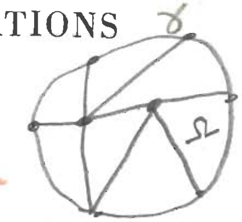
ENUMERATION OF TRIANGULATIONS OF THE DISK



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1. Introduction

LET Ω be a closed region in S^2 bounded by a simple closed curve γ. By a *triangulation T of Ω of type* $[n, m]$ we shall mean a simplicial complex (cf. (1) (5)) with polyhedron Ω, having $m+3$ 0-cells in its boundary, and n other 0-cells (called respectively *exterior* and *interior vertices*). The 1-cells of T will be called respectively *exterior* and *interior edges* according as they are or are not contained in γ; 2-cells will be called *faces*. The simplicial dissection induced in γ by T will be denoted by T_{ext}. The classes of i-cells of T and of T_{ext} will be called respectively T^i and $T_{\text{ext}}{}^i$ $(i = 0, 1, 2)$.

Let T and T' be triangulations of two such regions Ω and Ω' respectively. An *isomorphism* $\varphi : T \to T'$ is defined to be a one-to-one onto mapping $\varphi : \bigcup_{i=0}^{2} T^i \to \bigcup_{i=0}^{2} T'^i$ which carries i-cells onto i-cells, such that φa and φb are incident in T' if and only if a and b are incident in T.

As a simplex is completely determined by its vertices, we may denote *the* edge (face) incident with distinct vertices a, b (a, b, c) by $\langle ab \rangle$ $(\langle abc \rangle)$. We shall denote the *oriented* edge directed from vertex a to vertex b by $\langle ab \rangle^*$.

LEMMA. *Let T and T' be triangulations of Ω and Ω' respectively, and let $\varphi : T \to T'$ be an isomorphism. Then*

(1.1) *φ is determined by $\varphi | T^0$;*

(1.2) *φ induces homeomorphisms of Ω with Ω' which carry i-cells of T respectively onto i-cells of T' $(i = 0, 1, 2)$; conversely, any homeomorphism of Ω onto Ω' induces a triangulation T'' of Ω' and an isomorphism $\varphi'' : T \to T''$;*

(1.3) *$\varphi(T_{\text{ext}}{}^i) = T'_{\text{ext}}{}^i$ $(i = 0, 1)$;*

(1.4) *if $\langle ab \rangle$ is an edge in $T_{\text{ext}}{}^1$, then φ is determined by φa and φb.*

Proof. (1.1) and (1.2) are standard properties of simplicial mappings. (1.3) follows from the fact that edges incident with exactly one face in T

must be carried into edges of the same type in T'; hence the vertices incident with these edges must likewise correspond. (1.4) follows from a more general theorem which states that an isomorphism between non-separable maps on S^2 is completely determined by the images of one face, an edge in the boundary of that face, and the ends of that edge ((2) (5.1)).

It is our aim, ultimately, to enumerate the *isomorphism classes of triangulations of type* $[n, 0]$.

We define a *rooted triangulation* to be an ordered pair $(T, \langle pq \rangle^*)$, where T is a triangulation of some region Ω and $\langle pq \rangle$ is an exterior edge of T. Let $\langle pq \rangle, \langle p'q' \rangle$ be edges of triangulations T, T' of regions Ω, Ω' respectively. An *isomorphism* $\varphi : (T, \langle pq \rangle^*) \to (T', \langle p'q' \rangle^*)$ will be simply an isomorphism $\varphi : T \to T'$; φ will be said to be a *root-isomorphism* if $\varphi p = p'$ and $\varphi q = q'$. By (1.4), if such an isomorphism exists it is unique. We shall, henceforth, identify root-isomorphic rooted triangulations; by (1.2) it is no longer necessary to specify the region triangulated.

We define a *strong* triangulation to be one in which no interior edge is incident with *two* exterior vertices. In (7), Tutte has shown that the number (up to root-isomorphisms) of *rooted strong triangulations of type* $[n, m]$ is

$$\frac{3(m-1)!\,(m+2)!}{(3n+3m+3)!} \times \sum_{j=0}^{\min(m,n-1)} \frac{(m-3j)\,(m+j+2)\,(4n+3m-j+1)!}{j!\,(j+1)!\,(m-j+2)!\,(m-j)!\,(n-j-1)!} \quad \text{for} \quad m > 0$$

and

$$\frac{2(4n+1)!}{(3n+2)!\,(n+1)!} \quad \text{for} \quad m = 0.$$

We shall begin by enumerating (up to root-isomorphisms) *rooted triangulations of type* $[n, m]$ (all of which are strong for $m = 0$), showing their number to be

$$\frac{2(2m+3)!\,(4n+2m+1)!}{(m+2)!\,m!\,n!\,(3n+2m+3)!}.$$

Then we shall enumerate (again up to root-isomorphisms) rooted triangulations of type $[n, m]$ which are invariant under various types of isomorphisms, and finally compute the number of isomorphism classes of triangulations of type $[n, 0]$.

The author wishes to thank Professors W. T. Tutte and H. S. M. Coxeter for their kind assistance in connexion with this research.

I. ROOTED TRIANGULATIONS

2. Rooted triangulations of type $[n, m]$

Let $D_{n,m}$ denote the number (up to root-isomorphisms) of rooted triangulations of type $[n, m]$. The corresponding generating functions (tentatively defined as formal power series) are

$$D_{n.}(y) = \sum_{m=0}^{\infty} D_{n,m}\, y^m, \tag{2.1}$$

$$D_{.m}(x) = \sum_{n=0}^{\infty} D_{n,m}\, x^n, \tag{2.2}$$

$$\begin{aligned} D(x, y) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_{n,m}\, x^n y^m = \sum_{m=0}^{\infty} D_{.m}(x)\, y^m \\ &= \sum_{n=0}^{\infty} x^n D_{n.}(y). \end{aligned} \tag{2.3}$$

In the sequel we shall develop and solve an equation for $D(x, y)$ in a manner very similar to our treatment in (**2**) of the enumerating function for rooted non-separable planar maps.

3. An equation for $D(x, y)$

Let $(T, \langle p_1 p_2 \rangle^*)$ be any rooted triangulation of type $[n, m]$, wherein $\langle a p_1 p_2 \rangle$ is the face incident with $\langle p_1 p_2 \rangle$, and let U be the simplicial complex obtained from T by erasing $\langle p_1 p_2 \rangle$ and $\langle a p_1 p_2 \rangle$. By considering the possible forms of U we shall develop an equation for $D(x, y)$.

