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Guy

Many sequences

The second strong law ays

1 page from preprint then the whole of the published version

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127

THE SECOND STRONG LAW OF SMALL NUMBERS 3401

Richard K. Guy 5181

1602

You have probably already met The Strong Law of Small Numbers, either ~~5347~~ formally [15, 21, 22]: 1149

1856

> There aren't enough small numbers to meet the many demands made of them

1524

1519

or in some frustrated and semi–conscious formulation that occurred to you in the rough–and–tumble of everyday mathematical enquiry. It is the constant enemy of mathematical discovery: at once the Scylla, shattering sensible statement with spurious exceptions, and the Charybdis of capricious coincidences, causing careless conjectures: the dilemma to search for proof or for counter–example. It fooled Fermat (Example 1 of [21]) and we'll meet Euler's memorable example at the end of the article. 1071 719 1429 994 1475

5183

It's time to introduce The Second Strong Law of Small Numbers: 2379

1402

> When two numbers look equal, it ain't necessarily so!

1011

"How can this possibly be?" I hear you ask. By way of answer I invite you to examine 679 109

Example 36. Evaluate the polynomial $(n^4 - 6n^3 + 24n^2 - 18n + 24)/24$ for $n = 1, 2, 3, \ldots$ 103

Examples 1 to 35 are in [21]; there follow forty–four more. In each, you are invited to guess what pattern of numbers is emerging, and to decide whether the pattern will persist. Many of the examples are fraudulent, but some genuine theorems are mingled in, to keep you on your toes, and there may even be an unsolved problem or two. 1680 24/26

Examples 37 to 40 involve Pascal's triangle.

Example 37.

```
1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1
1 0 1 0 1 0 1
1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 1
1 1 0 0 0 0 0 0 1 1
1 0 1 0 0 0 0 0 1 0 1
1 1 1 1 0 0 0 0 1 1 1 1
1 0 0 0 1 0 0 0 1 0 0 0 1
1 1 0 0 1 1 0 0 1 1 0 0 1 1
1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1
1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1
1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1
1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1
1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1
1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

1317

Pascal's triangle (mod 2) has been a perennial topic. But have you tried reading the rows as binary numbers? 1, 3, 5, 15, 17, 51, 85, 255, 257, 771, 1285, 3855, 4369, 13107, 21845, 65535, 65537, ... Remember that there are zeros outside the triangle as well, so you can also include their doubles, 2, 6, 10, 30, 34, 102, ..., their quadruples, 4, 12, 20, 60, 68, ..., and so on, as well, if you like. Do you recognize these numbers?

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ARTICLES

3-20

The Second Strong Law of Small Numbers

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The University of Calgary
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You have probably already met The Strong Law of Small Numbers, either formally [**15**, **21**, **22**]

> There aren't enough small numbers to meet the many demands made of them

or in some frustrated and semiconscious formulation that occurred to you in the rough-and-tumble of everyday mathematical enquiry. It is the constant enemy of mathematical discovery: at once the Scylla, shattering sensible statement with spurious exceptions, and the Charybdis of capricious coincidences, causing careless conjectures: the dilemma to search for proof or for counterexample. It fooled Fermat (Example 1 of [**21**]) and we'll meet Euler's memorable example at the end of the article.

It's time to introduce The Second Strong Law of Small Numbers:

> When two numbers look equal, it ain't necessarily so!

"How can this possibly be?" I hear you ask. By way of answer I invite you to examine

Example 36 Evaluate the polynomial $(n^4-6n^3+24n^2-18n+24)/24$ for $n=1,2,3,\ldots$.

Examples 1 to 35 are in [**21**]; there follow forty-four more. In each, you are invited to guess what pattern of numbers is emerging, and to decide whether the pattern will persist. Many of the examples are fraudulent, but some genuine theorems are mingled in, to keep you on your toes, and there may even be an unsolved problem or two.

Examples 37 to 40 involve Pascal's triangle.

Example 37

Pascal's triangle (modulo 2) has been a perennial topic. But have you tried reading the rows as binary numbers? 1, 3, 5, 15, 17, 51, 85, 255, 257, 771, 1285, 3855, 4369, 13107, 21845, 65535, 65537, ... Remember that there are zeros outside the triangle as well, so you can also include their doubles, 2, 6, 10, 30, 34, 102, ..., their quadruples, 4, 12, 20, 60, 68, ..., and so on, as well, if you like. Do you recognize these numbers? 1317 3401

```
                               1
                              1 1
                             1 0 1
                            1 1 1 1
                           1 0 0 0 1
                          1 1 0 0 1 1
                         1 0 1 0 1 0 1
                        1 1 1 1 1 1 1 1
                       1 0 0 0 0 0 0 0 1
                      1 1 0 0 0 0 0 0 1 1
                     1 0 1 0 0 0 0 0 1 0 1
                    1 1 1 1 0 0 0 0 1 1 1 1
                   1 0 0 0 1 0 0 0 1 0 0 0 1
                  1 1 0 0 1 1 0 0 1 1 0 0 1 1
                 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
                1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
               1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
              1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
             1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1
            1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
           1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1
          1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1
         1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1
        1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1
       1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1
      1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1
     1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1
    1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
   1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
  1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

Example 38 Here we've drawn Pascal's triangle with each row starting off two places to the right of the previous start, i.e. with $\binom{n}{r}$ in row n and column $2n + r$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	1																								
1			1	1																					
2					1	2	1																		
3							1	3	3	1															
4									1	4	6	4	1												
5											1	5	10	10	5	1									
6													1	6	15	20	15	6	1						
7															1	7	21	35	35	21	7	1			
8																	1	8	28	56	70	56	28	8	1
9																			1	9	36	84	126	126	84
10																					1	10	45	120	210
11																							1	11	55

We've printed an entry in **bold** if it's divisible by its row number, and we've printed a column head in **bold** just if *all* the entries in the column are bold. What are these bold column heads?

Example 39 We've drawn Pascal's triangle again, but in contrast to the previous example, we've put an entry in **bold** just if it's not squarefree, i.e., just if it contains a square factor greater than 1.

0	1												
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	17	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

What are the row numbers (printed in **bold**) of the rows which contain at least one bold entry: 4, 6, 8, 9, 10, 12, ...?

