J. RECREATIONAL MATHEMATICS, Vol. 22(2) 124-132, 1990

CLASSIFICATION AND ENUMERATION OF PALINDROMIC SQUARES

MICHAEL KEITH
14 Quail Drive
Holland, Pennsylvania 18966

Introduction

In his interesting article, Rudolf Ondrejka presented a list of the first 151 integers whose squares, in base 10, are palindromes [1]. In this article, this list is extended by exhibiting the next forty-two palindromic squares (in Tables 1 and 2), which completes the enumeration of all 193 such numbers less than 10^{12}. Ashbacher, in the article immediately following this, also independently extended Ondrejka's list to include fifty-nine new entries beyond 10^{12} and a larger sporadic member shown as the thirty-fifth entry in Table 1. However, I also describe, in this article, a means of classifying all palindromic squares into six classes: the four trivial solutions; four infinite families (which account for the majority of such numbers); and "sporadic" solutions which do not fit into any family. This is a useful classification due to the fact that most palindromic squares belong to one of the four infinite families, which I characterize (almost) completely below. For example, of the 253 palindromic squares listed in Onjrejka's article, here, and in Ashbacher's article which follows, only thrity-five are sporadic (see Table 1). In a sense, the sporadic solutions are the only really "interesting" ones, although one of the four families contains an unsolved problem.

This classification scheme for palindromic squares is loosely analogous to the classification of the finite simple groups (see [2], for example), in which every finite simple group is either a member of one of seventeen infinite families or is a sporadic solution. However, unlike the sporadic finite simple groups, which number exactly twenty-six, I suspect that the number of sporadic palindromic squares is infinite.

Table 1. Complete List of Sporadic Palindromic Squares with Roots $< 2 \times 10^{12}$

Root (R)	*Square ($S = R^2$)*
26	676
264	69696
307	94249
836	698896
2285	5221225
2636	6948496
22865	522808225
24846	617323716
30693	942060249
798644	637832238736
1042151	1086078706801
1270869	1615108015161
2012748	4051154511504
2294675	5265533355625
3069307	9420645460249
11129631	123862676268321
12028229	144678292876441
12866669	165551171155561
30001253	900075181570009
64030648	4099923883299904
306930693	94206450305460249
2062386218	4253436912196343524
2481623254	6158453974793548516
10106064399	102132537636735231201
10207355549	104190107303701091401
13579355059	184398883818388893481*
22865150135	522815090696090518225*
30101273647	906086675171576680609*
30693069807	942064503484305460249*
83163115486	6916103777337773016196**
101116809851	10224609234443290642201*
111283619361	12384043938083934048321*
112247658961	12599536942224963599521*
128817084669	16593841302620314839561*
1349465117841	1821056104269624016501281***

* Found by the Author in 1989.
** Found by Graham Lyons in 1984.
*** Found by Charles Ashbacher in late 1989.

Table 2. The Non-Sporadic Palindromic Squares with Roots from 11101010111 to $< 10^{12}$

Root (R)	*Square ($S = R^2$)*
11101010111	123232425484524232321
11101110111	123234645696546432321
11110001111	123432124686421234321
11110101111	123434346696643434321
20000000002	400000000080000000004
20000100002	400004000090000400004
100000000001	10000000000200000000001
100001100001	10000220001410002200001
100010010001	10002002100400120020001
100011110001	10002222123632122220001
100100001001	10020010200400201002001
100101101001	10020230421612403202001
100110011001	10022014302620341022001
100111111001	10022234545854543222001
101000000101	10201000020402000010201
101001100101	10201222221612222210201
101010010101	10203022140604122030201
101011110101	10203244363836344230201
101100001101	10221210222622201212201
101101101101	10221432643834623412201
101110011101	10223234344844343232201
110000000011	12100000002420000000121
110001100011	12100242003630024200121
110010010011	12102202302620320220121
110011110011	12102444325852344420121
110100001011	12122010222622201022121
110101101011	12122252443834425222121
110110011011	12124214524842541242121
111000000111	12321000024642000012321
111001100111	12321244225852244212321
111010010111	12323222344844322232321
111100001111	12343210246864201234321
200000000002	40000000000800000000004

In what follows, the terms *root* and *square* refer to the number R and its square S under discussion. The symbol N is used for the number of digits in the number R.