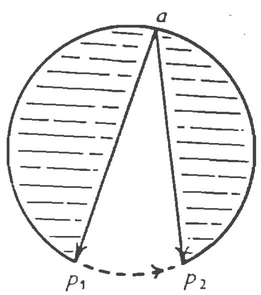

FIG. 1

Case 1. Suppose $a \in T_{\text{ext}}^{0}$ (cf. Fig. 1). Then, provided $\langle a p_j \rangle \notin T_{\text{ext}}^{1}$, the simple arc $a p_j$ in $\gamma - \langle p_1 p_2 \rangle$ can be completed by $\langle a p_j \rangle$ to form a simple closed curve enclosing a triangulated region with $\langle a p_j \rangle^*$ as root, say of type $[n_j, m_j]$ $(j = 1, 2)$; when $\langle a p_j \rangle \in T_{\text{ext}}^{1}$ the triangulation degenerates to an oriented edge, which we shall classify as the 'link-triangulation' of type $[0, -1]$ $(j = 1, 2)$. The following conditions must be satisfied:

$$m_1 + m_2 + 2 = m, \quad n_1 + n_2 = n. \tag{3.1}$$

Conversely, any pair of rooted triangulations of type $[n_j, m_j]$ $(j = 1, 2)$ (allowing the link-triangulation) satisfying (3.1) yields a unique rooted triangulation of this case. Thus these triangulations are enumerated by the generating function

$$y^2[y^{-1} + D(x, y)]^2 = [1 + yD(x, y)]^2.$$

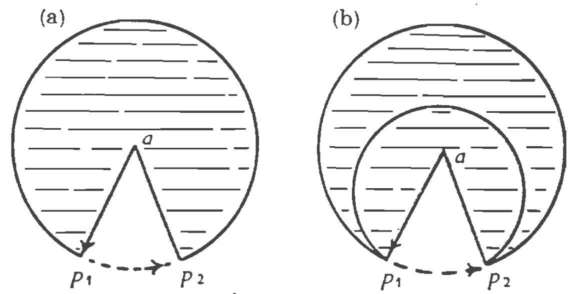

FIG. 2

Case 2. Suppose $a \notin T_{\text{ext}}^{0}$. Then U is a triangulation of type $[n-1, m+1]$ in which we may take $\langle a p_1 \rangle^*$ as root (cf. Fig. 2a). Conversely, let $(T', \langle a p_1 \rangle^*)$ be any rooted triangulation of type $[n-1, m+1]$ in which p_1 and p_2 are the two external vertices joined to a by external edges. Adjoining a new face $\langle a p_1 p_2 \rangle$ and a new edge $\langle p_1 p_2 \rangle$ to T' will produce a rooted triangulation $(T'', \langle p_1 p_2 \rangle^*)$ of type $[n, m]$ if and only if there is no edge $\langle p_1 p_2 \rangle$ in T'. We must therefore exclude from consideration

(i) triangulations T' of type $[n-1, 0]$, represented by the series† $xD(x, 0)$; and

(ii) triangulations T' in which p_1 and p_2 are joined by an internal edge; these are uniquely obtained by juxtaposing triangulations respectively of types $[n_1, m]$ and $[n_2, 0]$ (see Fig. 2b), where

$$n_1 + n_2 = n - 1, \tag{3.2}$$

and are therefore enumerated by the series†

$$xyD(x, 0)\, D(x, y).$$

Hence the triangulations in Case 2 are enumerated by

$$x\{y^{-1}[D(x, y) - D(x, 0)] - D(x, 0)\, D(x, y)\}.$$

Combining our results we obtain

$$D(x, y) = [1 + yD(x, y)]^2 + x\{y^{-1}[D(x, y) - D(x, 0)] - D(x, 0)\, D(x, y)\},$$

whence

$$y^3[D(x, y)]^2 + [2y^2 - y + x - xyD(x, 0)]\, D(x, y) + [y - xD(x, 0)] = 0. \tag{3.3}$$

† The coefficient of $x^n y^m$ in these series is the number of triangulations T' of type $[n, m]$ in which p_1 and p_2 are joined respectively by an external or an internal edge.

We could, of course, substitute our known value of $D(x,0)$ (from (7)) in (3.3) and attempt to solve for $D(x,y)$ directly. We shall show, however, that (3.3) determines $D(x,y)$ uniquely as a formal series in non-negative powers of x and y.

By comparing coefficients of powers of x in (3.3) we obtain

$$y^3[D_{0.}(y)]^2 + (2y^2 - y)\,D_{0.}(y) + y = 0, \tag{3.4}$$

$$\begin{aligned} y(1-2y)\,D_{n.}(y) &= y^3 \sum_{p=0}^{n} D_{n-p.}(y)\,D_{p.}(y) \\ &\quad - y \sum_{p=0}^{n-1} D_{n-p-1.}(y)\,D_{p.}(0) \\ &\quad + [D_{n-1.}(y) - D_{n-1.}(0)] \quad (n>0). \end{aligned} \tag{3.5}$$

Comparing coefficients of powers of y in (3.4) we obtain

$$\begin{aligned} D_{0,0} &= 1, \\ D_{0,1} &= 2D_{0,0}, \\ D_{0,m} &= 2D_{0,m-1} + \sum_{s=0}^{m-2} D_{0,s}\,D_{0,m-s-2} \quad (m>1). \end{aligned} \tag{3.6}$$

Hence $D_{0.}(y)$ is uniquely determined by (3.4). (The second root of the quadratic equation has a pole at $y = 0$.) Adding $2y^2 D_{n.}(y)$ to both sides of (3.5) we see that $D_{n.}(y)$ is uniquely determined provided $D_{r.}(y)$ are known for $r<n$. Hence, by induction, $D(x,y)$ is uniquely determined by (3.3) as that solution expressible as a series of products of non-negative powers of x and y. Given any function $K(x)$ which is analytic at $x = 0$, we can solve the quadratic equation

$$y^3[L(x,y)]^2 + [2y^2 - y + x - xyK(x)]\,L(x,y) + [y - xK(x)] = 0 \tag{3.7}$$

in the usual way. If one solution is analytic at $(x,y) = (0,0)$ then it must also satisfy

$$L(x,0) = K(x) \tag{3.8}$$

(which is (3.7) with $y = 0$), and its Taylor-series expansion about $(x,y) = (0,0)$ must, in fact, be $D(x,y)$. In the next section we shall *conjecture* that $D(x,0)$ is the function $g(x)$ obtained by Tutte in (7), and show that this can be extended to a function satisfying (3.3).

4. Solution of equation (3.3)

We set

$$x = uv^3, \tag{4.1}$$

$$v = 1-u, \tag{4.2}$$

$$K(x) = v^{-3}(1-2u), \tag{4.3}$$

and substitute in (3.7). The discriminant of the equation is

$$\begin{aligned} &[2y^2 - y + uv^3 - u(1-2u)\,y]^2 - 4y^3[y - u(1-2u)] \\ &\qquad = (y^2 - 2uv^2 y + u^2 v^4)\,(v^2 - 4y). \end{aligned}$$

Hence

$$y^3 L(x,y) = (1/2)\,[-2y^2 + yv(1+2u) - uv^3 \pm (y - uv^2)\,(v^2-4y)^{\frac{1}{2}}]. \tag{4.4}$$

Expanding by the binomial theorem, we obtain

$$\begin{aligned} (y-uv^2)\,(v^2-4y)^{\frac{1}{2}} &= (vy - uv^3)\left[1 - 2\sum_{s=0}^{\infty} \frac{(2s)!}{s!\,(s+1)!}\,y^{s+1}\,v^{-2s-2}\right] \\ &= -[2y^2 - yv(1+2u) + uv^3] \\ &\quad + 2u \sum_{m=0}^{\infty} \frac{(2m+4)!}{(m+2)!\,(m+3)!}\,y^{m+3}\,v^{-2m-3} \\ &\quad - 2\sum_{m=0}^{\infty} \frac{(2m+2)!}{(m+1)!\,(m+2)!}\,y^{m+3}\,v^{-2m-3}. \end{aligned}$$

Hence, selecting the negative sign in (4.4), we find one root of (3.7) to be

$$L(x,y) = 2\sum_{m=0}^{\infty} \frac{(2m+1)!}{m!\,(m+3)!}\,[(m+3)\,v^{-2m-3} - 2(2m+3)\,xv^{-2m-6}]\,y^m. \tag{4.5}$$

Clearly, $L(x,y)$ is absolutely convergent in a neighbourhood of

$$(x,y) = (0,0);$$

hence the Taylor expansion of $L(x,y)$ in powers of x and y about $(0,0)$ is $D(x,y)$. Now, applying Lagrange's Theorem ((**9**) 132) to

$$u = xv^{-3}$$

we obtain

$$v^{-t} = t\sum_{h=0}^{\infty} \frac{(4h+t-1)!}{h!\,(3h+t)!}\,x^h \quad (t>0) \tag{4.6}$$

(cf. also ((**2**) 4.14) and ((**7**) 5.9)).