Example 40 We finally draw Pascal's triangle with $\binom{n}{r}$ in the nth row and $(n+r)$th column:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1																
1		1	1														
2			1	2	1												
3				1	3	3	1										
4					1	4	6	4	1								
5						1	5	10	10	5	1						
6							1	6	15	20	15	6	1				
7								1	7	21	35	35	21	7	1		
8									1	8	28	56	70	56	28	8	1
9										1	9	36	84	126	126	84	36
10											1	10	45	120	210	252	210
11												1	11	55	165	330	462
12													1	12	66	220	495

The column totals are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. These numbers also seem to appear in the next three examples, as well as in Examples 70 and 80.

Example 41 The **ceiling** of, least integer not less than, $e^{(n-1)/2}$, for

$n =$	0	1	2	3	4	5	6	7	8	9...
is	1	1	2	3	5	8	13	21	34	55....

5181

Example 42 If $\sigma(n)$ is the sum of the divisors of n, then $\sigma(n)/n$ measures the **abundancy** of n. Every number n with abundancy $\geqslant j$ must have at least k factors, where, for

$j =$	2	3	4	5	6	7	8	9...
$k =$	2	3	4	6	9	14	22	35....

The differences of this last sequence are $1, 1, 2, 3, 5, 8, 13, \ldots$.

5347

Example 43 You may have suspected that some of the sequences in the last three examples are manifestations of the ubiquitous Fibonacci numbers ($u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_{n+1} + u_n$). According to the Lucas-Lehmer theory [33] the **rank of apparition** (the least n for which p divides u_n) of a prime p in the Fibonacci sequence is a divisor of $p - (p|5)$, where $(p|5)$ is the Legendre symbol, 0 for $p = 5$, and $+1$ or -1 according as $p \equiv \pm 1$ or ± 2, mod 5, otherwise. For example, the rank of apparition for the first few primes is

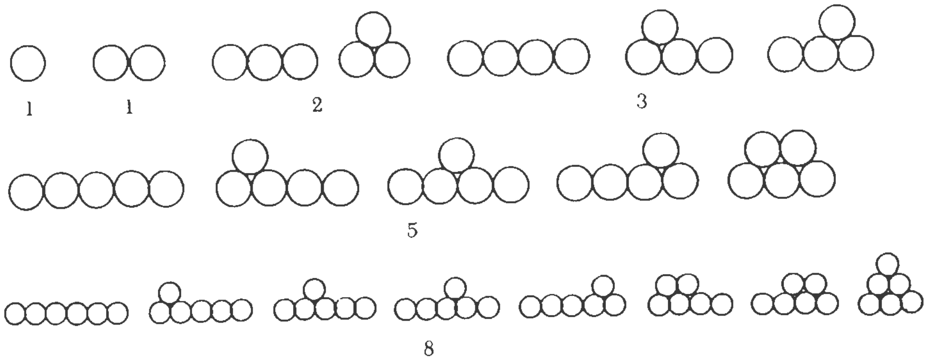
$p =$	2	3	5	7	11	13	17	19	23	29	31	37	41...	
	3	4	5	8	10	7	9	18	24	14	30	19	20....	1602

When a prime *does* first appear, does it always occur to the first power?

Example 44 Define a sequence by $c_1 = 1$, $c_2 = 2$ and c_{n+1} the least integer such that $c_{n+1} - c_{n-1}$ differs from all earlier positive differences $c_j - c_i$, $1 \leqslant i < j \leqslant n$, e.g.

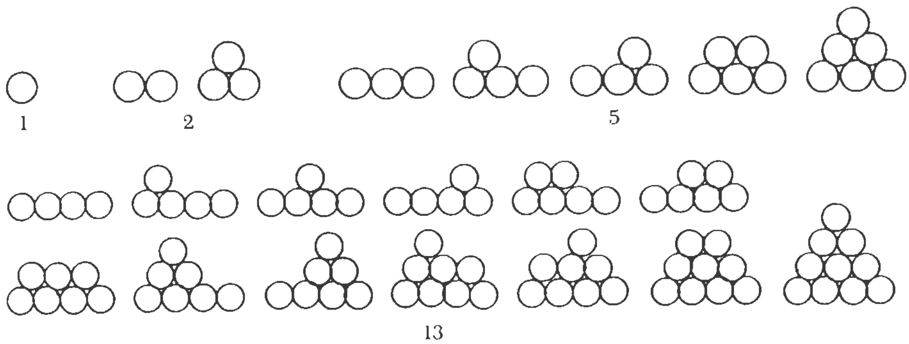
$\{c_1, c_2\} = \{1,2\}$	difference 1	$c_3 - c_1 = 2$	$c_3 = 3$	1149
$\{c_1, c_2, c_3\} = \{1,2,3\}$	differences 1,2	$c_4 - c_2 = 3$	$c_4 = 5$	
$\{c_1, \ldots, c_4\} = \{1,2,3,5\}$	differences 1,2,3,4	$c_5 - c_3 = 5$	$c_5 = 8$	
$\{c_1, \ldots, c_5\} = \{1,2,3,5,8\}$	differences 1,2,...,7	$c_6 - c_4 = 8$	$c_6 = 13$	

Example 45 In the following arrangements of pennies, each row forms a contiguous block, and each penny above the bottom row touches two pennies in the row below it. Count such arrangements by the total number of pennies:

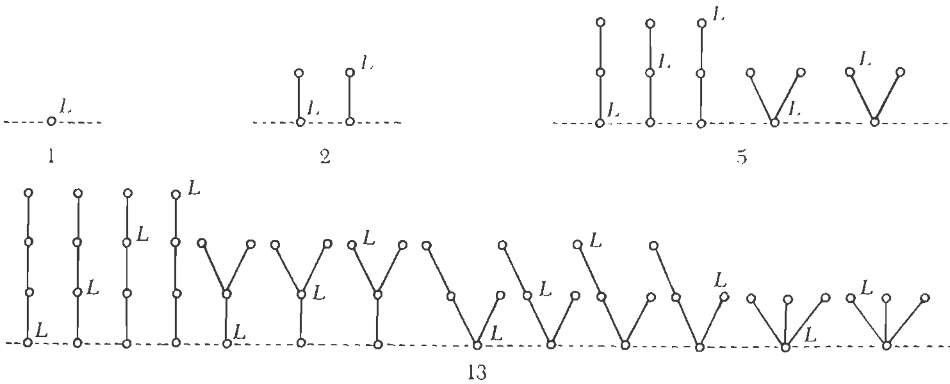


1524

Example 46 Alternatively, you could count the arrangements in the previous example by the number of pennies in the bottom row.

