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The Enumeration of Mating-Type Graphs*

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Abstract

Mating-type graphs (M-graphs), in which no two vertices have the same set of neighbours, are enumerated, as well as M-digraphs, defined similarly.

1 Introduction

In [1] Bull and Pease discuss a kind of graph that arises in the context of mating systems among animals. Let each vertex of a graph denote an individual animal and let two vertices be joined if the two animals are compatible, i.e., can mate. If two animals have identical compatibilities they are said to be of the same mating type, and in that case there is no point in their both being represetned in the graph. Thus, for economy, the different vertices can be taken instead to represent different mating types. It then follows that, in such a graph, no two vertices have identical neighbourhoods, i.e., the same set of adjacent vertices. This is the characteristic of "mating-type graphs", or "M-graphs" for short.

Definition 1.1 *An M-graph is a finite graph, without loops or multiple edges, with the property that no two vertices are adjacent to the same set of vertices.*

Bull and Pease gave the numbers of M-graphs on p vertices for $p = 1$ to 10. These numbers were obtained from the catalogue of graphs whose construction was described in [2]. In this paper I give a theoretical enumeration of M-graphs, deriving a generating function from which these numbers can be found without constructing the graphs. The cognate problem of enumerating M-digraphs (defined similarly) is also solved.

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2 Preliminaries

Any graph G has associated with it a unique M-graph, obtained by identifying any two vertices which have the same set of adjacent vertices. Conversely, we can obtain all the graphs which correspond in this way to a given M-graph M_1 by replacing each vertex u of M_1 by a number of independent vertices having the same adjacencies as u. The number of such vertices could be 1, but clearly must not be 0.

To enumerate, by number of vertices, all the graphs obtained in this way from M_1 is a simple exercise in Pólya's Theorem (see [6]) and is effected by substituting the figure counting series

$$x + x^2 + x^3 + x^4 + \ldots = x(1-x)^{-1}$$

in the cycle-index $Z(A_1)$ of the automorphism group A_1 of M_1. The result is standardly written as

$$Z(A_1; x(1-x)^{-1}).$$

For the present problem this is not enough. We need to find the sum of the cycle-indexes of the automorphism groups of the graphs thus obtained. This we get by substituting in $Z(A_1)$ the sum of the cycle-indexes for the set of graphs $\{E_n\}$ where E_n denotes the empty (i.e., edgeless) graph on n vertices, which is the replacement for a typical vertex of M_1. This cycle-index sum can be written as

$$\sum_{r \geq 1} Z(S_n)$$

and the result of substituting this in $Z(A_1)$ is written as

$$Z(A_1)[\sum_{r \geq 1} Z(S_n)]$$

the force of the substitution being that each variable s_i in $Z(A_1)$ is replaced by $\sum Z(S_r)$ in which every s_α is replaced by $s_{i\alpha}$.

If we now repeat this with every M-graph and sum the results, we obtain a cycle-index sum which can be written as

$$Z(\mathcal{M})[\sum_{r \geq 1} Z(S_r)] \qquad (2.1)$$

where $\mathcal{M}$ denotes the set of M-graphs and $Z(\mathcal{M})$ stands for the sum of the cycle-indexes of the automorphism groups of these graphs.

The expression (2.1) therefore gives the sum of the cycle-indexes for all inequivalent graphs obtained from M-graphs by the replacement of vertices by sets of vertices as described above. By what was stated earlier, every graph can be obtained uniquely in this way. Thus the result (2.1) must be the same as the sum of cycle-indexes for all graphs. This gives us our basic equation, namely

$$Z(\mathcal{M})[\sum_{r\geq 1} Z(S_r)] = Z(\mathcal{G}) \tag{2.2}$$

where $\mathcal{G}$ stands for the set of all (nonisomorphic) graphs and $Z(\mathcal{G})$ is the sum of the corresponding cycle-indexes.

The next step is to invert (2.2) to obtain useful results about $Z(\mathcal{M})$.

3 The Inversion

The method for doing this, as explained in [3], is to find a power series $\mu(x)$ such that

$$\{\sum_{r\geq 1} Z(S_r)\}[\mu(x)] = x$$

or, replacing the summation by an equivalent expression (see [4], page 52)

$$\{exp(s_1 + \frac{1}{2}s_2 + \frac{1}{3}s_3 + \ldots) - 1\}[\mu(x)] = x. \tag{3.1}$$

The usual way to find $\mu(x)$ is to compute its coefficients recursively (see [3] again); but the present problem has the interesting feature that the function $\mu(x)$ assumes a particularly simple form, which can be verified with ease. In fact, $\mu(x) = x - x^2$. For after the necessary substitutions have been made, the left-hand side of (3.1) becomes

$$\begin{aligned} exp(\sum_{r\geq 1} \frac{1}{r}(x^r - x^{2r})) - 1 &= exp(\sum_{k\geq 1} \frac{(-1)^{k+1}}{k} x^k) - 1 \\ &= exp(log(1 + x)) - 1 \\ &= x \end{aligned}$$