Hence, from (4.5),

$$\begin{aligned} D_{.m}(x) &= \frac{2(2m+1)!\,(2m+3)}{m!\,(m+2)!} \sum_{h=0}^{\infty} \frac{(4h+2m+2)!}{h!\,(3h+2m+3)!}\,x^h \\ &\quad - 4\sum_{h=0}^{\infty} \frac{(4h+2m+5)!}{h!\,(3h+2m+6)!}\,x^{h+1} \\ &= \frac{2(2m+3)!}{m!\,(m+2)!} \sum_{n=0}^{\infty} \frac{(4n+2m+1)!}{n!\,(3n+2m+3)!}\,x^n, \end{aligned}$$

and so

$$D_{n,m} = \frac{2(2m+3)!\,(4n+2m+1)!}{m!\,(m+2)!\,n!\,(3n+2m+3)!}. \tag{4.7}$$

In particular,

$$D_{n,0} = \frac{2(4n+1)}{(n+1)!\,(3n+2)!}, \tag{4.8}$$

as seen in (7), thus proving our conjecture. Also

$$D_{0,m} = \frac{2(2m+1)!}{m!\,(m+2)!}. \tag{4.9}$$

This latter result was first stated by Euler (4); its history is discussed in (3) and (6).

Applying Stirling's formula to (4.7) we obtain *for fixed* m, as $n \to \infty$,

$$D_{n,m} \sim \frac{2(2m+3)!\,e^{-(4n+2m+1)}\,(4n+2m+1)^{4n+2m+1}}{(m+2)!\,m!\,e^{-n}\,e^{-(3n+2m+3)}\,n^n(3n+2m+3)^{3n+2m+3}} \times \sqrt{\left(\frac{4n+2m+1}{2\pi n(3n+2m+3)}\right)} \tag{4.10}$$

$$= \frac{2(2m+3)!\,(4n)^{4n+2m+1}\,e^2[1+(4n)^{-1}(1+2m)]^{4n+2m+1}}{(m+2)!\,m!\,n^n(3n)^{3n+2m+3}\,[1+(3n)^{-1}(3+2m)]^{3n+2m+3}} \times \sqrt{\left(\frac{4n+2m+1}{2\pi n(3n+2m+3)}\right)}$$

$$\sim \frac{1}{48}\frac{(2m+3)!}{(m+2)!\,m!}\left(\frac{16}{9}\right)^m n^{-5/2}\left(\frac{256}{27}\right)^{n+1}\sqrt{\left(\frac{3}{2\pi}\right)}.$$

In particular, for $m = 0$,

$$D_{n,0} \sim \frac{1}{16}n^{-5/2}\left(\frac{256}{27}\right)^{n+1}\sqrt{\left(\frac{3}{2\pi}\right)}, \tag{4.11}$$

as shown in (7).

II. TRIANGULATIONS WITH ROTATIONAL SYMMETRY

5. Automorphisms

An isomorphism of a triangulation with itself will be called an *automorphism*. We define an *automorphism* $\varphi_{\text{ext}} : T_{\text{ext}} \to T_{\text{ext}}$ to be a one-to-one mapping $\varphi_{\text{ext}} : T_{\text{ext}}{}^0 \cup T_{\text{ext}}{}^1 \to T_{\text{ext}}{}^0 \cup T_{\text{ext}}{}^1$ carrying vertices onto vertices, such that $\varphi_{\text{ext}}\, a$ and $\varphi_{\text{ext}}\, b$ are incident if and only if a and b are incident.

The choice of a root $\langle pq \rangle^*$ in T induces an orientation in T_{ext}. An isomorphism $\varphi : (T, \langle pq \rangle^*) \to (T', \langle p'q' \rangle^*)$ will be said to be *orientation-preserving* or *-reversing* according to its action on the induced orientation of T_{ext}. Clearly a root-isomorphism is orientation-preserving. We state without proof the following

LEMMA. *Let T be a triangulation of type $[n, m]$.*

(5.1) *Each automorphism $\varphi : T \to T$ induces an automorphism*

$$\varphi_{\text{ext}} : T_{\text{ext}} \to T_{\text{ext}}.$$

(5.2) *The automorphisms of T and T_{ext} respectively form groups, which we denote by $\mathfrak{A}(T), \mathfrak{A}_{\text{ext}}(T)$. $\mathfrak{A}_{\text{ext}}(T)$ is isomorphic to the (automorphism) group of the $(m+3)$-gon,* i.e. *the dihedral group*

$$D_{m+3} = \{R_1, R_2 : R_1{}^2 = R_2{}^2 = (R_1 R_2)^{m+3} = I\}.$$

(5.3) *We can define a monomorphism* ext $: \mathfrak{A}(T) \to \mathfrak{A}_{\text{ext}}(T)$ *by* $\varphi \to \varphi_{\text{ext}}$. *Thus $\mathfrak{A}(T)$ is isomorphic to a subgroup of D_{m+3}.*

(5.4) *The orientation-preserving automorphisms of T form an invariant subgroup $\mathfrak{A}^+(T)$ of $\mathfrak{A}(T)$.*

(5.5) *As the property of being orientation-preserving or -reversing is invariant under* ext, *and the orientation-preserving automorphisms of the $(m+3)$-gon form a cyclic subgroup $\mathfrak{C}_{m+3} = \{R_1 R_2 : (R_1 R_2)^{m+3} = I\}$ of D_{m+3}, $\mathfrak{A}^+(T)$ is isomorphic to a subgroup of $\mathfrak{C}_{m+3}$, and hence the order of $\mathfrak{A}^+(T)$ divides $m+3$.*

6. Rooted triangulations of type $[n, m; r]$

A rooted triangulation $(T, \langle pq \rangle^*)$ of type $[n, m]$ will be said to be *of type* $[n, m; r]$ if r divides the order of $\mathfrak{A}^+(T)$ (cf. ((2) § 6)). It follows that m must be of the form $rw - 3$, where $w > \delta_{r,2}$ ($\delta_{r,2}$ being the Kronecker delta). Let ${}_rE_{n,m}$ be the number, up to root-isomorphisms, of rooted triangulations of type $[n, m; r]$.

If ${}_rF_{n,m}$ is the number, up to root-isomorphisms, of rooted triangulations $(T, \langle pq \rangle^*)$ of type $[n, m]$ such that the order of $\mathfrak{A}^+(T)$ is *exactly* r, then it follows that

$${}_rE_{n,m} = \sum_{k=1}^{\infty} {}_{kr}F_{n,m} \tag{6.1}$$

and hence, by the Möbius inversion theorem ((8) 36),

$${}_rF_{n,m} = \sum_{k=1}^{\infty} \mu(k)\, {}_{kr}E_{n,m}, \tag{6.2}$$

where μ is the Möbius function.