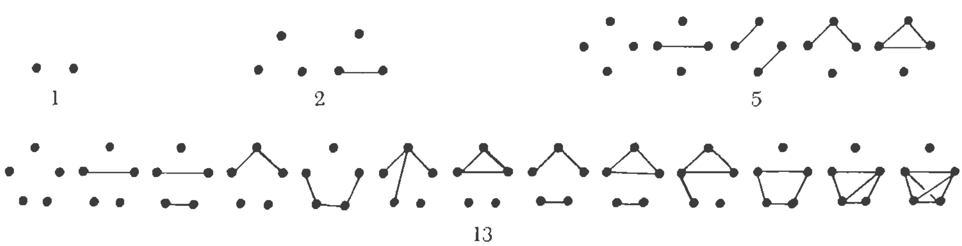


Example 47 The number of rooted trees with n vertices, just one of which is labelled.

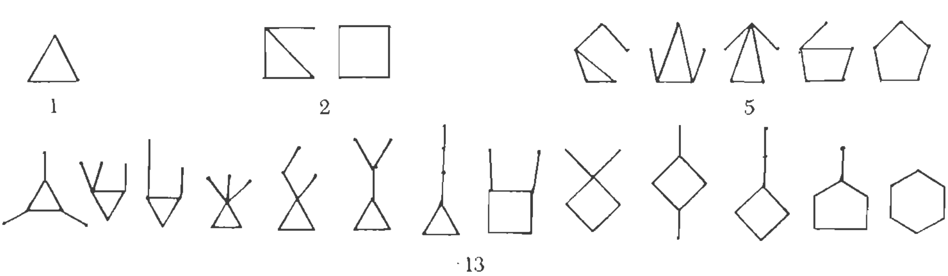


Example 48 The number of disconnected graphs with $n + 1$ vertices.

719



Example 49 The number of connected graphs on $n + 2$ vertices with just one cycle.



For many other examples involving graphs, see [22], which does not, however, include Examples 47–49.

Example 50 The coefficients in the power series solution

$$y = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{5x^5}{5!} + \frac{13x^6}{6!} + \cdots$$

of the differential equation $D^2 y = e^x y$.

Example 51 The sequence $a_n = a_{n-1} + na_{n-2}$ $(n \geqslant 1)$ with $a_{-1} = a_0 = 1/2$

$$a_1 = \frac{1}{2} + 1 \times \frac{1}{2} = 1$$

$$a_2 = 1 + 2 \times \frac{1}{2} = 2$$

$$a_3 = 2 + 3 \times 1 = 5$$

$$a_4 = 5 + 4 \times 2 = 13.$$

Example 52 The sequence $b_n = (n-1)2^{n-2} + 1$, $n \geqslant 1$.

$$b_1 = 0 \times 2^{-1} + 1 = 1$$

$$b_2 = 1 \times 2^0 + 1 = 2$$

$$b_3 = 2 \times 2^1 + 1 = 5$$

$$b_4 = 3 \times 2^2 + 1 = 13.$$

Example 53 How many distinct sums, $f(n)$, may there be of n different ordinal numbers? Obviously, $f(1) = 1$. However, $f(2) = 2$, because ordinal addition is not commutative. For example, $1 + \omega = \omega \neq \omega + 1$. You might guess that $f(3)$ could be as large as $3! = 6$, but in fact you can't have more than 5 distinct sums of 3 different ordinals. The answers

for $n =$	1	2	3	4	5	6	7	8...
are $f(n) =$	1	2	5	13	33	81	193	449...

perhaps the same sequence as Example 52. Or perhaps not.

Example 54 The values of the polynomial $9n^2 - 231n + 1523$ for $n = 0, 1, 2, \ldots$ are 1523, 1301, 1097, 911, 743, 593, 461, 347, 251, 173, 113, 71, 47, 41, 53, 83, 131, 197,.... Try also the polynomial $47n^2 - 1701n + 10181$.

Example 55 What are the next three terms in the sequence

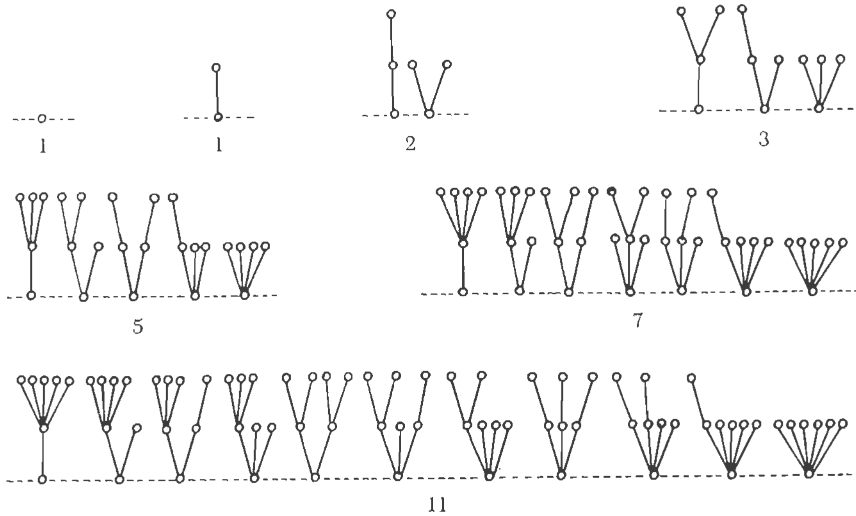
$$(1), 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, \ldots?$$

Example 56 The integer part of the nth power of 3/2

n	0	1	2	3	4	5	6
$(3/2)^n$	1	1.5	2.25	3.375	5.0625	7.59375	11.390625
	0	1	2	3	5	7	11

2379

Example 57 The number of trees with n edges, and height at most 2.



1402

Example 58 The number of partitions of n

$n =$	0	1	2	3	4	5	6	7	8	9...
$p(n) =$	1	1	2	3	5	7	11	15	22	30....

Example 59 If we form successive differences of the partition function:

1 1 2 3 5 7 11 15 22 30 42 56 77 101 135 176 231 297 385 490 627 ..
0 1 1 2 2 4 4 7 8 12 14 21 24 34 41 55 66 88 105 137 ...
1 0 1 0 2 0 3 1 4 3 7 3 10 7 14 11 22 17 32 ...
-1 1 -1 2 -2 3 -2 3 -2 5 -4 7 -3 7 -3 11 -5 15 ...

we see that the third-order differences alternate in sign.

