A property of p-rough numbers

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It is a well-known elementary arithmetical result that the product of any k consecutive integers $n(n+1)\ldots(n+k-1)$ is divisible by $k!$. Probably the most direct proof of this result is to observe that when n is a positive integer

$$\frac{n(n+1)\ldots(n+k-1)}{k!} \quad = \quad \binom{n+k-1}{k},$$

and the binomial coefficent on the right-hand side gives the number of ways of choosing k items from $n+k-1$ items. When n is a nonpositive integer the result is either trivial (because the product in the numerator vanishes) or follows easily from the positive case. An alternative proof of the result, due to Gauss [1], is to show that for any prime factor p of $k!$, the power to which p divides $k!$ is less than or equal to the power to which p divides the product $n(n+1)\ldots(n+k-1)$.

A natural question to ask is are there integer values of r, other than 1, such that the product $n(n+r)(n+2r)\ldots(n+(k-1)r)$ of any k integers in arithmetic progression is divisible by $k!$? The answer is given by the following result.

Proposition 1. Let r be a nonzero integer and let k be a positive integer. The product $n(n+r)(n+2r)\ldots(n+(k-1)r)$ is divisible by $k!$ for all integer n if and only if r is coprime to every prime factor of $k!$. $\square$

The proposition can be proved by following Gauss' approach in the case $r=1$. Rather than prove the proposition in general we shall examine a particular case of the proposition to illustrate the method of proof.

Let $k=10$. We have the prime factorization $k! = 10! = 2^8 3^4 5^2 7$. Thus in this case Proposition 1 states that

$$\frac{n(n+r)\ldots(n+9r)}{10!} \quad \in \quad \mathbb{Z} \quad \text{for all integer } n$$

if and only if r is coprime to the primes 2, 3, 5 and 7.

1) First we prove the forward implication by proving the contrapositive statement: if r is not coprime to every prime factor of 10! then the product $n(n+r)\ldots(n+9r)$ is not divisible by 10! for all integer n.

Suppose then r is divisible by one of the prime factors 2, 3, 5 or 7 of 10!, say, for example (and without loss of generality), 5 divides r. Then, choosing $n = 1$, we see that the product $n(n+r)\dots(n+9r)$ has the form $5m+1$, and so is not divisible by 5, and hence is certainly not divisible by 10!.

2) We next prove the backward implication. Suppose now r is coprime to every prime factor of 10!, that is, r is coprime to the primes 2, 3, 5 and 7. We will show that

$$\frac{n(n+r)\dots(n+9r)}{10!} \quad \in \quad \mathbb{Z} \quad \text{for all } n. \tag{1}$$

We do this by showing that the power to which a given prime divides 10! is less than or equal to the power to which that prime divides the numerator $P(n) := n(n+r)\dots(n+9r)$ in (1). Of course, we only need to do this for the primes 2, 3, 5 and 7 appearing in the prime factorization of 10!. We consider each prime factor in turn. In what follows it will be notationally convenient to let $\bar{p}$ denote some unspecified multiple of a prime p, (which, in a slight abuse of notation, may have different values in the same equation).

(i) The prime 7:

Since by assumption r is coprime to 7, the set of numbers $\{0, r, 2r, \dots, 6r\}$ forms a complete set of residues modulo 7. Hence the numerator $P(n)$ has the form $n(n+1+\bar{7})(n+2+\bar{7})\dots(n+6+\bar{7}) * (n+7)(n+8)(n+9)$. Clearly, among the first 7 factors of this product there will be one factor that is divisible by 7 (since one of the seven consecutive numbers $n, n+1, \dots, n+6$ is guaranteed to be divisible by 7). Therefore, 7 divides the numerator $n(n+r)\dots(n+9r)$ in (1) for each n and each r coprime to 2, 3, 5 and 7.

(ii) The prime 5:

Since by assumption r is coprime to 5, both sets of numbers $\{0, r, \dots, 4r\}$ and $\{5r, 6r, \dots, 9r\}$ are a complete set of residues modulo 5. Hence the numerator $P(n)$ has the form

$$\{n(n+1+\bar{5})(n+2+\bar{5})(n+3+\bar{5})(n+4+\bar{5})\}*\{(n+\bar{5})(n+1+\bar{5})(n+2+\bar{5})(n+3+\bar{5})(n+4+\bar{5})\}.$$

Clearly, one of the first five factors of this product is divisible by 5 (since one of the five consecutive numbers $n, n+1, \dots, n+4$ is guaranteed to be divisible by 5), and, similarly, one of the final five factors of this product is divisible by 5. Hence the numerator $n(n+r)\dots(n+9r)$ in (1) is divisible by 5^2 for each n and each r coprime to 2, 3, 5 and 7.