Let $(T, \langle pq \rangle^*)$ be a rooted triangulation of type $[n, m]$ such that $\mathfrak{A}^+(T)$ has order r, and let the orientation of T_{ext} be given. Then there are $(m+3)/r$ different (up to orientation-preserving isomorphisms) possible rootings of T which induce the given orientation. Hence the number, $G_{n,m}$, of rooted triangulations of type $[n, m]$ up to orientation-preserving isomorphisms is given by

$$(6.3)\qquad G_{n,m} = \sum_{r=1}^{\infty}[r/(m+3)]\,{}_rF_{n,m} = [1/(m+3)]\sum_{r=1}^{\infty} r \sum_{k=1}^{\infty}\mu(k)\,{}_{kr}E_{n,m}$$
$$= [1/(m+3)]\sum_{s|(m+3)}\sum_{k|s}\mu(k)\,(s/k)\,{}_sE_{n,m}$$
$$= [1/(m+3)]\sum_{s|(m+3)}\varphi(s)\,{}_sE_{n,m},$$

where φ is the Euler function ((8) 27). In particular,

$$(6.4)\qquad G_{n,0} = (1/3)[D_{n,0} + 2\,{}_3E_{n,0}].$$

Thus, if we define an *oriented* triangulation T to be one in which T_{ext} is oriented, then the number (up to isomorphisms which preserve the orientation) of oriented triangulations of type $[n, m]$ is $G_{n,m}$.

The generating functions ${}_rE(x,y)$, ${}_rE_{n.}(y)$, ${}_rE_{.m}(x)$ are defined analogously to $D(x,y)$, $D_{n.}(y)$, $D_{.m}(x)$. In the following two sections we shall develop and solve an equation for ${}_rE(x,y)$ (cf. ((2) §§ 7, 8)).

As in (2) we define for any mapping $\lambda: U \to U$ the mapping $\tilde{\lambda}_r$, which associates with each u in U the *ordered* set

$$\tilde{\lambda}_r u = \{u, \lambda u, \lambda^2 u, \lambda^3 u, \ldots, \lambda^{r-1} u\} \quad (r = 1, 2, \ldots).$$

7. An equation for ${}_rE(x,y)$

Clearly ${}_1E(x,y) = D(x,y)$. Assume now that $r > 1$. Let $(T, \langle p_1 p_2\rangle^*)$ be a rooted triangulation of type $[n, m; r]$, wherein $\langle a p_1 p_2\rangle$ is the face incident with $\langle p_1 p_2\rangle$. Let λ be the generator of $\mathfrak{A}^+(T)$ which induces in T_{ext} a rotation through $w = (m+3)/r$ edges in the direction of the orientation. Let U be the simplicial complex obtained from T by removing the edges of $\tilde{\lambda}_r\langle p_1 p_2\rangle$ and the faces of $\tilde{\lambda}_r\langle a p_1 p_2\rangle$. By considering the possible forms of U we shall develop an equation for ${}_rE(x,y)$.

Case 1. Suppose $a \notin T_{\text{ext}}{}^0$, and that two elements of $\tilde{\lambda}_r a$ coincide. Let b be the smallest positive integer such that $a = (\lambda)^b a$. Then removal of $\langle p_1 p_2\rangle$, $(\lambda)^b\langle p_1 p_2\rangle$, $\langle a p_1 p_2\rangle$, and $(\lambda)^b\langle a p_1 p_2\rangle$ from T leaves a simplicial complex V which consists of two simply connected triangulated regions (allowing either or both to be degenerate as the link-triangulation) with $\{a\}$ as their intersection (cf. Fig. 3a). Suppose, for the moment, that $b > 1$, and hence

$$(7.1)\qquad 1 < b < r-1.$$

Then λa, $(\lambda)^{b+1} a$ lie in different regions, but they must coincide; hence $\lambda a = a$, contrary to our hypothesis that $b > 1$. Thus all elements of $\tilde{\lambda}_r a$ coincide, and U consists of a sequence of r identical rooted triangulations of type $[n', m']$ (allowing the link-triangulation as type $[0, -1]$) satisfying

the conditions

$$(7.2)\qquad rn' + 1 = n,\qquad r(m'+2) - 3 = m.$$

Conversely, any triangulation of type $[n', m']$ (again allowing the link-triangulation, provided $r > 2$) satisfying (7.2) determines a triangulation

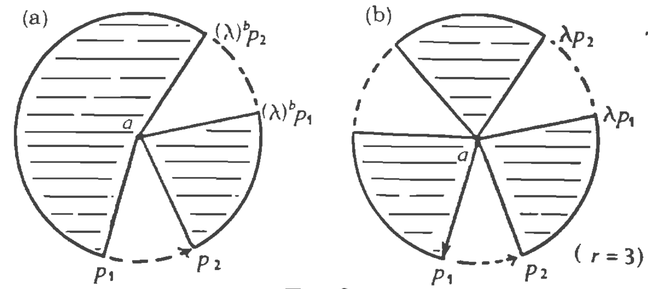

FIG. 3

of type $[n, m; r]$ of this case (cf. Fig. 3b). Thus the generating function for rooted triangulations of type [, ; r] of this form is

$$xy^{2r-3} D(x^r, y^r) + xy^{r-3}(1 - \delta_{r,2}).$$

(We cannot admit the degenerate link-triangulation when $r = 2$ as this would allow two edges joining the same pair of vertices in T.)

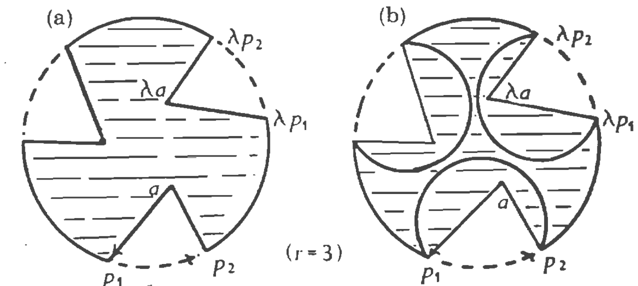

FIG. 4

Case 2. Suppose $a \notin T_{\text{ext}}{}^0$, but that all elements of $\tilde{\lambda}_r a$ are distinct. Then U is a rooted triangulation of type $[n', m'; r]$, where

$$(7.3)\qquad n' + r = n,\qquad m' - r = m$$

(cf. Fig. 4a). Conversely, let $(T', \langle a p_1\rangle^*)$ be any rooted triangulation of type $[n', m'; r]$ in which p_1 and p_2 are the external vertices joined to a by an external edge; let λ' be the generator of $\mathfrak{A}^+(T')$ which induces in T'_{ext} a rotation through $(m'+3)/r$ edges in the direction opposite to the orientation.