Example 60 If you expand the product $(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$, you get, successively

$$1-x$$
$$1-x-x^2+x^3$$
$$1-x-x^2+x^4+x^5-x^6$$
$$1-x-x^2+2x^5-x^8-x^9+x^{10}$$

and a coefficient 2 has appeared. Indeed, at stage 10, a coefficient 3 appears. However, further calculation appears to cancel these out, leaving

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots.$$

Are there any coefficients other than 0, ± 1 in the final result?

Example 61 For each integer exponent, n, is there an integer $m > 1$ such that the sum of the decimal digits of m^n is equal to m?

$$2^1, 9^2 = 81, 8^3 = 512, 7^4 = 2401, 28^5 = 17210368, 18^6 = 34012224, 18^7 = 61222032,$$
$$46^8, 54^9, 82^{10}, 98^{11}, 108^{12}, 20^{13}, 91^{14}, 107^{15}, 133^{16}, 80^{17}, 172^{18}, 80^{19}, 90^{20}, 90^{21}, \ldots .$$

Example 62 A **Niven number** has been defined as one which is divisible by the sum of its decimal digits, such as 21 and 133. Is $n!$ always a Niven number?

$$4! = 24, 5! = 120, 6! = 720, 7! = 5040, 8! = 40320, 9! = 362880, 10! = 3628800, \ldots .$$ 5349

Example 63 Can you choose a sequence of real numbers from the interval $(0, 1)$ so that the first two lie in different halves, the first three in different thirds, the first four in different quarters, and so on? For example,

$$0.71, 0.09, 0.42, 0.85, 0.27, 0.54, 0.925, 0.17, 0.62, 0.355, 0.78, 0.03, 0.48, \ldots .$$

If you run into difficulty, you are allowed to adjust earlier members of the sequence, if you like.

Example 64 Surely every odd number (greater than 1, if you don't want to count 1 as a prime) is expressible as a prime plus twice a square?

$$3 + 2 \cdot 0^2, 3 + 2 \cdot 1^2, 5 + 2 \cdot 1^2, 7 + 2 \cdot 1^2, 3 + 2 \cdot 2^2, 11 + 2 \cdot 1^2,$$
$$7 + 2 \cdot 2^2, 17 + 2 \cdot 0^2, 11 + 2 \cdot 2^2, 3 + 2 \cdot 3^2, 5 + 2 \cdot 3^2, 23 + 2 \cdot 1^2, \ldots .$$

Indeed, some numbers, such as 61, have several such representations.

Example 65 Is $n!$ always expressible as the difference of two powers of 2?

$$0! = 1! = 2^1 - 2^0, 2! = 2^2 - 2^1, 3! = 2^3 - 2^1, 4! = 2^5 - 2^3, 5! = 2^7 - 2^3, \ldots .$$

Example 66 It's well known that $4! = 5^2 - 1$, $5! = 11^2 - 1$ and $7! = 71^2 - 1$, but not so well known that if you take the *next* square bigger than $n!$ the difference is always a square:

$$6! = 27^2 - 3^2, 8! = 201^2 - 9^2, 9! = 603^2 - 27^2,$$
$$10! = 1905^2 - 15^2, 11! = 6318^2 - 18^2, \ldots .$$

Example 67 The values of $\sin^2(k\pi/12)$, for $k = 0, 1, \ldots 6$ are

$k =$	0	1	2	3	4	5	6
$\sin^2(k\pi/12) =$	0	$(2-\sqrt{3})/4$	$1/4$	$1/2$	$3/4$	$(2+\sqrt{3})/4$	1

It's also well known that

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{2n(2n-2)\cdots 4 \cdot 2}\,\frac{\pi}{2}.$$

If you calculate the integral by the trapezoidal rule, using 6 equal subintervals, you will get the answer

$$\left\{2^{2n-1} + (2+\sqrt{3})^n + (2-\sqrt{3})^n + 3^n + 2^n + 1\right\}\pi/12\cdot 4^n,$$

which is exact for $n = 1, 2, 3, 4, 5, 6$ and 7.

Example 68 The continued fraction for π^2/e^γ is

$$\frac{\pi^2}{e^\gamma} = 5 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\cdots.$$

Example 69 Define a sequence by $P(1) = P(2) = 1$, and for $n > 2$, $P(n) = P(P(n-1)) + P(n - P(n-1))$. The first 32 terms are 1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 8, 8, 8, 8, 9, 10, 11, 12, 12, 13, 14, 14, 15, 15, 15, 16, 16, 16, 16, 16. Note that $P(2) = 1$, $P(4) = 2$, $P(8) = 4$, $P(16) = 8$, and $P(32) = 16$.

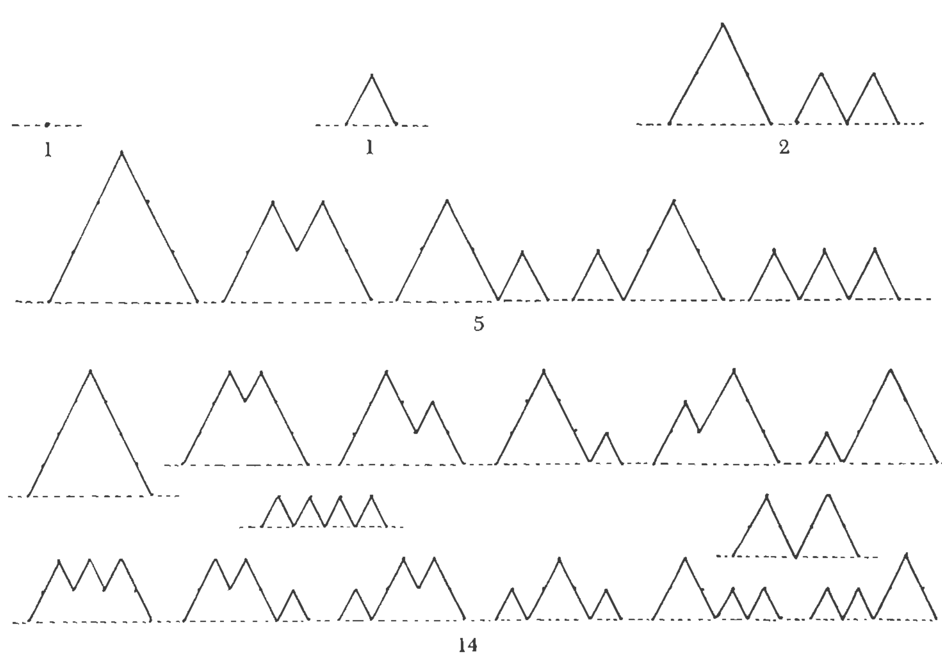
Example 70 A similar sequence starts with $Q(1) = Q(2) = Q(3) = 1$, and the same recurrence for $n > 3$, $Q(n) = Q(Q(n-1)) + Q(n - Q(n-1))$. The first 34 terms are 5350 1, 1, 1, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 7, 7, 8, 8, 8, 8, 8, 9, 10, 11, 11, 12, 12, 12, 13, 13, 13, 13, 13, 13. Notice that $Q(2) = 1$, $Q(3) = 1$, $Q(5) = 2$, $Q(8) = 3$, $Q(13) = 5$, $Q(21) = 8$ and $Q(34) = 13$.

