Arctanh(z) and the Legendre polynomials

Peter Bala, March 19 2024

Gauss's continued fraction for the function arctanh(z) is

z/(1 - 1^2*z^2/(3 - 2^2*z^2/(5 - 3^2*z^2/(7 - ...)))) ... **(1)**

valid for complex z not in either of the intervals (-oo, -1] or [1, oo).

In this note we find expressions in terms of Legendre polynomials for both the numerator and denominator polynomials of the n-th convergent of Gauss's continued fraction.

This allows us to give rapidly converging series for some well-known constants.

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We begin by replacing z with 1/z in (1) and then making use of equivalence transformations to obtain the continued fraction representation

arctanh(1/z) =

1/(z - 1/(3*z - 2^2/(5*z - 3^2/(7*z - ... - (n - 1)^2/((2*n - 1)*z - ...))))) ... **(2)**

valid for complex z not in the closed interval [-1, 1].

Let N(n,z)/D(n,z) denote the n-th convergent to the continued fraction (2):

N(n, z)/D(n, z) = 1/(z - 1/(3*z - 2^2/(5*z - 3^2/(7*z - ... - (n - 1)^2/((2*n - 1)*z))))).

The first four convergents (numbered 1 through 4) are

1/z, 3*z/(3*z^2 - 1), z*(4*z^2 - 15)/(3*(3*z^2 - 5)) and

5*z*(21 - 11*z^2)/(3*(3*z^4 - 30*z^2 + 35)).

By the elementary theory of continued fractions, both the sequence of numerator polynomials {N(n, z)} and the sequence of denominator polynomials {D(n, z)} satisfy the 3-term recurrence

u(n, z) = (2*n – 1)*z*u(n-1, z) - (n – 1)^2*u(n-2, z) ... **(3)**

for n >= 3, with the initial values

N(1,z) = 1, N(2, z) = 3*z

and

D(1, z) = z, D(2, z) = 3*z^2 – 1.

The following theorem gives explicit expressions for the polynomials N(n, z) and D(n, z) in terms of Legendre polynomials.

Theorem. *Let P(n,z) denote the n-th Legendre polynomial. Then*

(i) *D(n, z) = n!*P(n, z)*

(ii) *N(n, z) = D(n, z) * Sum_{k = 1..n} 1/(k*P(k-1, z)*P(k, z))*

Proof.

The Legendre polynomials satisfy the 3-term recurrence

n*P(n, z) = (2*n – 1)*z*P(n-1, z) – (n – 1)*P(n-2, z) ... **(4)**

with P(1, z) = z and P(2, z) = (3*z^2 – 1)/2. Thus (i) holds for n = 1 and n = 2.

Multiplying (4) by (n - 1)! we see that the polynomial sequence {n!*P(n, z)} satisfies the same recurrence (3)

u(n, z) = (2*n – 1)*z*u(n-1, z) - (n – 1)^2*u(n-2, z)

satisfied by the denominator polynomials D(n, z), and with the same initial conditions.

Thus the polynomial sequences {D(n, z)} and {n!*P(n, z)} are identical, completing the proof of (i).

(ii) Define

$$A(n, z) = D(n, z) \cdot \sum_{k=1}^{n} \frac{1}{k \cdot P(k-1, z) \cdot P(k, z)} \qquad \ldots \textbf{(5)}$$

We calculate the initial values

$$A(1, z) = 1 = N(1, z)$$

and

$$A(2, z) = 3 \cdot z = N(2, z).$$

We show that the sequence $\{A(n, z)\}$ also satisfies the 3-term recurrence (3) satisfied by the sequence $\{N(n, z\}\}$, hence proving that $A(n, z) = N(n, z)$ for all n.

From (5),

$$\begin{aligned} A(n+1, z) &= D(n+1, z) \cdot \sum_{k=1}^{n+1} \frac{1}{k \cdot P(k-1, z) \cdot P(k, z)} \\ &= D(n+1, z) \cdot \sum_{k=1}^{n} \frac{1}{k \cdot P(k-1, z) \cdot P(k, z)} \\ &\quad + D(n+1, z)/((n + 1) \cdot P(n, z) \cdot P(n+1, z)) \\ &= (D(n+1, z)/D(n, z)) \cdot A(n, z) \\ &\quad + D(n+1, z)/((n + 1) \cdot P(n, z) \cdot P(n+1, z)). \end{aligned}$$

Substituting the value $D(n, z) = n! \cdot P(n, z)$ from part (i) and multiplying both sides of the resulting identity by $P(n, z)$ we find that

$$P(n, z) \cdot A(n+1, z) = (n + 1) \cdot P(n+1, z) \cdot A(n, z) + n!. \qquad \ldots \textbf{(6)}$$

Hence

$$P(n+1, z) \cdot A(n+2, z) = (n + 2) \cdot P(n+2, z) \cdot A(n+1, z) + (n + 1)! \qquad \ldots \textbf{(7)}$$

Multiply (6) by n + 1, subtract the result from (7) and then replace n with n - 2. Making use of the recurrence equation (4) for the Legendre polynomials we find after a short calculation that $A(n, z)$ satisfies the same 3-term recurrence (3)

$$A(n, z) = (2 \cdot n - 1) \cdot z \cdot A(n-1, z) - (n - 1)^2 \cdot A(n-2, z)$$

satisfied by the numerator polynomials N(n, z), completing the proof of part (ii).

Corollary 1.

arctanh(1/z) = lim_{n -> oo} N(n, z)/D(n, z)

*= Sum_{k > = 1} 1/(k*P(k, z)*P(k-1, z))*

valid for complex z not in the closed interval [-1, 1].

This result allows us to give rapidly converging series for values of some well-known constants, for example,

i*atanh(1/i) = Pi/4 = Sum_{n >= 1} i/(n*P(n, i)*P(n-1, i)),

2*arctanh(1/2) = log(3) = 2*Sum_{n >= 1} 1/(n*P(n, 2)*P(n-1, 2))

and

2*arctanh(1/3) = log(2) = 2*Sum_{n >= 1} 1/(n*P(n, 3)*P(n-1, 3)).

The last result is due to Burnside.

Corollary 2.

The n-th convergent of Gauss's continued fraction (1)

*z/(1 – 1^2*z^2/(3 – 2^2*z^2/(5 – ... (n – 1)^2*z^2/(2*n-1))))*

is equal to N(n, 1/z)/D(n, 1/z).

The finite continued fraction has a Taylor expansion around z = 0 equal to

z + z^3/3 + z^5/5 + z^7/7 + ... + z^(2*n-1)/(2*n – 1) + O(z^(2*n+1)).

Thus the rational function N(n, 1/z)/D(n, 1/z) is a Padé approximant to arctanh(z): more precisely, N(2*n+1, 1/z)/D(2*n+1, 1/z) is the [2*n+1, 2*n] Padé approximant to arctanh(z) and N(2*n, 1/z)/D(2*n, 1/z) is the [2*n-1, 2*n] Padé approximant to arctanh(z).