(iii) The prime 3:

Since by assumption r is coprime to 3, the three sets of numbers $\{0, r, 2r\}$, $\{3r, 4r, 5r\}$ and $\{6r, 7r, 8r\}$ are each a complete set of residues modulo 3. Hence the numerator $P(n)$ has the form

$$\{n(n+1+\bar{3})(n+2+\bar{3})\}*\{(n+\bar{3})(n+1+\bar{3})(n+2+\bar{3})\}*\{(n+\bar{3})*(n+1+\bar{3})(n+2+\bar{3})\}*(n+9).$$

Each term in braces is divisible by 3 since one of the three consecutive numbers $n, n+1, n+2$ has a factor of 3. Hence 3^3 divides $P(n)$.

Furthermore, since r is coprime to 9, the set of numbers $\{0, r, \ldots, 8r\}$ is a complete set of residues modulo 9. Hence the numerator $P(n)$ has the form $\{n(n+1+\bar{9})\ldots(n+8+\bar{9})\} * (n+9)$. Clearly, 9 divides one of the first nine factors of this product. This gives an extra factor of 3 dividing $P(n)$ from the three previously found. Hence the numerator $n(n+r)\ldots(n+9r)$ in (1) is divisible by $3^{3+1} = 3^4$ for each n and each r coprime to 2, 3, 5 and 7.

(iv) The prime 2:

The number r is coprime to 2 by assumption and hence also coprime to 4 and 8. First we take each of the ten factors in the product $P(n)$ modulo 2 to find that floor$(10/2) = 5$ of the factors are divisible by 2. Then we take each of the ten factors in the product $P(n)$ modulo 4 to find that floor$(10/4) = 2$ of the factors are divisible by 4. This gives two extra factors of 2 dividing $P(n)$ from the five just found. Finally, we consider each of the ten factors in the product $P(n)$ modulo 8 to find that floor$(10/8) = 1$ of the factors is divisible by 8. This gives an extra factor of 2 dividing $P(n)$ from those already found. In total we have shown that $2^{5+2+1} = 2^8$ divides $P(n)$ for each n and each r coprime to 2, 3, 5 and 7.

Therefore, combining the conclusions of (i) through (iv), we see that $2^8 3^4 5^2 7 = 10!$ divides $P(n) = n(n+r)\ldots(n+9r)$ for every integer n and every number r coprime to the primes 2, 3, 5 and 7, thus proving (1).

p-rough numbers

Let p be a prime We define a p-rough number to be an integer coprime to all the primes less than p. For example, the set of 3-rough numbers is the set of odd numbers. For each prime p, the set of positive p-rough numbers is a semigroup with unity (a monoid) under the operation of multiplication and has the unique factorization property.

In the terminology of p-rough numbers the particular case of Proposition 1 considered above states that

$$\frac{n(n+r)\dots(n+9r)}{10!} \quad \in \quad \mathbb{Z} \quad \text{for all integer } n$$

if and only if r is 11-rough.

Taking in turn $k = 7, 8$ and 9 in Proposition 1 gives the companion results

(i)

$$\frac{n(n+r)\dots(n+8r)}{9!} \quad \in \quad \mathbb{Z} \quad \text{for all integer } n$$

if and only if r is 11-rough.

(ii)

$$\frac{n(n+r)\dots(n+7r)}{8!} \quad \in \quad \mathbb{Z} \quad \text{for all integer } n$$

if and only if r is 11-rough.

(iii)

$$\frac{n(n+r)\dots(n+6r)}{7!} \quad \in \quad \mathbb{Z} \quad \text{for all integer } n$$

if and only if r is 11-rough.

Finally we ask is there any combinatorial interpretation of the integers $n(n+r)\dots(n+(k-1)r)/k!$, where r is an integer coprime to $k!$?

REFERENCES

[1] C. F. Gauss, Disquisitiones Arithmeticae, Section 1V, Art. 127