Examples 40–52 and 70 perhaps contain manifestations of the Fibonacci numbers. Almost as ubiquitous are the **Catalan numbers,** $(2n)!/n!(n+1)!$,

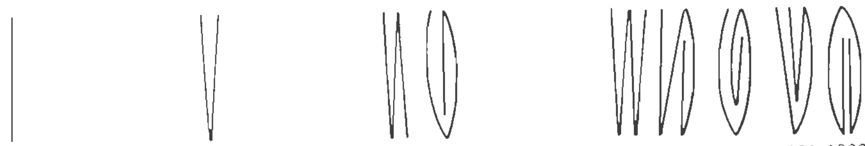
$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots.$$

How many of Examples 71 to 79 are genuine?

Example 71 The number of mountain ranges you can draw with n upstrokes and n downstrokes:

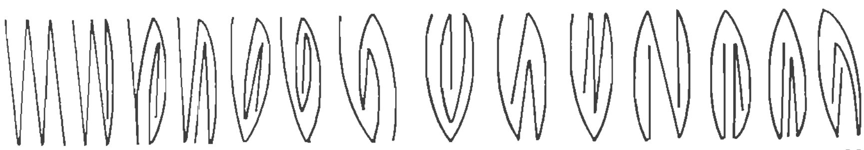


Example 72 The number of ways of making n folds in a strip of $n+1$ postage stamps, where we don't distinguish between front and back, top and bottom, or left and right:



0 01 012 021 0123 0132 0231 0321 1032

1 1 2 5



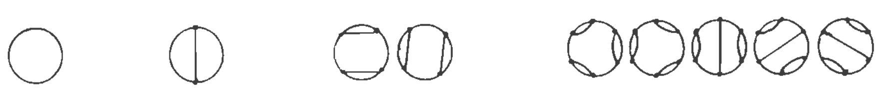
01234 01243 01342 01432 02341 02431 03214 03421 04123 04321 10243 10342 10432 14032

14

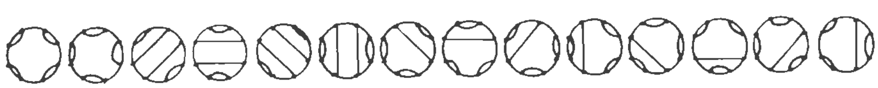
Example 73 The number of different groups, up to isomorphism, of order 2^n is,

for $n =$	0	1	2	3	4...
no. of groups =	1	1	2	5	14....

Example 74 The number of ways $2n$ people at a round table can shake hands in pairs without their hands crossing.

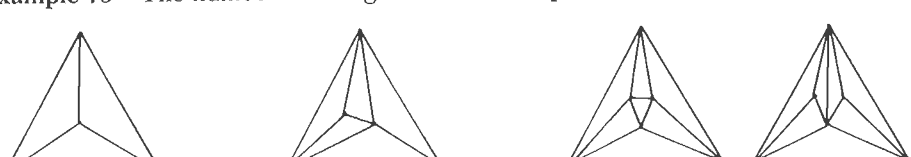


1 1 2 5

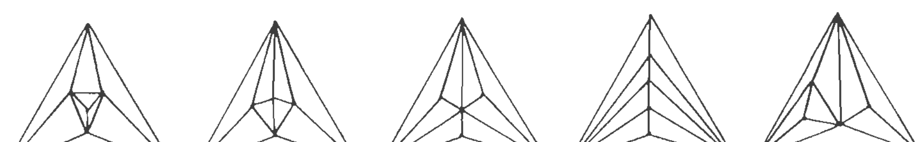


14

Example 75 The number of triangulations of the sphere with $n+4$ points.



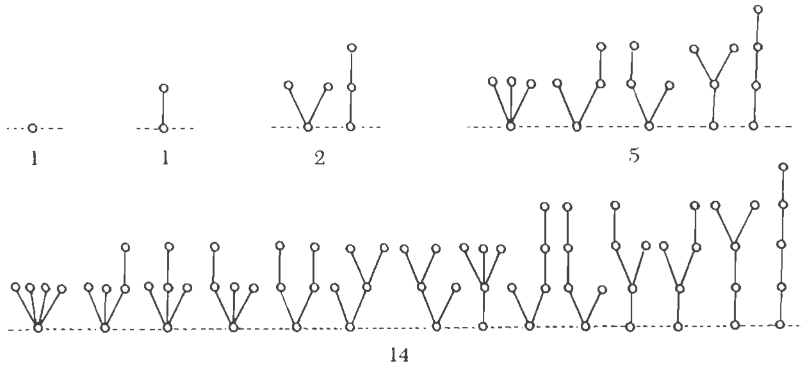
1 1 2



5

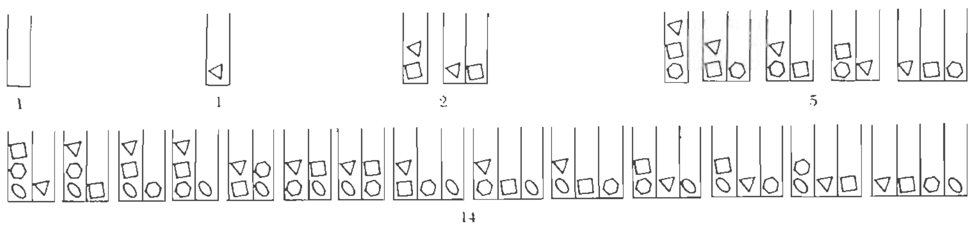
We leave the reader to verify that there are just 14 distinct triangulations of the sphere with 8 points.

Example 76 The number of rooted plane trees with n edges. "Plane" means that we distinguish between left and right.



Example 77 The number of ways of distributing n different objects in indistinguishable boxes, with at most three objects in a box.

