to multiply a partition by an integer k means replacing each 'part' by the same multiple). In this way we are able to define a certain d-dimensional polytope $P = P^{m,n}$ from which the entire sequence of numbers N_k is obtainable, where d is the number of 'degrees of freedom' one has in counting the said SST's: it is not hard to argue that (cf. [DO]) $d = (m-1)(n-m-1)$. [This method of calculating Kostka numbers, i.e. of counting the number of SST's for any given shape and content, appears already in [DO], from which our interpretation of any Kostka number as the number of integer points in a certain polytope follows transparently; however, this observation does not seem to appear in the literature explicitly]. More precisely, P is naturally embedded in $\mathbb{R}^d$ (and is determined by explicitly known faces) such that N_k precisely equals the number of integer points in $k.P$ (the 'dilatation' of P by the factor k), or what is the same thing it counts the number of rational points in P with denominator (a submultiple of) k. From the 'Ehrhardt theory' (cf. [St] or [Hi]) one knows then that the arithmetic function $k \rightsquigarrow N_k$ is a quasi-polynomial. However, we know (as is to be seen below in detail) that these functions (one for each pair of values of (m, n)) are indeed polynomials when $m = 2$, and strongly suspect that our underlying argument should carry over for all m. One also knows (á priori) that the degree of the polynomial must be $d = (m-1)(n-m-1)$); more precisely, one knows the 'leading term estimate' for this function (from [DO] initially), that the value of N_k is asymptotic to k^d times the volume of the polytope P. While we report below, our further progress mostly for the $m = 2$ case, let us mention here that we're unable to find a suficiently closed formula even for the volume of P (our implicit formula for $m = 2$ below is in the form of an alternating sum which we've been unable to simplify yet).

We have found it possible to supply a closed (albeit slightly cumbersome) formula only for the $m = 2$ case (this was obtained jointly with S. R. Ghorpade of I.I.T. Bombay). To state the result, we need to slightly modify our notation in the (present) $m = 2$ case, so that our formulae become independent of the dichotomy of the 2 values of $\gcd(m, n)$, i.e. do not need distinction between the 2 cases of even and odd values of n. Thus, we make the convention of grading the ring Q by (non-negative) half-integers in case n is even; with this convention, the Ghorpade formula reads:

$$N_k = -\frac{1}{2} \sum_j{}' \, (-)^j . \binom{n}{j} . \binom{(n-2j)k + n - j - 2}{n-3}, \tag{1}$$

where the $'$ in the sum means that j is to run only from 0 to n_1 where n_1 stands for $\lfloor \frac{n-1}{2} \rfloor$. It should be noted that each term in this sum is a polynomial in k of degree $d = (m-1)(n-m-1) = n-3$ (remember $m = 2$); it is highly desirable to recast this sum in a simpler form, which avoids lots of 'cancellation of terms' that is implicit in the formula.

At least for successive low values of n (with $m = 2$) this enables us express the Hilbert series of Q as a nice rational function which sheds a good deal of light on the structure of Q. More precisely, let us write G_n for the generating function $\sum_{k\in\mathbb{Z}} \dim Q_k.q^k$, remembering that by the 'half-integer' convention for k introduced above this misses from being the full GF (= generating function) of Q when n is even (in that case only alternate terms are present in this GF); to repair that damage, it makes sense to define the 'full GF' $\tilde{G}_n$ (this notation defined only for the n even case) to be the sum $\sum_{k\in\frac{1}{2}\mathbb{Z}} \dim Q_k.q^{2k}$. [It may be noted that the two GF's, for n even, determine each other in a somewhat complicated manner.] Then we have

$$G_n = \frac{\nu_n'}{(1-q)^{d+1}}, \quad \text{and} \quad \tilde{G}_n = \frac{\tilde{\nu}_n}{(1-q)^{d+1}},$$

where the 'numerator' ν_n (respectively, $\tilde{\nu}_n$ is a polynomial in q of degree (exactly) d (resp. $d-1$); we list them now:

$$\begin{aligned}
\nu_3 &= 1\\
\nu_4 &= 1+q\\
\nu_5 &= 1+3q+q^2\\
\nu_6 &= 1+11q+11q^2+q^3\\
\nu_7 &= 1+31q+90q^2+31q^3+q^4\\
\nu_8 &= 1+85q+544(q^2+q^3)+85q^4+q^5\\
\nu_9 &= 1+225(q+q^5)+2997(q^2+q^4)+6559q^3, \text{ etc.;}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\nu}_4 &= 1\\
\tilde{\nu}_6 &= 1+q+q^2\\
\tilde{\nu}_8 &= 1+8q+22q^2+8q^3+q^4\\
\tilde{\nu}_{10} &= 1+34q+295q^2+565q^3+295q^4+34q^5+q^6, \text{ etc.}
\end{aligned}$$

It is very interesting to ponder on these tabulations, both for the numerology, and the theoretical implications/suggestions contained therein. We do so under the following 2 heads separately:

REMARK 1: These two tabulations are clearly equivalent to two 'arrays' (= bisequences, or infinite matrices – which happen to be row-finite) of integers, which is obtained by listing coefficients of various powers of q in successive q-polynomials ν_n (and $\tilde{\nu}_n$) as entries of the n^{th} row of the array. By looking at various columns one gets infinitely many interesting 'integer sequences', and it is curious to note that none of the sequences we find in this manner seems to have appeared in the existing literature. (One may take their absence in the famous compendium [SP] of N. J. A. Sloane, as a safe proof of this assertion.) Further, it is good to know that the sums of coefficients in each ν_n (as also in $\tilde{\nu}_n$) – let's call them $\sigma(n)$ (resp. $\tilde{\sigma}(n)$) possesses a nice conceptual interpretation: viz., the volume of the polytope $P = P^{m,n}$ (for $m=2$) equals precisely $\nu(n)/d!$ for odd n and $\tilde{\nu}(n)/d!$ for even n (and the meaning of $\nu(n)/d!$ for even n is that it equals $2^d \times \tilde{\nu}(n)/d!$, i.e. is the volume of 2.P). Thus, we find that our combinatorics of determining the Hilbert function 'numerators' ν_n and $\tilde{\nu}_n$ amounts to certain canonical q-refinements of these two crucial integer sequences (of which the latter $\tilde{\sigma}$ is gotten form the former σ by dividing alternate terms by successive odd powers of 2); let us now look at a listing of the two integer sequences to larger range:

n	3	4	5	6	7	8	9	10	11	12	13
$\sigma(n)$	1	2	5	24	154	1280	13005	156800	2189725	34793472	620169186
$\tilde{\sigma}(n/2)$	.	1	.	3	.	40	.	1225	.	67956	.

The next 5 values are given by

$\tilde{\sigma}(14) = 5986134$, $\sigma(15) = 266267950740$, $\tilde{\sigma}(16) = 2^5.3.11.13.29.1933$,
$\sigma(17) = 3.11.13.17.31.7129.100279$, $\tilde{\sigma}(18) = 3^3.11.13.127.277663$.

As the reader may suspect, we tried to look at the prime factorization for the numbers involved, only to see little pattern; except that *by and large* the largest prime factor is much larger than the second-largest (we found a near-exception to this in $\nu(21)$ for which the 2 top primes are 4593569 and 9595199) and 'repeated primes' appear only for very small primes. (Thus no $\sigma(n)$ for $n < 25$ is divisible by p^2 for a prime $p > 17$ and the top primes for $n = 23$ and 25 are both 18-digits long.)

REMARK 2: The apparent 'symmetry' in the 2 sets of polynomials (ν and $\tilde{\nu}$ above) is related, of course, to the celebrated result of Hochster causing the ring R be Gorenstein (hence also Cohen-Macaulay); that is so, because the 2-step process of taking invariants in S to get Q (via R) can be cut short to identify Q directly as a suitable invariant subring of S under a *reductive algebraic group* (viz. invariants under the joint action of two groups $\mathbf{PGL}_m$ and the diagonal torus in $\mathbf{SL}_n$, noting that the two actions commute). Now it seems reasonable to expect, in the present case of $m = 2$, that a 'good' s.o.p. (system of parameters, in the standard sense of Commutative Algebra, cf. Stanley [St']) for Q is furnished by picking a suitable subset of a basis for the 'first' component (Q_1 for odd n and Q_{frac12} for even n – this uneasy distinction being due to our clumsy convention for better maneuvring elsewhere); and if Q' is the subring generated by our s.o.p. one expects also the finite free-module structure of Q over Q' is read out precisely by the q-series ν_n (for odd n) or by $\tilde{\nu}_n$ (for even n).

While we have little to add in general (even towards the limited combinatorial objective like that for $m = 2$) for higher m, the case $(m, n) = (3, 6)$ deserves special commentary. [The case (3,5) needs little commentary in view of the general observation that our problem, for any given (m, n) (satisfying $n > m$), is symmetric in m and $n - m$; thus, 'classifying' 5 points in $\mathbf{P}^1$ is the 'same' as doing that in $\mathbf{P}^2$ – which may look mysterious to the uninitiated (cf. [DO]). Actually this duality is a ready consequence of the easy symmetry satisfied by $\mathbf{Grass}_m(n)$ (under the exchange of m and $n - m$).] We find the GF for the Hilbert series of Q given by

$$\frac{1+q^2}{(1-q)^5} = \frac{1-q^4}{(1-q)^5.(1-q^2)}.$$

This has profound implications already: Let us first note that on each component of $Q = Q^{m,n}$ (for arbitrary m, n) the symmetric group $\mathbf{S}_n$ possesses a natural action, and Q_1 is the 5-dimensional irreducible $\mathbf{S}_6$-module corresponding to the partition $\lambda = < 2, 2, 2 >$. [It is one of those rare instances of an irreducible module M for the group $W = \mathbf{S}_n$, which is not the so-called natural or reflection representation of W (viz. that corresponding to the partition $< n-1,\ 1 >$) and yet the invariants for W in the algebra $\mathrm{Sym}^\bullet(M)$ is again a polynomial algebra; this is because of the existence of an outer automorphism of W that conjugates the representation at hand with the said reflection module, but one needs to verify this with the character table (and the known action of the said exotic automorphism on the conjugacy classes).] Then Q is a rank-2 free-module over the subalgebra Q' generated by Q_1, with a patently nice (and rather canonical) basis $\{1, \Delta\}$, where Δ is a very special element of Q_2 which is the difference of two terms $\xi_{\mathbf{I}}$ and $\xi_{\mathbf{J}}$ coming from the two unique SST's of shape $< 4, 4, 4 >$ (out of a total of 16 ($=$ dim Q_2) SST's of that shape) which do not lie individually in Q_2 (but their sum does); then the square of Δ lies in Q' in accordance with the GF obtained above; and besides Δ possesses another distinguishing feature, by way of a very fundamental and simple geometric property – viz. its vanishing is a necessary and sufficient condition for the 6 points (in our space of 6-point configurations inside $\mathbf{P}^2$) to lie on a conic!

Thus, while we're unable to say very much on the geometry of Y, it seems we're able to make a beginning at least, by unraveling some of the ring-theoretic mysteries of Q and the associated combinatorics built into the problem.

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