Sean A6991

Wilson

letter

12 Oct '93

√ 894 A6991

12 October 1993

Neil James Alexander Sloane
% Room 2C-376, Mathematics Research Center
AT&T Bell Telephone Laboratories Inc.
600 Mountain Avenue
Murry Hill, New Jersey 07974-0636
908+582-3000, ext. 2005

Subject: A Handbook of Integer Sequences

Dear Dr. Sloane,

Please consider the following sequences for inclusion in your forthcoming second edition of the above this fall.

"A number k is called congruent if integers x and y exist such that both $x^2 + ky^2$ and $x^2 - ky^2$ are squares. The expression $x^2 - ky^2$ may also be a negative square as it is always possible to obtain a positive solution from such a negative one. No certain method for ascertaining the congruence of a given number is known even in these modern times when mighty analytic tools are at our disposal. It is also difficult to find a solution even when we know that k is congruent." [Beiler]

The first sequence represents the square free or "primitive congruent" numbers. They are as follows: 5, 6, 7, 13, 14, 15, 21, 22, 23, 29, 30, 31, 34, 37, 38, 39, 41, 46, 47, 53, 55, 61, 62, 65, 69, 70, 71, 77, 78, 79, 85, 86, 87, 93, 94, 95, 101, 102, 103, 109, 110, 111, 118, 119, 127, 133, 134, 137, 138, 141, 142, 143, 145, 149, 151, 154, 157, 158, 159, 161, 165, 166, 167, 173, 174, 181, 182, 183, 190, 191, 194, 197, 199, 205, 206, 210, 213, 214, 215, 219, 221, 222, 223, 226, 229, 230, 231, 237, 238, 239, 246, 247, 253, 254, 255, 257, 262, 263, 265, 269, 271, 277, 278, 285, 286, 287, 291, 293, 295, 299, 301, 302, 303, 309, 310, 311, 313, 317, 318, 319, 323, 326, 327, 330, 334, 335, 341, 349, 353, 357, 358, 359, 365, 366, 367, 371, 373, 374, 381, 382, 383, 386, 389, 390, 391, 395, 397, 398, 399, 406, 407, 410, 413, 415, 421, 422, 426, 429, 430, 431, 434, 437, 438, 439, 442, 445, 446, 447, 453, 454, 455, 457, 461, 462, 463, 465, 469, 470, 471, 478, 479, 485, 487, 493, 494, 501, 502, 503, 505, 509, 510, 511, 514, 517, 518, 519, 526, 527, 533, 534, 535, 541, 542, 546, 551, 557, 559, 561, 565, 566, 574, 581, 582, 583, 589, 590, 591, 598, 599, 602, 606, 607, 609, 613, 614, 615, 622, 629, 631, 638, 645, 646, 647, 651, 653, 654, 655, 658, 661, 662, 663, 669, 670, 671, 674, 677, 678, 679, 685, 687, 689, 694, 695, 697, 701, 703, 709, 710, 718, 719, 721, 723, 727, 731, 733, 734, 741, 742, 743, 749, 751, 757, 758, 759, 761, 766, 767, 773, 777, 781, 782, 789, 790, 791, 793, 797, 798, 799, 805, 806, 807, 813, A6991

814, 815, 821, 822, 823, 829, 830, 831, 838, 839, 853, 854, 861, 862, 863, 866, 869, 870, 871, 877, 878, 879, 885, 886, 887, 889, 890, 894, 895, 901, 902, 903, 905, 910, 911, 915, 919, 926, 934, 935, 941, 942, 943, 949, 951, 957, 958, 959, 966, 967, 973, 974, 982, 983, 985, 987, 989, 991, 995, 997, 998, $\cdots$.

The complete set can be generated by taking the members of the above infinite set and multiplying them by all the integers squared. This will produce the following sequence, although I would advise against its inclusion. It is as follows: 5, 6, 7, 13, 14, 15, 20, 21, 22, 23, 24, 28, 29, 30, 31, 34, 37, 38, 39, 41, 45, 46, 47, 52, 53, 54, 55, 56, 60, 61, 62, 63, 65, 69, 70, 71, 77, 78, 79, 80, 84, 85, 86, 87, 88, 92, 93, 94, 95, 96, 101, 102, $\cdots$.