If p_1 and p_2 are not joined by an edge in T', then adjoining edges $\langle(\lambda')^u p_1, (\lambda')^u p_2\rangle$ and faces $\langle(\lambda')^u a, (\lambda')^u p_1, (\lambda')^u p_2\rangle$ $(u = 0, 1, \ldots, r-1)$ produces a triangulation of type $[n, m; r]$ in this case, subject only to conditions (7.3). Triangulations T' in which p_1 and p_2 are joined by an external edge must be of type $[n', 0; 2]$ and are excluded from consideration by requiring $m \geqslant 0$. Triangulations T' in which p_1 and p_2 are joined by an internal edge (cf. Fig. 4b) are

$$\sum_{n_1} \sum_{n_2} {}_rE_{n_1,m} D_{n_2,0}$$

in number, where summation is effected subject to the condition

$$n_1 + rn_2 = n - r, \tag{7.4}$$

and are therefore represented by the generating function

$$\sum_{n=r}^{\infty} \sum_{m=0}^{\infty} \sum_{n_2=0}^{[n/r]-1} {}_rE_{n-r(1+n_2),m} D_{n_2,0} x^n y^m = x^r {}_rE(x,y) D(x^r, 0).$$

Hence the generating function for triangulations in this case is

$$\sum_{n=r}^{\infty} \sum_{m=0}^{\infty} {}_rE_{n-r,m+r} x^n y^m - x^r {}_rE(x,y) D(x^r, 0)$$

$$= x^r y^{-r}\left[{}_rE(x,y) - \sum_{m=0}^{r-1} {}_rE_{.m}(x) y^m\right] - x^r {}_rE(x,y) D(x^r, 0)$$

$$= x^r y^{-r}[{}_rE(x,y) - (1-\delta_{r,2})\, {}_rE_{.r-3}(x)\, y^{r-3} - \delta_{r,2}\, {}_2E_{.1}(x)\, y]$$

$$- x^r {}_rE(x,y) D(x^r, 0)$$

by (5.5).

Case 3. Suppose $a \in T_{\text{ext}}{}^0$. Then, as in § 3, Case 1, removal of $\langle p_1 p_2\rangle$ and $\langle a p_1 p_2\rangle$ from T leaves two distinct triangulated regions (possibly degenerate). We shall denote by T_j the residual triangulation having p_j as a vertex. Let k_j be the number of elements in $\tilde{\lambda}_r\langle a p_1 p_2\rangle \cap T_j{}^2$ and assume, without limiting generality, that $k_1 \not> k_2$. We prove that $k_1 = 0$, i.e. that $k_2 = r-1$. For suppose some face $(\lambda)^b \langle a p_1 p_2\rangle$ lay in $T_1{}^2$. Then, of the two triangulations which remain after $(\lambda)^b \langle p_1 p_2\rangle$ and $(\lambda)^b \langle a p_1 p_2\rangle$ are erased from T, one must lie entirely within T_1; hence, by symmetry, either $k_1 < k_1$ or $k_2 < k_1$, both of which are absurd. Thus $k_1 = 0$, i.e. all elements of $\tilde{\lambda}_r\langle a p_1 p_2\rangle \backslash \{\langle a p_1 p_2\rangle\}$ lie together in one or other of the residual triangulations T_j $(j = 1, 2)$. Except for the sub-case where both T_1 and T_2 are degenerate, which is enumerated by $\delta_{r,3}$, there are two possible sub-cases, according as k_1 or $k_2 = 0$. Hence U is of one of the forms shown in Fig. 5, consisting of a central region triangulated of type $[n', m'; r]$ (possibly degenerate, but only when $r = 2$) and r copies of a triangulation

of type $[n'', m'']$ (possibly degenerate), subject only to

$$n' + rn'' = n, \quad m' + r(m''+2) = m. \tag{7.5}$$

Thus triangulations in this case are enumerated by

$$2[y^r {}_rE(x,y) + \delta_{r,2}\, y]\,[y^r D(x^r, y^r) + 1] + \delta_{r,3}.$$

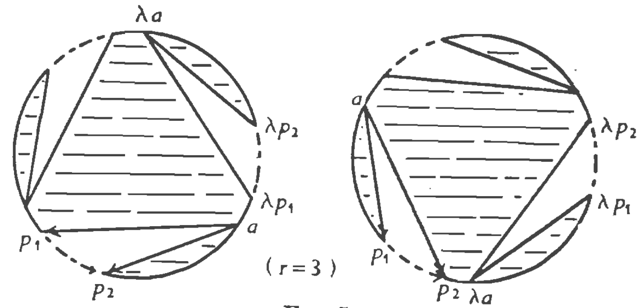

FIG. 5

Combining our results we obtain

$${}_rE(x,y) = xy^{2r-3} D(x^r, y^r) + (1-\delta_{r,2})\, xy^{r-3}$$
$$+ x^r y^{-r}[{}_rE(x,y) - (1-\delta_{r,2})\, {}_rE_{.r-3}(x)\, y^{r-3} - \delta_{r,2}\, {}_2E_{.1}(x)\, y]$$
$$- x^r {}_rE(x,y) D(x^r, 0)$$
$$+ 2[y^r {}_rE(x,y) + \delta_{r,2}\, y]\,[y^r D(x^r, y^r) + 1] + \delta_{r,3},$$

whence

$$y^3\{y^r - x^r + x^r y^r D(x^r, 0) - 2y^{2r}[y^r D(x^r, y^r) + 1]\}\, {}_rE(x,y)$$
$$= [xy^{2r} + 2\delta_{r,2}\, y^6]\,[y^r D(x^r, y^r) + 1] - (1-\delta_{r,2})\, x^r y^r\, {}_rE_{.r-3}(x)$$
$$- \delta_{r,2}\, xy^4[1 + x_2 E_{.1}(x)] + \delta_{r,3}\, y^6. \tag{7.6}$$

In the next section we shall solve equation (7.6) directly; as in our solution of equation (7.4) in (**2**), no preliminary conjecture is required in this case. We shall make use of the fact that, since the power series $D(x,y)$ is absolutely convergent in some neighbourhood of $(x,y) = (0,0)$, and since for every r the power series ${}_rE(x,y)$ is majorized by $D(x,y)$, both $D(x,y)$ and ${}_rE(x,y)$ represent analytic functions in some neighbourhood of $(x,y) = (0,0)$.

8. Solution of equation (7.6)

We set

$$x^r = uv^3 \tag{8.1}$$

and again define v to be $1-u$. We shall henceforth assume y, x (and hence u) small enough to ensure that the various functions are analytic.

By (4.4) (with the negative sign) the coefficient of ${}_rE(x,y)$ in (7.6) is equal to $-y^3(uv^2-y^r)(v^2-4y^r)^{\frac{1}{2}}$. But, it follows from (4.4) (with the negative sign) that $D(x^r, uv^2) = v^{-3}$. Thus, setting $y^r = uv^2$ in (7.6) we obtain

$$0 = (xu^2v^4 + 2\delta_{r,2}u^3v^6)(1+uv^{-1}) - (1-\delta_{r,2})u^2v^5\,{}_rE_{.r-3}(x)$$
$$- u^2v^4\delta_{r,2}[x + uv^3\,{}_2E_{.1}(x)] + \delta_{r,3}u^2v^4,$$

yielding

(8.2)
$$ {}_2E_{.1}(x) = xv^{-4} + 2v^{-2},$$
$$ {}_3E_{.0}(x) = xv^{-2} + v^{-1},$$
$$ {}_rE_{.r-3}(x) = xv^{-2} \qquad (r > 3).$$

Substitution in (7.6) yields (again by (4.4))

$$-y^3(uv^2-y^r)(v^2-4y^r)^{\frac{1}{2}}\,{}_rE(x,y)$$
$$= (uv^2-y^r)\{[(x/2)+\delta_{r,2}y^2][(v^2-4y^r)^{\frac{1}{2}}-v] + \delta_{r,2}xy^2v^{-1} - \delta_{r,3}y^3\},$$

whence

(8.3)
$$y^3\,{}_rE(x,y) = [(x/2)+\delta_{r,2}y^2][v(v^2-4y^r)^{-\frac{1}{2}}-1]$$
$$- \delta_{r,2}xy^2v^{-1}(v^2-4y^r)^{-\frac{1}{2}} + \delta_{r,3}y^3(v^2-4y^r)^{-\frac{1}{2}}.$$

But, by the binomial theorem,

$$(v^2-4y^r)^{-\frac{1}{2}} = v^{-1}\sum_{p=0}^{\infty}\binom{2p}{p}v^{-2p}y^{rp}.$$

Hence

(8.4)
$$y^3\,{}_rE(x,y) = [(x/2)+\delta_{r,2}y^2]\sum_{p=1}^{\infty}\binom{2p}{p}v^{-2p}y^{rp}$$
$$+ (\delta_{r,3}-\delta_{r,2}xv^{-1})\sum_{p=0}^{\infty}\binom{2p}{p}v^{-2p-1}y^{r(p+1)},$$

i.e.