1680



Example 78 The probability, for $n = 0, 1, 2, \ldots,$ that a function $f(k)$, with domain $k = 0, 1, \ldots, n$ and range $[0,1]$, is convex, is

$$\frac{1}{(0!)^2}, \frac{1}{(1!)^2}, \frac{2}{(2!)^2}, \frac{5}{(3!)^2}, \frac{14}{(4!)^2}, \ldots.$$

Example 79 The incomplete Bessel function of order one has power series expansion

$$I_1(2x)/x = 1 + \frac{x^2}{2!} + \frac{2x^2}{4!} + \frac{5x^6}{6!} + \frac{14x^8}{8!} + \cdots.$$

Example 80 Examples 37 to 40 involved Pascal's triangle, whose entries are the binomial coefficients. We can use a similar array to expand $(1 + x + x^2)^n$, giving **trinomial coefficients**. Each entry is the sum of the three nearest in the previous row.

									1									
								1	1	1								
							1	2	3	2	1							
						1	3	6	7	6	3	1						
					1	4	10	16	19	16	10	4	1					
				1	5	15	30	45	51	45	30	15	5	1				
			1	6	21	50	90	126	141	126	90	50	21	6	1			
		1	7	28	77	161	266	357	393	357	266	161	77	28	7	1		
	1	8	36	112	266	504	784	1016	1107	1016	784	504	266	112	36	8	1	
1	9	45	156	414	882	1554	2304	2907	3139	2907	2304	1554	882	414	156	45	9	1

The central trinomial coefficient, a_n, 2426

$$1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, \ldots$$

almost trebles in size at each step: if we calculate $3a_n - a_{n+1}$ we get

$$2, 0, 2, 2, 6, 12, 30, 72, 182, \ldots$$

which are **pronic numbers**, $m(m+1)$, for $m = 1, 0, 1, 1, 2, 3, 5, 8, 13, \ldots$.

Answers

36. This is the polynomial $\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}$ of Example 5 of [**21**], and represents the number of pieces you can cut a circular cake into by slicing between every pair of points chosen from n around the circumference. It is also the number of regions that 4-dimensional space is chopped into by $n-1$ hyperplanes in general position. The sequence is #427 in [**45**]: 1, 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, 794, 1093, 1471, ... 127
37. This very beautiful setting for Example 1 of [**21**] was observed 20 years ago by William Watkins, now co-editor of *Coll. Math. J.* Gauss has told us that the number of sides in a regular polygon which can be constructed with straightedge and compass is of shape $2^m \prod F_n$, where the F_n are distinct Fermat primes $2^{2^n} + 1$. Only five such, $0 \leqslant n \leqslant 4$, are known and some people believe that no others will ever be found. So the pattern breaks down at row 32. Fermat thought that $2^{32} + 1$ was prime, but Euler discovered the factorization 641×6700417. 3401
38. This is the Mann-Shanks primality test [**36**]. Surprising, if not practical. Can you prove it?
39. This is an observation of Gerry Myerson: that the bold numbers are the composite numbers. However, this breaks down in row 13, because $\binom{13}{5} = 3^2 11 \cdot 13$ and $\binom{13}{6} = 2^2 3 \cdot 11 \cdot 13$ are not squarefree.
40. This well-known relation between Pascal's triangle and Fibonacci numbers is easily seen to persist, since each entry is the sum of the entries in the previous two columns of the previous row, so each total is the sum of the two previous totals.
41. This is adapted from an inequality of Larry Hoehn, of Clarksville TN. The coincidence is quite surprising, since $\sqrt{e} \approx 1.64872$ and the golden ratio $(1 + \sqrt{5})/2 \approx 1.61803$ are not remarkably close. For $n = 10, 11, 12, \ldots$ the terms $91, 149, 245, \ldots$ begin to diverge from the Fibonacci sequence $89, 144, 233, \ldots$.
42. In [**32**], Richard Laatsch shows that the sequence continues $55, 89, 142, 230, \ldots$ with differences

    20 30 53 88 143 236 387 641 1061 1763 2737 4903 8202 13750 23095... 5347

    which stay close to the Fibonacci numbers

    21 34 55 89 144 233 377 610 987 1597 2584 4181 6765 10946 17711...

    for awhile, but eventually tend to infinity more rapidly.
43. See sequence #912 in [**45**]. This is still a notorious open question: there are extensive tables [**30, 34, 49, 50**]. During revision of this article, Dick Lehmer kindly ran a program on a 75 Vax, and found no counterexample with p less than a million.
44. The sequence continues $17, 26, 34, 45, 54, 67, \ldots$ and is denser than the Fibonacci sequence. It is #254 in [**45**], but the reference there is misleading. The sequence doesn't solve *Amer. Math. Monthly* problem E1910 [1966, 775; partial solution 1149

1968, 80–81] because the differences are not unique: e.g., $17-8=26-17=54-45$. Nor is it the auxiliary sequence $\{r_n\}=\{4, 5, 9, 10, 11, 16, 18, 22, 23, 24, 25, 27, 28, 29,\dots\}$, used to construct the Sierpiński sequence, #425 in [45]. There's another open question here: find the smallest possible asymptotic growth for a sequence of integers such that every positive integer occurs *uniquely* as a difference. 1856

45. This also fails to continue with the Fibonacci sequence. The numbers of arrangements with $7, 8, 9,\dots$ pennies are $12, 18, 26,\dots$. These arrangements were studied by Auluck [2]; see sequence #253 in [45], and compare Example 34 in [21]. 1524
46. These are indeed the odd-ranking Fibonacci numbers, u_{2n-1}, sequence #569 in [45], which have the property 1519