The Trivial (Tr) Solutions

There are but four trivial solutions. The roots *and* the squares are single digits:

R	S
0	0
1	1
2	4
3	9

The Four Families

To illustrate the four families, here is a list of palindromic squares with 6- or 7-digit roots:

Root	*Square*	*Type*
100001	10000200001	B
101101	10221412201	B
110011	12102420121	B
111111	12345654321	B
200002	40000800004	E
798644	637832238736	S
1000001	1000002000001	T
1001001	1002003002001	T
1002001	1004006004001	T
1010101	1020304030201	T
1012101	1024348434201	T
1042151	1086078706801	S
1100011	1210024200121	T
1101011	1212225222121	T
1102011	1214428244121	T
1109111	1230127210321	A
1110111	1232346432321	T
1111111	1234567654321	T
1270869	1615108015161	S
2000002	4000008000004	E
2001002	4004009004004	E
2012748	4051154511504	S
2294675	5265533355625	S
3069307	9420645460249	S

The *Type* designation identifies each one as a member of one of the four families (B, T, E, or A) or as a sporadic solution (S). The four families of palindromic squares will now be defined and described in detail.

Some further notation is necessary. In the following, [d], where d is a digit, represents a string of zero or more d's. The symbol (d) represents a string of one or more d's. In either case, the use of an "x" instead of a digit represents a string of arbitrary (and not necessarily all the same) digits. A primed string, such as [x]', represents the same string as [x] but with the order of the digits reversed.

The Binary Root (B) Family

This family is defined for all even $N \geqslant 2$. R is of the form $R = 1\,[x]\,[x]'1$, where all x's - 0 or 1. The family name is due to the fact that all digits of the root R are either 0 or 1. Note that R, as well as S, is a palindrome.

At first glance (in the above table for $N = 6$, for instance), it would appear that all combinations of 0's and 1's are possible for the string [x]. If this were so, the number of elements of the B family for $N = 2, 4, 6, \ldots$, would be 1, 2, 4, 8, 16, 32, etc. However, a computer search reveals that in fact the number of elements in the B family is 1, 2, 4, 8, 15, 26, etc. What is going on here?

To understand this, consider what happens when we square R with all x's equal to zero. The multiplication looks like:

```
            10....01
            10....01
           ----------
            10....01
          00....00
        00....00
            etc.
    00....00
  10....01
  ------------------
  10....020....01
```

S is, of course, a palindrome. Because R is palindromic, S will remain palindromic if we change 0's to 1's in R subject to the condition that no carries are generated when adding up the partial products. Now, note that each 0 we change to a 1 in [x] will cause an increase of 2 in the middle column. Therefore, we can change *at most three* 0's or 1's (increasing the middle digit of the product from 2 to 8). It is easy to verify that this necessary condition is also sufficient; therefore, the members of the B family are given by

$$R = 1\,[x]\,[x]'1, \quad \text{all x's} = 0 \text{ or } 1, \text{with at most three equal to } 1.$$

The total number of B elements for a given value of N is thus equal to the number of binary strings of length $(N - 2)/2$ with at most three ones. This can be calculated by summing up the first four elements of row $(N - 2)/2$ of Pascal's triangle, or, more prosaically, by the formula

$$|B| = (N^3 - 6N^2 + 32N)/48.$$

The Ternary Root (T) Family

This family, so called because all the digits of R are 0, 1, or 2, is defined for all odd $N \geqslant 3$. R is of the form $R = 1\,[x]\,y\,[x]'1$, where all x's = 0 or 1 and y = 0, 1, or 2. Note that R is again palindromic.

By a similar analysis as above, we can establish the following additional necessary and sufficient conditions for S to be palindromic:

(a) if $y = 2$, then [x] contains at most one 1; and
(b) if $y < 2$, then [x] contains at most three 1's.