(8.5)
$$ {}_rE(x,y) = x\sum_{p=0}^{\infty}\binom{2p+1}{p}v^{-2(p+1)}y^{rp+(r-3)}$$
$$+ \delta_{r,2}\left\{\sum_{p=1}^{\infty}\binom{2p}{p}v^{-2p}y^{2p-1} - x\sum_{p=0}^{\infty}\binom{2p}{p}v^{-2(p+1)}y^{2p-1}\right\}$$
$$+ \delta_{r,3}\sum_{p=0}^{\infty}\binom{2p}{p}v^{-2p-1}y^{3p}.$$

Hence

(8.6)
$$ {}_2E_{.2p-1}(x) = \binom{2p}{p}v^{-2p} + x\binom{2p}{p-1}v^{-2(p+1)} \qquad (p>0),$$

(8.7)
$$ {}_rE_{.r(p+1)-3}(x) = \delta_{r,3}\binom{2p}{p}v^{-2p-1} + x\binom{2p+1}{p}v^{-2(p+1)} \qquad (p \geqslant 0).$$

Applying (4.6) with x replaced by x^r we obtain

(8.8)
$$ {}_2E_{.2p-1}(x) = 2p\binom{2p}{p}\sum_{h=0}^{\infty}\frac{(4h+2p-1)!}{h!\,(3h+2p)!}x^{2h}$$
$$+ 2(p+1)\binom{2p}{p-1}\sum_{h=0}^{\infty}\frac{(4h+2p+1)!}{h!\,(3h+2p+2)!}x^{2h+1},$$

(8.9)
$$ {}_rE_{.r(p+1)-3}(x) = \delta_{r,3}(2p+1)\binom{2p}{p}\sum_{h=0}^{\infty}\frac{(4h+2p)!}{h!\,(3h+2p+1)!}x^{3h}$$
$$+ 2(p+1)\binom{2p+1}{p}\sum_{h=0}^{\infty}\frac{(4h+2p+1)!}{h!\,(3h+2p+2)!}x^{rh}.$$

Thus all of the numbers ${}_rE_{n,m}$ are zero except the following:

(8.10)
$$ {}_2E_{2s+j,2p-1} = \frac{2(2p)!\,(4s+2p+2j-1)!}{p!\,(p-1)!\,s!\,(3s+2p+2j)!} \qquad (j=0,1;\ p>0;\ s\geqslant 0),$$

(8.11)
$$ {}_3E_{3s,3p} = \frac{(2p+1)!\,(4s+2p)!}{p!\,p!\,s!\,(3s+2p+1)!} \qquad (p\geqslant 0;\ s\geqslant 0),$$

(8.12)
$$ {}_rE_{rs+1,r(p+1)-3} = \frac{(2p+2)!\,(4s+2p+1)!}{p!\,(p+1)!\,s!\,(3s+2p+2)!} \qquad (p\geqslant 0;\ s\geqslant 0;\ r>2).$$

We note that ${}_rE_{rs+1,r(p+1)-3}$ is *independent of r*.

In Table 1 values of $G_{n,m}$ computed by means of formula (6.3) are tabulated for small n, m.

TABLE 1

Values of $G_{n,m}$ for $n+m<9$

n \ m	0	1	2	3	4	5	6	7	8
0	1	1	1	4	6	19	49	150	442
1	1	2	5	16	48	164	599	1,952	
2	1	6	21	88	330	1,302	5,005		
3	5	26	119	538	2,310	9,882			
4	24	147	735	3,568	16,500				
5	133	892	4,830	24,596					
6	846	5,876	33,253						
7	5,661	40,490							
8	39,556								

9. Asymptotic behaviour of ${}_rE_{n,m}$

Applying Stirling's formula to (8.10), (8.11), and (8.12) it can be shown that for fixed p and r and large s,

$$(9.1)\qquad {}_rE_{rs+1,r(p+1)-3} \sim \frac{1}{16}\frac{(2p+1)!}{p!\,p!}\sigma(s,p),$$

$$(9.2)\qquad {}_2E_{2s,2p-1} \sim \frac{9}{256}\frac{(2p)!}{p!\,(p-1)!}\sigma(s,p),$$

$$(9.3)\qquad {}_2E_{2s+1,2p-1} \sim \frac{1}{16}\frac{(2p)!}{p!\,(p-1)!}\sigma(s,p),$$

$$(9.4)\qquad {}_3E_{3s,3p} \sim \frac{3}{128}\frac{(2p+1)!}{p!\,p!}\sigma(s,p),$$

where

$$\sigma(s,p) = s^{-3/2}\left(\frac{256}{27}\right)^{s+1}\left(\frac{16}{9}\right)^{p}\Big/\sqrt{\left(\frac{3}{2\pi}\right)}.$$

Since $(256/27)^{s/r} < 256/27$ for $s<r$, it follows that ${}_rE_{n,m}/D_{n,m} \to 0$ as $n\to\infty$ for fixed m and $r>1$. Hence

$$\frac{\sum_{r|(m+3)} \varphi(r)\,{}_rE_{n,m}}{D_{n,m}} \to 0 \quad \text{as} \quad n\to\infty.$$

It follows from (6.3) that, for fixed m,

$$(9.5)\qquad G_{n,m} \sim \frac{D_{n,m}}{m+3} \quad \text{as} \quad n\to\infty;$$

i.e. almost all triangulations T of type $[n,m]$ are rotationally asymmetrical ($A^+(T) = I$).

III. TRIANGULATIONS WITH REFLECTIONAL SYMMETRY

10. Rooted triangulations of types $[n,m]^+$ and $[n,m]^-$

Let T be a triangulation of type $[n,m]$. We shall say that T is *of type* $[n,m]^+$ or *of type* $[n,m]^-$ according as $\mathfrak{A}^+(T)$ has index 1 or 2 in $\mathfrak{A}(T)$. A rooted triangulation $(T,\langle pq\rangle^*)$ of type $[n,m]^-$ will be said to be a *K-rooting* (*L-rooting*) of T if $\langle pq\rangle$ (p) is invariant under the operation of some element of $\mathfrak{A}(T)$ other than the identity.

We state without proof the following

(10.1) LEMMA. *Let T be a triangulation of type $[n,m]^-$. There exist, up to automorphisms, either two K-rootings, two L-rootings, or one K-rooting and one L-rooting of T.*

Let $K_{n,m}$ and $L_{n,m}$ respectively represent the numbers, up to root-isomorphisms, of K- and L-rooted triangulations of type $[n,m]^-$. Then the number, up to root-isomorphisms, of rooted triangulations of type $[n,m]^-$ is $(1/2)\,[K_{n,m}+L_{n,m}]$.