$$u_{2n-1}=u_{2n-3}+2u_{2n-5}+3u_{2n-7}+\cdots+(n-1)u_1+1$$

which can be seen to be the number of ways that a row of n pennies may be surmounted by an arrangement with $n-k$ in its bottom row, in any one of k possible positions, where $k=1,2,\dots,n-1$ or it's not surmounted at all $(k=n)$.
47. These are *not* the alternate Fibonacci numbers, e.g., the numbers of such trees with $5, 6, 7,\dots$ vertices are $35, 95, 262,\dots$ See sequence #570 in[45] or p. 134 in [43]. 107
48. Nor are these. The next few members of the sequence are 44, 191, 1229, 13588, 288597,.... See sequence #574 in [45], or [24]. 719
49. Neither is this the sequence of alternate Fibonacci numbers, but continues 33 (one short!), 89 (correct!), 240 (7 too many), 657, 1806, 5026,.... See sequence #568 in [45] or page 150 in [43]. 1429
50. Nor is this, which continues $36, 109, 359, 1266, 4731, 18657, 77464,\dots$; see sequence #572 in [45] and Tauber's paper [48]. 994
51. Nor again, since $a_5=13+5\times 5=38$, $a_6=116$, $a_7=382,\dots$; see sequence #573 in [45]. 148
52. Neither are these, $b_5=4\cdot 2^3+1=33$, $b_6=5\cdot 2^4+1=81$, $b_7=6\cdot 2^5+1=193$, $b_8=7\cdot 2^6+1=449$, alternate Fibonacci numbers, but they *do* feature (for a while) in the next Example: 5183
53. which I got from John Conway. If $g(k)=k\cdot 2^{k-1}+1$, then

$$f(n)=\max_{0<k<n} f(n-k)g(k),$$

and, for $n\leqslant 8$, $f(n)$ is indeed equal to $g(n-1)$. Thereafter the situation gets more complicated, but a simple rule eventually emerges: for $n=9, 10, 11, 12, 13$, $f(n)=33^2, 33\cdot 81, 81^2, 81\cdot 193, 193^2$, and, for $n\geqslant 14$, $f(n)=81f(n-5)$, except that $f(19)=193^3$. 5348
54. Several readers of [21] said that I should have included Euler's famous formula, n^2+n+41, which gives primes for $0\leqslant n\leqslant 39$, not noticing that Example 21 was just that, except for the disguise of omitting the tell-tale 41 $(n=0)$. For some astonishing examples of The Strong Law in this connection, see the papers of Stark [46, 47]. The present polynomial is a slight adaptation of one due to Sidney Kravitz, and is found by replacing n in Euler's formula by $38-3n$. Surprisingly, this still gives primes for $0\leqslant n\leqslant 39$, although thirteen of them are not among the original forty; $n=40$ and 41 give $6683=41\times 163$ and $7181=43\times 167$.

The polynomial $47n^2-1701n+10181$ was discovered recently by Gilbert Fung. If you work modulo p for primes $2\leqslant p\leqslant 43$, you'll find that it's never divisible by such primes. It takes prime values for $0\leqslant n\leqslant 42$, beating Euler's record by two. Notice that the discriminant of Euler's polynomial is -163, and that of Kravitz is $-3^2 \times 163$, while Fung's polynomial has to have a *positive* discriminant, 979373.

55. Such questions are hardly fair, since arguments can be advanced for continuing sequences in any way you wish. Some answers are more plausible than others, however, and the one that Persi Diaconis hoped you would miss is $59, 60, 61, \ldots$, the orders of the simple groups!
56. Another futile attempt to fool you into thinking of the primes. The next member is 17, then $25, 38, \ldots$; see sequence #245 in [45].
57. This is not the same sequence as the previous example, but see the next!
58. To see the correspondence between this and the previous example, note that the number of vertices at height one is the number of parts, and the valences of these vertices are the sizes of the parts. Sequence #244 in [45]; see also page 122 in [43] and page 836 in [1]. 1402
59. This example was sent by Gerry Myerson. It can be proved that the differences of any order are positive from some point on, but that point recedes rather rapidly as you take higher order differences. The next few third differences are $-4, 17, -2, 24, -4, 32, 1, 38, 5, \ldots$ and are positive from now on. The fourth differences alternate in sign until the 67th, after which they are positive.
60. This is Euler's famous pentagonal numbers theorem:

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

See theorem 353 in [26], for example.

61. Norman Megill of Waltham, MA, finds such m for each $n \leqslant 104$. For $n = 105$, however, no such m exists.
62. This question was asked by Sam Yates. Carl Pomerance suggested that counterexamples might be expected by the time n has reached 500, and indeed Yates found that 432! is not a Niven number, since the sum of its digits is $3^2 \times 433$, and 433 is prime.
63. The given sequence can be continued, $0.97, 0.22, 0.66, 0.32$, but Berlekamp and Graham [3] have shown that no such sequence exists with more than 17 members!
64. This special case of the Hardy-Littlewood problem was mentioned by Ron Ruemmler of Edison, NJ, who believes that the first exception is 5777, and asks if it is also the last! It is known from the work of Hooley [27], Miech [37], and Polyakov [42] that the density of exceptions is zero.
65. Ignace Kolodner got this from Harold N. Shapiro in an NYU Problem Seminar in 1949. It's left to the reader to prove that $n!$ is never again the difference of two powers of two.
66. This was observed by Larry Hoehn of Clarksville, TN. It fails for 12!, but $13! = 78912^2 - 288^2$, $14! = 295260^2 - 420^2$, $15! = 1143536^2 - 464^2$, $16! = 4574144^2 - 1856^2$. It's doubtful if this often occurs from here on (note that you must take the *next* square bigger than $n!$), but it may be hard to prove anything.
67. This is also correct for $n = 8, 9, 10$, and 11, but for $n = 12$ we get $(1352079\pi)/2^{24}$ instead of $(1352078\pi)/2^{24}$, out by 3 parts in four million! The trapezoidal rule gives the right answer if you use k subintervals, provided $2n$ is less than $4k$: see [28], for example. David Bloom suggested that "four million" should read "sixteen million": I intended the *relative* error, $\approx 2.958/4000000$: the *actual* error is $\approx 2.996/16000000$: more examples of the Strong Law!
68. If this pattern, noticed by James Conlan [8], were to continue, we would have $(5 + \sqrt{37})e^{\gamma} = 2\pi^2$. Close, but no cigar!

69. The sequence that hit the national presses on both sides of the Atlantic, e.g. [6], publicizing the Conway-Mallows encounter. I have an earlier manuscript of Conway in which he has written (in another notation) "$P(2^k) = 2^{k-1}$ (easy), $P(2n) \leqslant 2P(n)$ (hard), $P(n)/n \to \frac{1}{2}$ (harder)." It was the proof of a precise form of this last statement that almost won Mallows even more money than Conway intended. Papers mentioning this sequence include [16, 35].
70. Yes, the Fibonacci pattern continues [40]. David Newman showed this to David Bloom as a conjecture in 1986.