Condition (b) had been noted much earlier by Simmons, who stated that if R is a palindrome with 9 or fewer 1's and the remainder of its digits are 0, then S is necessarily a palindrome since a carry cannot be generated [3].

The total number of T family members for a given value of N is

$$|T| = (N^3 - 9N^2 + 59N - 51)/24.$$

The Even Root (E) Family

This is the only family in which the R values are even. They are of the form 2[0]2 if N is even, or 2[0]x[0]2, where x is either 0 or 1, if N is odd. Therefore, the total number of E family members is simply:

$$|E| = 1 \text{ if } N \text{ is even.}$$
$$= 2 \text{ if } N \text{ is odd.}$$

The Asymmetric Root (A) Family

This family, defined for all odd $N \geqslant 7$, is by far the most interesting of the four. R is of the form 1(x)0[9]9[0]1(x)'1, where all x's = 0 or 1, and where the [9] and [0] strings are the same length. This family gets its name from the fact that all its R values are asymmetric (that is, non-palindromic). Note that there are two independent parameters which can be varied: the length of the (x) string, which we denote by x, and the total number of 9's in R (equivalently, the length of the [9] and [0] strings plus 1), which we denote by z. The question is: for each choice of x and z, which (x) strings generate palindromic squares?

This appears to be a very difficult question, and we only have a partial answer. By actually doing the multiplication, it is easy to see that if (x) is all zeros, the square is not palindromic. Therefore, (x) must contain at least one 1 digit. But the middle digit of the product, when (x) is all zeros, is a 4; therefore, at most two (x) digits can be 1's.

So (x) contains either one or two 1's, but not, it turns out, in all combinations. Here is a table of the number of (x) strings that actually generate palindromic squares, for each choice of x, z, $\leqslant 12$, produced by computer search.

Number of 9's (z)

Number of x's (x)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	4	2	1	0	0	0	0	0	0	0	0	0	0	0
4	4	3	2	1	0	0	0	0	0	0	0	0	0	0
5	5	4	1	2	1	0	0	0	0	0	0	0	0	0
6	4	3	4	1	2	1	0	0	0	0	0	0	0	0
7	5	5	5	2	1	2	1	0	0	0	0	0	0	0
8	4	3	2	3	2	1	2	1	0	0	0	0	0	0
9	5	6	5	6	1	2	1	2	1	0	0	0	0	0
10	4	3	4	3	4	1	2	1	2	1	0	0	0	0
11	5	6	4	4	5	2	1	2	1	2	1	0	0	0
12	4	3	4	3	4	3	2	1	2	1	2	1	0	0
13	5	6	5	6	5	6	1	2	1	2	1	2	1	0
14	4	3	4	3	2	3	4	1	2	1	2	1	2	1

These numbers are quite mysterious, although we do have the following theorems and conjectures (where $A(x,z)$ denotes an entry in the above array):

Theorem 1: $A(x,z) = 0$ for $x < z$.

Conjecture 1: For $z \leqslant x < 2z$,
$A(x,z) = 1$ if $x + z$ is even,
$= 2$ if $x + z$ is odd.

Conjecture 2: For $x \geqslant 4z$,
$A(x,z) = 3$ if x is even and z is even,
$= 4$ if x is even and z is odd,
$= 5$ if x is odd and z is odd, and
$= 6$ if x is odd and z is even.

The **boldface numbers** in the above array correspond to Conjecture 1, and the boldface ***italic numbers*** correspond to Conjecture 2. Even if both of these conjectures are true, the wedge of underlined numbers in between is still unexplained (although we conjecture that they are all between 1 and 6 inclusive).

Lacking a general formula, this table at least lets us compute the value of $|A|$ (that is, the total number of A-family members) for the first few values of N, by adding up along the minor diagonals (all entries with $x + z = (N - 3)/2$):

$$|A| = 1, 2, 5, 6, 9, 10, 10, 15, 15, 16, 18, 24, 18, 26, \ldots.$$

These correspond, respectively, to

$$N = 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, \ldots.$$

Some examples:

There is one A-family member with a 7-digit R:

$$1109111^2 = 1230127210321.$$

There are two A-family members with 9-digit R's:

$$110091011^2 = 12120030703000121$$
$$111091111^2 = 12341234943214321.$$

There are nine A-family members with 15-digit R's, one of which, for example, is:

$$111000091000111^2 = 12321020202032923020202012321.$$

Finding an exact formula for either $A(x,z)$ or $|A|$ remains an open problem.