Now that $\mu(x)$ is found the rest of the inversion process is easy. We act on $\mu(x)$ with each side of (2.2) and obtain, for the left-hand side

$$(Z(\mathcal{M})[\sum_{r\geq 1} Z(S_r)])[\mu(x)]$$

which, since the substitution is associative, is the same as

$$Z(\mathcal{M})[(\sum_{r\geq 1} Z(S_r))[\mu(x)]] = Z(\mathcal{M})[x] \quad (3.2)$$

Now the right-hand side of (3.2) is the cycle-index sum $Z(\mathcal{M})$ with each s_i replaced by x^i. This will give x^n for each cycle-index in the sum which has total weight n, and hence the right-hand side reduces to the required generating function for M-graphs, of the form

$$M(x) = \sum_{p\geq 1} m_p x^p$$

where m_p is the number of M-graphs having p vertices.

The right-hand side of (2.2), after the substitution of $\mu(x)$, becomes

$$Z(\mathcal{G})[x - x^2] \quad (3.3)$$

i.e., the cycle-index sum $Z(\mathcal{G})$ with every s_i replaced by $x^i - x^{2i}$. Hence if we can compute $Z(\mathcal{G})$ we can find the required generating function for M-graphs.

4 The Generating Function

To compute $Z(\mathcal{G})$ we follow the method for enumerating graphs on a given number p of vertices, as given, for example, in [4] or [5].

This is a straight-forward application of Pólya's Theorem and yields the result

$$Z(S_p^{(2)}; 1 + y) \quad (4.1)$$

where y marks edges, and $S_p^{(2)}$ is the group of permutations induced on the set of unordered pairs of p vertices by the group S_p of all permutations of the vertices themselves. It can be shown (op. cit.) that the term

$$\frac{1}{\Pi i^{\rho_i} \rho_i!} \Pi s_i^{\rho_i}$$

in $Z(S_p)$ gives the term

$$\frac{1}{\prod i^{\rho_i}\rho_i!}\prod_n (s_n s_{2n}^{n-1})^{\rho_{2n}} \prod_n s_{2n+1}^{n\rho_{2n+1}} \prod_n s_n^{n\binom{\rho_n}{2}} \prod_{1\le i<j\le p} s_{lcm(i,j)}^{\rho_i\rho_j^{gcd(ij)}}$$

in $Z(S_p^{(2)})$, where $lcm(i,j)$ and $gcd(i,j)$ denote the lowest common multiple and greatest common division, respectively, of i and j. There is such a term for every partition $\rho = (1^{\rho_1}2^{\rho_2}3^{\rho_3}\ldots)$ of p. (For this notation and others below, see [7]).

If we want the total number of graphs on p vertices (that is, not broken down by numbers of edges as given in (4.1) by the coefficients of various powers of y) we put $y = 1$, so that each s_i is replaced by 2. The total number of graphs on p vertices is then given by

$$\sum_\rho h_\rho 2^{X_\rho}$$

where

$$h_\rho = \frac{1}{\prod_i^{\rho_i}\rho_i!},$$

$$X_\rho = \sum_n n\rho_{2n} + \sum_n n\rho_{2n+1} + \sum_n n\binom{\rho_n}{2} + \sum_{1\le i<j\le p} \rho_i\rho_j\, gcd(i,j)$$

and the summation is over all partitions ρ of p.

To obtain the sum of the cycle-indexes of the graphs on p vertices, we have merely to retain, for each term in (4.2), the monomial for the permutation of the vertices, viz $s_1^{\rho_1}s_2^{\rho_2}\ldots s_p^{\rho_p}$. Thus, on summing for $p > 0$, we derive the formula

$$Z(\mathcal{G}) = \sum_{p>0}\{\sum_\rho h_\rho 2^{X_\rho} s_1^{\rho_1}s_2^{\rho_2}\ldots s_p^{\rho_p}\}.$$

Making the substitution of $x - x^2$ in this expression, we obtain the generating function for M-graphs in the form

$$\sum_{p>0}\{\sum_\rho h_\rho 2^{X_\rho}\prod_{i=1}^{p}(x^i - x^{2i})^{\rho_i}\}$$

which can be rewritten as

$$M(x) = \sum_{p>0}\{x^p\sum_\rho h_\rho 2^{X_\rho}\prod_{i=1}^{p}(1 - x^i)^{\rho_i}\} \qquad (4.2)$$

The numbers of M-graphs for $p = 1$ to 12 have been calculated from (4.3). The numbers for $p = 1$ to 10 agree with those given in [1]. The values for $p = 11$ and 12 are 912908876 and 154636289460 respectively.

5 M-Digraphs

The same method will enumerate M-digraphs, defined as digraphs in which no two vertices have identical out- and in-neighbours. By replicating vertices of an M-digraph, we can obtain any digraph, and hence we arrive at the equation

$$Z(\mathcal{N})[\sum_{r\geq 1} Z(S_r)] = Z(\mathcal{D}) \tag{5.1}$$

where $\mathcal{N}$ represents the set of all M-digraphs and $\mathcal{D}$ the set of all digraphs. This is the analogue of (2.2). Since the expression in brackets is the same as before, the analysis is the same, and we obtain

$$Z(\mathcal{D})[x - x^2] \tag{5.2}$$

as the analogue of (3.3).