The non-included members cited above might be entitled the "incongruent" or "noncongruent" set of the integers. "The remaining 49 integers [in the first 100 integers], among which are 1 and 2, are incongruent. Thus $x^2 + y^2 = u^2$ and $x^2 - y^2 = v^2$ are impossible and also $x^2 + 2y^2 = u^2$ and $x^2 - 2y^2 = v^2$." [Beiler] They are as follows: 1, 2, 3, 4, 8, 9, 10, 11, 12, 16, 17, 18, 19, 25, 26, 27, 32, 33, 35, 36, 40, 42, 43, 44, 48, 49, 50, 51, 57, 58, 59, 64, 66, 67, 68, 72, 73, 74, 75, 76, 81, 82, 83, 89, 90, 91, 97, 98, 99, 100, $\cdots$.

Likewise, by removing the squared factors, a reduced or "primitive incongruent" set is obtained and provides a much more condensed set of integers. They are as follows: 1, 2, 3, 10, 11, 17, 19, 26, 33, 35, 42, 43, 51, 57, 58, 59, 66, 67, 73, 74, 82, 83, 89, 91, 97, 105, 106, 107, 113, 114, 115, 122, 123, 129, 130, 131, 139, 146, 155, 163, 170, 177, 178, 179, 185, 186, 187, 193, 195, 201, 202, 203, 209, 211, 217, 218, 227, 233, 235, 241, 249, 251, 258, 259, 266, 267,

273, 274, 281, 283, 290, 298, 305, 307, 314, 321, 322, 329, 331, 337, 339, 345, 346, 347, 354, 355, 362, 370, 377, 379, 385, 393, 394, 401, 402, 403, 409, 411, 417, 418, 419, 427, 433, 435, 443, 449, 451, 458, 466, 467, 473, 474, 481, 483, 489, 491, 497, 498, 499, 506, 515, 521, 523, 530, 537, 538, 545, 547, 553, 554, 555, 562, 563, 570, 571, 579, 586, 587, 595, 601, 610, 611, 617, 618, 619, 626, 633, 634, 635, 641, 642, 643, 649, 659, 665, 667, 673, 681, 682, 683, 690, 691, 697, 698, 699, 705, 707, 713, 714, 715, 730, 737, 739, 745, 746, 753, 754, 755, 762, 763, 769, 770, 771, 778, 779, 785, 786, 787, 794, 795, 803, 811, 817, 818, 826, 827, 834, 835, 842, 843, 849, 851, 858, 859, 865, 874, 883, 899, 906, 907, 913, 914, 921, 922, 923, 929, 930, 937, 946, 947, 955, 962, 969, 970, 971, 977, 978, 979, 986, 993, 994, $\cdots$.

Adding the above sequence with the first sequence will produce a sequence of square free integers, which is the 𝕾𝕾ℕ 223. "Primitive" or square free integers whose characteristics are not yet known are as follows: 543, 573, 597, 623, 627, 706, 717, 801, 893, 897, 898, 917, 933, 938, 939, 965, ⋯.

Many "primitive' congruent numbers come in groups of three, and the first of the three are as follows: 5, 13, 21, 29, 37, 69, 77, 85, 93, 101, 109, 141, 157, 165, 181, 213, 221, 229, 237, 253, 285, 301, 309, 317, 357, 365, 381, 389, 397, 429, 437, 445, 453, 461, 469, 501, 509, 517, 533, 581, 589, 613, 645, 653, 661, 669, 677, 741, 757, 789, 797, 805, 813, 821, 829, 861, 869, 877, 885, 901, 941, 957, ⋯. Please notice that all of the above are ≡ 5 modulus 8. Therefore; the quotient of the above sequence is: 0, 1, 2, 3, 4, 8, 9, 10, 11, 12, 13, 17, 19, 20, 22, 26, 27, 28, 29, 31, 35, 37, 38, 39, 44, 45, 47, 48, 49, 53, 54, 55, 56, 57, 58, 62, 63, 64, 66, 72, 73, 76, 80, 81, 82, 83, 84, 92, 94, 98, 99, 100, 101, 102, 103, 107, 108, 109, 110, 112, 117, 119, ⋯.