Let $H_{n,m}$ and $R_{n,m}$ respectively denote the number, up to root-isomorphisms, of rooted triangulations of type $[n,m]^+$ and the number, up to all isomorphisms, of triangulations of type $[n,m]$. Then clearly

$$(10.2)\qquad \begin{aligned} G_{n,m} &= 2H_{n,m}+(1/2)\,[K_{n,m}+L_{n,m}],\\ R_{n,m} &= H_{n,m}+(1/2)\,[K_{n,m}+L_{n,m}]\\ &= (1/2)\,G_{n,m}+(1/4)\,[K_{n,m}+L_{n,m}]. \end{aligned}$$

The generating functions $K(x,y)$, $L(x,y)$ and $K_{.m}(x)$, $L_{.m}(x)$ are defined analogously to $D(x,y)$ and $D_{.m}(x)$.

If T is a triangulation of type $[n,m]^-$, where m is even, then each orientation-reversing automorphism of T leaves one vertex and one edge of T_{ext} invariant: thus $K_{n,m} = L_{n,m}$ for even m. We define the common y-even part of $K(x,y)$ and $L(x,y)$ to be $J(x,y^2)$.

In the following four sections we shall develop equations which relate $K(x,y)$, $L(x,y)$, and $J(x,0)$, and show how an equation uniquely determining $J(x,0)$ can be obtained from them.

11. First equation for $K(x,y)$ and $L(x,y)$

Let $(T,\langle p_1p_2\rangle^*)$ be a K-rooted triangulation of type $[n,m]^-$ wherein $\langle ap_1p_2\rangle$ is the face incident with $\langle p_1p_2\rangle$, and consider the simplicial complex U obtained from T by erasing $\langle p_1p_2\rangle$ and $\langle ap_1p_2\rangle$.

Case 1. $a\in T_{\text{ext}}{}^0$ (cf. Fig. 1). The dissection here obtained is similar to that of §3, Case 1, except that the two triangulations obtained are mirror images, say of type $[n',m']$ (allowing the link-triangulation), where n' and m' satisfy the conditions

$$(11.1)\qquad \begin{aligned} 2n' &= n,\\ 2(m'+1) &= m. \end{aligned}$$

Conversely, any pair of mirror images of a triangulation of type $[n',m']$ (allowing the link-triangulation) yields a unique triangulation in this case of type $[n,m]^-$, subject only to (11.1). The generating function for these triangulations is thus $1+y^2\,D(x^2,y^2)$.

Case 2. $a\notin T_{\text{ext}}{}^0$ (cf. Fig. 2). An argument analogous to that in §3, Case 2, shows that the generating function for triangulations in this case is

$$x\{y^{-1}[L(x,y)-J(x,0)]-J(x,0)\,K(x,y)\}.$$

Combining our results, we obtain as our first equation

$$K(x,y) = 1 + y^2 D(x^2, y^2) + x\{y^{-1}[L(x,y) - J(x,0)] - J(x,0)\, K(x,y)\},$$

i.e.

$$(11.2) \qquad [1 + xJ(x,0)]\, K(x,y) - xy^{-1} L(x,y) + [xy^{-1} J(x,0) - 1 - y^2 D(x^2, y^2)] = 0.$$

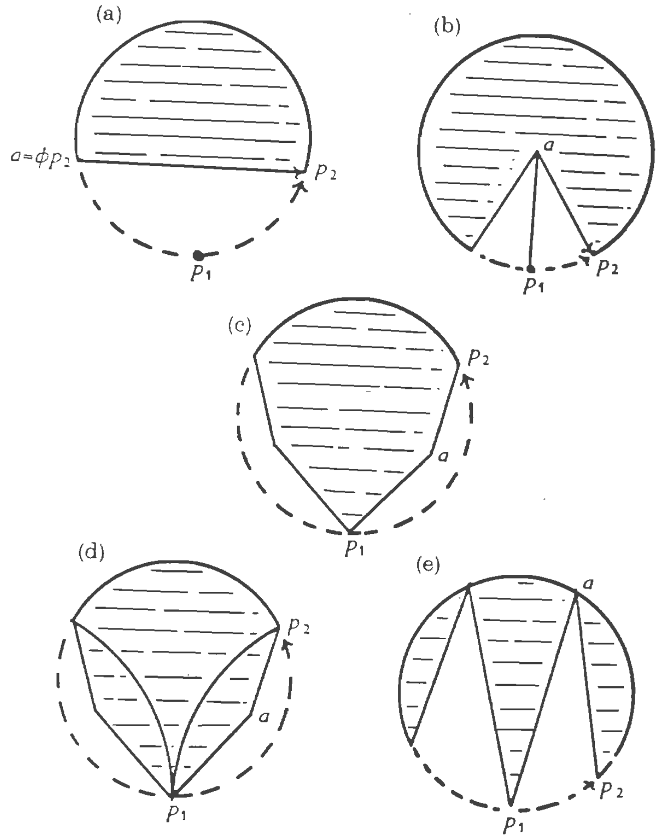

FIG. 6

2. Second equation for $K(x,y)$ and $L(x,y)$

Let $(T, \langle p_1 p_2 \rangle^*)$ be an L-rooted triangulation of type $[n, m]^-$, wherein $\langle p_1 p_2 \rangle$ is the face incident with $\langle p_1 p_2 \rangle$; let φ be the corresponding orientation-reversing automorphism. Consider the simplicial complex U obtained by removing $\langle p_1 p_2 \rangle$, $\langle p_1, \varphi p_2 \rangle$, $\langle a p_1 p_2 \rangle$, and $\langle \varphi a p_1 \varphi p_2 \rangle$ from T.

Case 1. Suppose $a = \varphi p_2$ (cf. Fig. 6a). U consists of the isolated vertex p_1 and a K-rooted triangulation (possibly the degenerate link-triangulation) of type $[n, m-1]^-$. The generating function in this case is thus $+yK(x,y)$.

Case 2. Suppose that $\varphi a = a$ and $a \notin T_{\text{ext}}{}^0$ (cf. Fig. 6b). Here $\langle ap_1 \rangle$ is invariant under φ. U consists of the edge $\langle ap_1 \rangle$ and the vertex p_1 together with an L-rooted (non-degenerate) triangulation of type $[n-1, m]^-$. The generating function in this case is thus $xL(x,y)$.

Case 3. Suppose that $\varphi a \neq a$ and $a \notin T_{\text{ext}}{}^0$ (cf. Fig. 6c). By an argument similar to that in § 3, Case 2, it can be shown that the generating function for this case is

$$x^2 y^{-2}[L(x,y) - J(x,0) - yL_{.1}(x)] - x^2 D(x^2, 0)\, L(x,y).$$

The term $yL_{.1}(x)$ is subtracted to ensure that U have at least five external edges; the term $x^2 D(x^2, 0)\, L(x,y)$ is analogous to $D(x,0)\, D(x,y)$ in § 3 (cf. Fig. 6d).