Nine of the final ten examples are intended to look like the Catalan numbers; sequence #577 in [45]. At first it is a matter of some surprise that

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

is always an integer. In connection with some recent correspondence [41], John Conway makes the more general observation that

$$\frac{(m,n)(m+n-1)!}{m!n!}$$

is an integer, where (m,n) is the g.c.d. of m and n, because

$$\frac{m(m+n-1)!}{m!n!} = \binom{m+n-1}{m-1} \quad \text{and} \quad \frac{n(m+n-1)!}{m!n!} = \binom{m+n-1}{n-1}$$

are both integers. This also answers a question in B33 of [20], where Neil Sloane gave the example $n = 4m+3$.

Catalan numbers occur in many widely different looking contexts: see [18], with nearly 500 references, and [31], with a list of 31 structures, both obtainable from H. W. Gould, Department of Mathematics, West Virginia University, Morgantown, WV, 26506. An article with a good bibliography is [5]. Several "proofs without words," showing the equivalence of several of the structures, will appear in [9].

71. This is a genuine example of the Catalan numbers. The mountain ranges are the same as paths from $(0,0)$ to (n,n) which do not cross $y = x$, or incoming tied ballots in which one candidate is never behind, or sequences of zeros and ones, or of ± 1s, subject to appropriate sum conditions, e.g., random one-dimensional walks in which you never go to the left of the origin; see [13].
72. This sequence, #576 in [45], is not, and continues 39 (not 38, as stated in [14]), 120, 358, 1176, 3527, 11622, 36627, 121622, 389560, ..., see [29].
73. The numbers of groups of orders 2^5 and 2^6 are 51 and 267 [23]. This sequence, #581 in [45], continues 2328 [51], 56092 [52]. 679
74. This is genuine Catalan again: see [39].
75. But this one, sequence #580 in [45], has been calculated for only four more terms [4, 12, 19]. Of the 109

$$1, 1, 2, 5, 14, 50, 233, 1249, 7595 \text{ triangulations,}$$

only $0, 0, 1, 1, 2, 5, 12, 34, 130$ contain no vertex of valence 3.
76. is a genuine manifestation of the Catalan numbers [7, 25], but
77. is not: sequence #579 in [45] continues 46, 166, 652, 2780, 12644, 61136, 312676, 1680592, ... [38].
78. The probability for general n is indeed $c_n/(n!)^2$ [10].
79. In [10] we asked what was the exponential generating function for the Catalan

numbers. Louis W. Shapiro observes that

$$\sum_{n=0}^{\infty} c_n \frac{x^{2n}}{(2n)!} = I_1(2x)/x$$

where I_1 is the modified Bessel function of order one: see formula 9.6.10. on page 375 of [1]. In the paper [44] he obtains results for lattice paths which stay below given points, arranged with increasing abscissas and ordinates, somewhat analogous to the convex functions of [10].

Before we say goodbye to the Catalan numbers, here's an observation which may not be widely known. It originated in a discussion with John Conway only six months ago. What *is* well known is that the Catalan numbers are associated with parenthesization. By that most people mean the numbers of possible orders of n nonassociative operations, usually indicated by $n-1$ pairs of parentheses:

$n=0$	a	$n=1$ ab		$n=2$ $(ab)c$ or	$a(bc)$
$n=3$	$((ab)c)d$	$(a(bc))d$	$a((bc)d)$	$a(b(cd))$	$(ab)(cd)$
$n=4$	$(((ab)c)d)e$	$((a(bc))d)e$	$(a((bc)d))e$	$(a(b(cd)))e$	$((ab)(cd))e$
	$((ab)c)(de)$	$(a(bc))(de)$	$(ab)((cd)e)$	$(ab)(c(de))$	$a(((bc)d)e)$
	$a((b(cd))e)$	$a((bc)(de))$	$a(b((cd)e))$	$a(b(c(de)))$	

and so on. But they are also the numbers of ways of arranging n pairs of parentheses as a pattern, just for their own sake:

$n=0$ $n=1$ ( ) $n=2$ (( )) or ( )( )

$n=3$ ((( ))) (( ))( ) ( )(( )) (( )( )) ( )( )( )

$n=4$ (((( )))) ((( )))( ) ( )((( ))) ((( )( ))) ((( ))( )) (( )(( ))) (( ))(( ))

(( )( ))( ) ( )(( )( )) (( ))( )( ) ( )(( ))( ) ( )( )(( )) (( )( )( )) ( )( )( )( )

An examination of the symmetries in the two cases makes it unlikely that you'll find a direct combinatorial comparison. One-one correspondences between the former manifestation and other Catalan manifestations are well known. The latter are easily seen to be in correspondence with the pairs of people shaking hands in Example 73, and with the mountains in Example 70.

80. Jack Good [17] has given an asymptotic formula for the central trinomial coefficient:

$$a_n \sim \frac{3^{n+\frac{1}{2}}}{2\sqrt{\pi n}}\left\{1 - \frac{3}{16n} + \frac{1}{512n^2} + O(n^{-3})\right\}$$

which shows that the left side of the "identity"

$$¿\quad 3a_n - a_{n+1} = u_{n-1}(u_{n-1}+1) \quad ?$$

grows like $c \times 3^n \times n^{-3/2}$, whereas the right side grows like $\tau^{2n}/5$, where τ is the golden ratio, $\tau^2 = (3+\sqrt{5})/2$. Further calculation shows that $a_{10} = 8953$, $3a_9 - a_{10} = 464$, while $u_8(u_8+1) = 21 \times 22 = 462$. The asymptotic formula is good to the nearest integer for quite large values of n.

This example was sent by Donald Knuth. Euler [11] was one of the earlier discoverers of The Strong Law of Small Numbers, and called this

exemplum memorabile inductionis fallacis.

On the same page he gives the Fibonacci formula that's often attributed to Binet.

Coda I showed this example to George Andrews during the recent Bateman Retirement Conference at Allerton Park, Illinois. Half-an-hour later he came back with what Euler really should have said. He defines the trinomial coefficients centrally by

$$(1+x+x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{n+j}$$

and proves that, if F_n is the nth Fibonacci number, then

$$F_n(F_n+1) = 2\sum_{\lambda=-\infty}^{\infty}\left(\binom{n}{10\lambda+1}_2 - \binom{n}{10\lambda+2}_2\right).$$

For $-1 \leqslant n \leqslant 7$, the only nonzero term on the right is $\lambda = 0$, which accounts for Euler's observation, since

$$3\binom{n}{0}_2 - \binom{n+1}{0}_2 = 2\binom{n}{0}_2 - 2\binom{n}{1}_2.$$

Andrews will publish the q-analog of this theorem shortly.

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