Just as with the "primitive" congruent numbers, so do the "primitive" incongruent numbers come in groups of three, and the first of the three are as follows: 1, 57, 105, 113, 129, 177, 185, 201, 345, 401, 417, 497, 553, 617, 633, 641, 681, 697, 713, 753, 769, 785, 921, 969, 977, ⋯. Please notice that all of the above are ≡ 1 modulus 8 and this is opposite the "primitives." This should be no surprize. see 5th Ed. Hardy&Wright. Therefore; the quotient of the above sequence is: 0, 7, 13, 14, 16, 22, 23, 25, 43, 50, 52, 62, 69, 77, 79, 80, 85, 87, 89, 94, 96, 98, 115, 121, 122, ⋯. Both of these groupings of three would have many more members if one was to include numbers which are not square free.

These congruent numbers are intimately connected with primitive Pythagorean triangles and their generators. These two co-prime integer generators are *m* and *n* with $m > n$. Therefore; for increasing *m*s and *n*s, the generated primitive conguent numbers which equal $m \cdot n \cdot (m^2 - n^2)/p_\alpha^2$ produce a sequence, less any repeats that do occur, which is as follows: 6, 30, 15, 21, 210, 5, 330, 70, 231, 546, 14, 390, 154, 65, 34, 110, 2730, 3570, 190, 286, 1155, 5610, 1254, 2310, 429, 1785, 1995, 759, 4290, 221, 10374, 8970, 39, 7854, 1330, 1610, 462, 6630, 3135, 4830, 6090, 255, 741, 161, 7, 165, 1131, 465, 9690, 4641, 25806, 7395, 22134, 561, 646, 2990, 4466, 4030, 1190, 13566, 6555, 1482, 5510, 12369, 43890, 1406, 5865, 1365, 1595, 1705, 15015, 9435, 3705, 18354, 357, 15834, 71610, 3885, 4305, 10626, 66990, 8866, 10010, 84570, 72930, 51414, 19866, 966, 1311, 68034, 21390, 98670, 26565, 11914, 6279, 9430, 14835, 2530, 138, 16530, 30030, 31746, 29274, 24510, 6486, 609, 3534, 1122, 1443, 6006, 41, 602, 3102, 6, 78, 52026, 84630, 114114, 15470, 158730, 175890, 19006, 17290, 128310, 1794, 1326, 174, 2139, 3990, 18870, 22386, 1419, 4935, 4134, 651, 4921, 51051, 55965, 58695, 6545, 6251, 41055, 1113, 5394, 957, 140070, 45066, 214890, 60639, 261870, 1885, 29986, 145, 227766, 46110, 124410, 11571, 26970, 178710, 257070, 285090, 311610, 6270, 255990, 51330, 59334, 3255, 6882, 55614, 266910, 75981, 36890, 21855, 806, 86955, 36146, 229710, 38409, 56730, 2046, 1110, 1886, 2470, 23970, 510, 25194, 24486, 798, 1770, 10614, 434, $\cdots$.

The cited numbers are for all "primitve" congruent numbers generated for m & $n \leq 2^5$.

Reference: BE3: Albert H. Beiler, Recreations In The Theory Of Numbers, The Queen of Mathematics Entertains, pgs. 155-7, Dover Publ., NY, 1964.

Oystein Ore, Number Theory and Its History, pgs. 191&202, Dover Publ., NY, 1948.

SI1: Waclaw Sierpinski, Elementary Theory of Numbers, 2nd Ed., Chpt. 2.9., pgs. 62-66, Vol. 31, North-Holland Mathematical Library, Elsevier Science Publ., NY, 1988.

Sequentially yours,

Robert G. Wilson v

Ph.D., ATP/CF&GI

cc:Edward Shapiro

RGWv:T4400C

sloane31