Proceeding as before but using the cycle-index of $S_p^{[2]}$ - the group of permutations of the *ordered* pairs of vertices induced by S_p, we obtain the formula

$$\sum_p \{x^p \sum_\rho h_\rho 2^{Y_p} \prod_{i=1}^{p} (1 - x^i)^{\rho_i}\} \tag{5.3}$$

where

$$Y_p = \sum_m (m-1)\rho_m + \sum_m m\rho_m(\rho_m - 1) + 2 \sum_{1\leq i<j\leq p} \rho_i \rho_j \, gcd(i,j)$$

(for details of the cycle-index $Z(S_p^{[2]})$ see [4] or [5]).

Formula (5.3) has been used to compute the numbers of M-digraphs up to $p = 8$. The numbers are

p	1	2	3	4	5	6	7	8
-	1	2	12	183	8884	1495984	872987584	1787227218134

6 Further Remarks

The simplicity of the above results - in particular, the fact that the function $\mu(x)$ assumes the particularly simple form $x-x^2$, strongly suggests that there is a simplier interpretation of the results obtain. Without going into details we can readily convince ourselves that this is the case, and that what we have here is a variation on the well-known principle of exclusion and inclusion.

Let us take every graph, and for each vertex a either leave it as it is or replace it by two nonadjacent vertices having the same adjacencies as a. By Pólya's theorem the graphs obtained in this way are enumerated, by number of vertices, by the generating function

$$Z(\mathcal{G}; x + x^2) \tag{6.1}$$

with x marking vertices.

Let G be an M-graph with p vertices, and let G_a be the graph on $p+1$ vertices obtained by doubling the vertex a. If our aim is to count M-graphs, then G_a is not wanted. Now G_a will appear as a graph in its own right (with no duplicated vertices) and we can cancel this occurrence of it by regarding the graph G_a as having a negative sign attached to it. This would be the case if we replaced $x + x^2$ by $x - x^2$ in (6.1).

Let G_{ab} be a graph obtained from the M-graph G by duplicating exactly two vertices a and b. Again we want the occurrence of G_{ab} in its own right be cancelled, but this time both the graphs G_a and G_b (with the obvious notation) will occur negatively. Thus the occurrence of G_{ab} will be over-compensated for, and one occurrence will have to be re-instated. To ensure this we must regard G_{ab} as being counted positively. Proceeding in this was, we see that graphs to which an even number of extra vertices have been added must be counted positively, while those with an odd number of extra vertices should be counted negatively. All this is catered for by the change from $Z(\mathcal{G}; x + x^2)$ to $Z(\mathcal{G}; x - x^2)$.

This explanation is over-simplified. For example, it takes no account of the group action on the graphs. This line of investigation will not be pursued further here; suffice it to say that formula (3.3) exhibits in a somewhat more complicated manner than usual the pattern of cancellations and reinstatements that are characteristic of applications of the principle of exclusion and inclusion.

7 Labelled M-Graphs

The enumeration of labelled M-graphs, though of lesser interest, is worth a brief mention. It is easily performed from first principles.

Consider how to obtain graphs on n labelled vertices by replicating vertices in M-graphs with k labelled vertices. To do this we take the set of n labels and divide them into k nonempty subsets. These subsets are then used to label the vertices of the M-graph, each subset providing the labels for the replications of the vertex to which it is attached. The number of subsets is the Stirling number of the second kind $S(n,k)$, [8] page 48, and hence the total number of labelled graphs on n vertices will be

$$\sum_{k\geq 0} a_k S(n,k)$$

where a_k is the number of labelled M-graphs with k vertices. But the number of labelled graphs on n vertices is known; it is $2^{\frac{1}{2}n(n-1)}$ (see [5] page 3). Hence we have

$$2^{\frac{1}{2}n(n-1)} = \sum_{k\geq 0} a_k S(n,k) \tag{7.1}$$

By virtue of the inverse relation which holds between the two kinds of Stirling numbers, (8.1) can be inverted to give

$$a_k = \sum_{n\geq 0} 2^{\frac{1}{2}n(n-1)} s(n,k) \tag{7.2}$$

where the $s(n,k)$ denote Stirling numbers of the first kind. This completes the enumeration of labelled M-graphs.

The enumeration of labelled M-digraphs is similar, the only difference being the replacement of $2^{\frac{1}{2}n(n-1)}$ by $2^{n(n-1)}$. Thus if b_k is the number of M-digraphs on k labelled vertices, we have

$$b_k = \sum_{n\geq 0} 2^{n(n-1)} s(n,k) \tag{7.3}$$

Specific values for a_k and b_k can be easily found from (8.2) and (8.3). A few values are given in the table below.

k	1	2	3	4	5	6	7	8
a_k	1	1	4	32	588	21476	1551368	218218610
b_k	1	3	54	3750	1009680	-	-	-

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