Sporadic Solutions

Table 1 gives a complete list of all sporadic solutions with $R < 2 \times 10^{12}$, including ten new ones not listed in [1]. They seem to be fairly evenly distributed, so we conjecture that there are an infinite number of sporadics. We also conjecture that there are no other infinite families other than the four described above. Can someone supply a proof of either of these conjectures?

There is a nice example of a false conjecture contained in Table 1. It appears that there is another infinite family—note the R values 307, 30693, 3069307, 306930693, 30693069307. There is one more solution in this series:

$$3069306930693^2 = 9420645034800084305460249,$$

but, in fact, the next one,

$$306930693069307^2 = 94206450348005140084305460249,$$

is *not* palindromic! So this is not another infinite series, merely a very interesting curiosity. Does it have anything to do with the fraction 31/101, which equals .30693069 . . . ?

Table 2 extends Ondrejka's list of non-sporadic palindromic squares.

Summary

Shown on the next page is a list of the number of palindromic squares of each type for each value of N.

N	Tr	B	T	E	A	S	*Total*
1	4						4
2		1		1	1	1	3
3			3	2		3	8
4		2		1		2	5
5			6	2		3	11
6		4		1		1	6
7			11	2	1	5	19
8		8		1		5	14
9			20	2	2	1	25
10		15		1		2	18
11			35	2	5	7	49
12		26		1		4	31
13			58	2	6	1+?	67+
14		42		1		?	43+
15			91	2	9	?	102+
16		64		1		?	65+
Totals:	4	162	224	22	23	35+	470+

Ondrejka mentions the eighty-year-old conjecture that 1010101010101010l is the smallest integer that generates a palindromic square that is also pandigital. In searching up to 12-digit *R*'s, we have not discovered a counterexample. To finally answer this question, "only" the four question marks in the above tabulation need to be resolved.

References:

1. R. Ondrejka, A Palindrome (151) of Palindromic Squares, *Journal of Recreational Mathematics, 20*:1, pp. 68-71, 1988.
2. D. Gorenstein, *Finite Simple Groups,* Plenum Press, New York, 1982.
3. G. J. Simmons, Palindromic Powers, *Journal of Recreational Mathematics, 3*:3, pp. 93-98, April 1970.

J. RECREATIONAL MATHEMATICS, Vol. 22(2) 133-135, 1990

MORE ON PALINDROMIC SQUARES

CHARLES ASHBACHER

Box 121

Hiawatha, Iowa 52233

In a recent article, Rudolph Ondrejka presented a table of the first 151 positive integers n, where n^2 is a palindromic number [1]. He noted that there are two general types of these squares: those in which n is also palindromic and those in which n is not. Simmons had demonstrated that there are infinitely many members of the former set [2, 3], and Ondrejka demonstrated that there are infinitely many members in the latter set. Keith, in the article immediately preceding this one, classifies all palindromic squares into six classes [4]. He also defines four infinite families which account for almost all palindromic squares. The fifty-nine entries in Table 1 were found independently by me and all fall into one of the families defined by Keith.

Ondrejka closed his paper by presenting three problems for computer search:

1. Palindromic squares with an even number of digits are rare. Only four are known:

$$(836)^2 = 698896;$$
$$(798644)^2 = 637832238736;$$
$$(64030648)^2 = 4099923883299904; \text{ and}$$
$$(8316115486)^2 = 69161037773377301 6196.$$

Is there a fifth member of this set and are these squares members of an infinite set? These are all members of the sporadic family of palindromic squares mentioned by Keith. The list, on page 125 of this issue of *JRM*, of the thirty-five known members of this family, includes the four members of the above set. The search for this particular type of palindromic square must include the fact that the leading digits of n must be greater than 3162277660 . . . to ensure that the square has $2n$ digits. Neither Keith nor I have found any new members of this particular set.