Case 4. Suppose that $a \neq \varphi p_2$ but $a \in T_{\text{ext}}{}^0$ (cf. Fig. 6e). Here U consists of a central L-rooted triangulation of type $[n', m']^-$ (possibly degenerate) and a pair of identical rooted triangulations of type $[n'', m'']$ (possibly degenerate), where n', m', n'', m'' satisfy the conditions

$$(12.1) \qquad n' + 2n'' = n, \qquad m' + 2m'' + 4 = m.$$

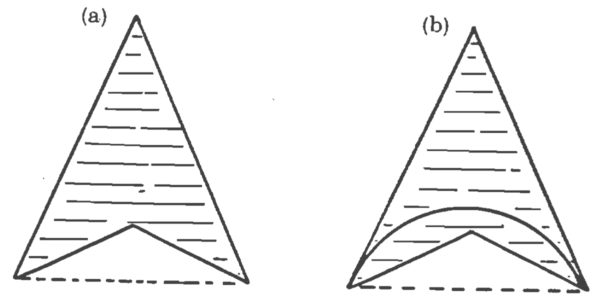

FIG. 7

Thus the generating function here is

$$y[1 + yL(x,y)]\,[1 + y^2 D(x^2, y^2)].$$

Before collecting our above results we note that $L_{.1}(x)$ is given by

$$(12.2) \qquad K(x,0) = 1 + x[L_{.1}(x) - J(x,0)^2]$$

(cf. Figs. 7a, 7b). Thus

$$(12.3) \quad L(x,y) = 1 + yK(x,y) + xL(x,y) + x^2 y^{-2}[L(x,y) - J(x,0)] - xy^{-1}[xJ(x,0)^2 + J(x,0) - 1] - x^2 D(x^2, 0)\, L(x,y) + y[1 + yL(x,y)]\,[1 + y^2 D(x^2, y^2)].$$

We define

$$(12.4) \qquad \bar{D}(x,y) = 1 + yD(x,y).$$

(3.3) can now be rewritten as

(12.5) $$y^2[\bar{D}(x,y)]^2+[x-y-xyD(x,0)]\bar{D}(x,y)+(y-x)=0.$$

(12.3) and (12.5) together yield, after rearrangement,

(12.6) $$\begin{aligned} yK(x,y)&+[x+(x^2y^{-2}-1)\bar{D}(x^2,y^2)^{-1}]L(x,y)\\ &+1+y\bar{D}(x^2,y^2)-x^2y^{-2}J(x,0)\\ &+xy^{-1}[xJ(x,0)^2+J(x,0)-1]=0. \end{aligned}$$

13. Solution of equations (11.2), (12.6); an equation for $J(x,0)$

Solving (11.2) and (12.6) for $K(x,y)$ and $L(x,y)$ in terms of x, y, and $J(x,0)$, we obtain

(13.1) $$K(x,y):L(x,y):1::\alpha(x,y):\beta(x,y):\gamma(x,y),$$

where

(13.2) $$\begin{aligned}\alpha(x,y)=-x[y^2\bar{D}(1+xJ_0)+J_0(-x^2\bar{D}+x^2-y^2)]\\ +y\bar{D}[x^2J_0(1+xJ_0)-y^2],\end{aligned}$$

(13.3) $$\begin{aligned}\beta(x,y)=y^2\bar{D}[x(1+xJ_0)(xJ_0^2+J_0-1)-y^2\bar{D}(2+xJ_0)]\\ +y\bar{D}[x^2J_0(1+xJ_0)-y^2],\end{aligned}$$

(13.4) $$\gamma(x,y)=y[xy^2\bar{D}+(1+xJ_0)(xy^2\bar{D}+x^2-y^2)],$$

and $J_0=J(x,0)$, $\bar{D}=\bar{D}(x^2,y^2)$. It follows that

(13.5) $$[x^2J_0(1+xJ_0)-y^2]\bar{D}=[xy^2\bar{D}+(1+xJ_0)(xy^2\bar{D}+x^2-y^2)]J(x,y^2).$$

For x and y sufficiently small, $J(x,y^2)\neq 0$. If we set

(13.6) $$y^2=x^2J_0(1+xJ_0)$$

the equation

(13.7) $$xy^2\bar{D}+(1+xJ_0)(xy^2\bar{D}+x^2-y^2)=0$$

must be satisfied; hence

(13.8) $$\bar{D}(x^2,y^2)=x^{-1}J_0^{-1}(2+xJ_0)^{-1}[xJ_0^2+J_0-1].$$

Substituting (13.6) and (13.8) into the equation obtained by replacing x by x^2 and y by y^2 in (12.5) we obtain

(13.9) $$[xJ_0^2+J_0-1]\{P^2(P-1)^2+2xP-(P^2-1)[1+x^2D(x^2,0)]\}=0,$$

where P is defined to be $1+xJ_0$. But if we had $xJ_0^2+J_0-1=0$, then we would have

$$J_0=\sum_{n=0}^{\infty}\frac{(2n)!\,(-1)^n}{n!\,(n+1)!}x^n,$$

which is inadmissible by virtue of the presence of negative coefficients: it is also inconsistent with the solution which could be obtained from

(13.5) by comparing the coefficients of powers of x. The remaining factor in (13.9) yields a quartic equation in P, which could, of course, be solved for an explicit expression for the root which has value 1 at $x=0$. For ease of computation, however, we shall rewrite the equation obtained in the form

(13.10) $$J_0=1+xJ_0+x^2J_0[1+(xJ_0/2)][J_0^2-D(x^2,0)].$$

The coefficients in J_0 can be computed from (13.10) with relative ease. The first terms in the expansion are

(13.11) $$\begin{aligned}J_0=1+x+x^2+3x^3+8x^4+23x^5+68x^6+215x^7\\ +680x^8+2226x^9+7327x^{10}+\dots\end{aligned}$$

The expansions of $K(x,y)$ and $L(x,y)$ could now be obtained from (13.1). As we are interested primarily in the numbers $K_{n,0}=L_{n,0}$, we shall not perform this computation.

The numbers $R_{n,0}$ for $n<9$ were computed using (10.2), and are shown in Table 2. For $n<5$ the triangulations of type $[n,0]$ are shown in Fig. 8.

TABLE 2

Values of $R_{n,0}$ for $n<9$

n	0	1	2	3	4	5	6	7	8
$G_{n,0}$	1	1	1	5	24	133	846	5,661	39,556
$K_{n,0}$	1	1	1	3	8	23	68	215	680
$R_{n,0}$	1	1	1	4	16	78	457	2,938	20,118

14. Asymptotic behaviour of $K_{n,0}=L_{n,0}$

Let $U=\sum_{n=0}^{\infty}u_nx^n$ and $V=\sum_{n=0}^{\infty}v_nx^n$ be arbitrary power series (having real coefficients). We shall write $U\ll V$ if $u_n\leqslant v_n$ $(n=0,1,\dots)$.

We confine our attention to the root J_0 of (13.10) such that $1\ll J_0$. For this root,

$$J_0\ll 1+xJ_0+x^2J_0^3+x^3J_0^4/2.$$

Hence

(14.1) $$\begin{aligned}J_0^2&\ll(1+xJ_0+x^2J_0^3+x^3J_0^4/2)+J_0(xJ_0+x^2J_0^3+x^3J_0^4/2)\\ &\ll 1+2xJ_0^2+2x^2J_0^4+x^3J_0^6\ll(1+xJ_0^2)^3\end{aligned}$$

and

(14.2) $$M\ll 1+xM^3,\quad\text{where}\quad M=1+xJ_0^2.$$

Let

$$N=\sum_{n=0}^{\infty}\frac{(3n)!}{n!\,(2n+1)!}x^n.$$

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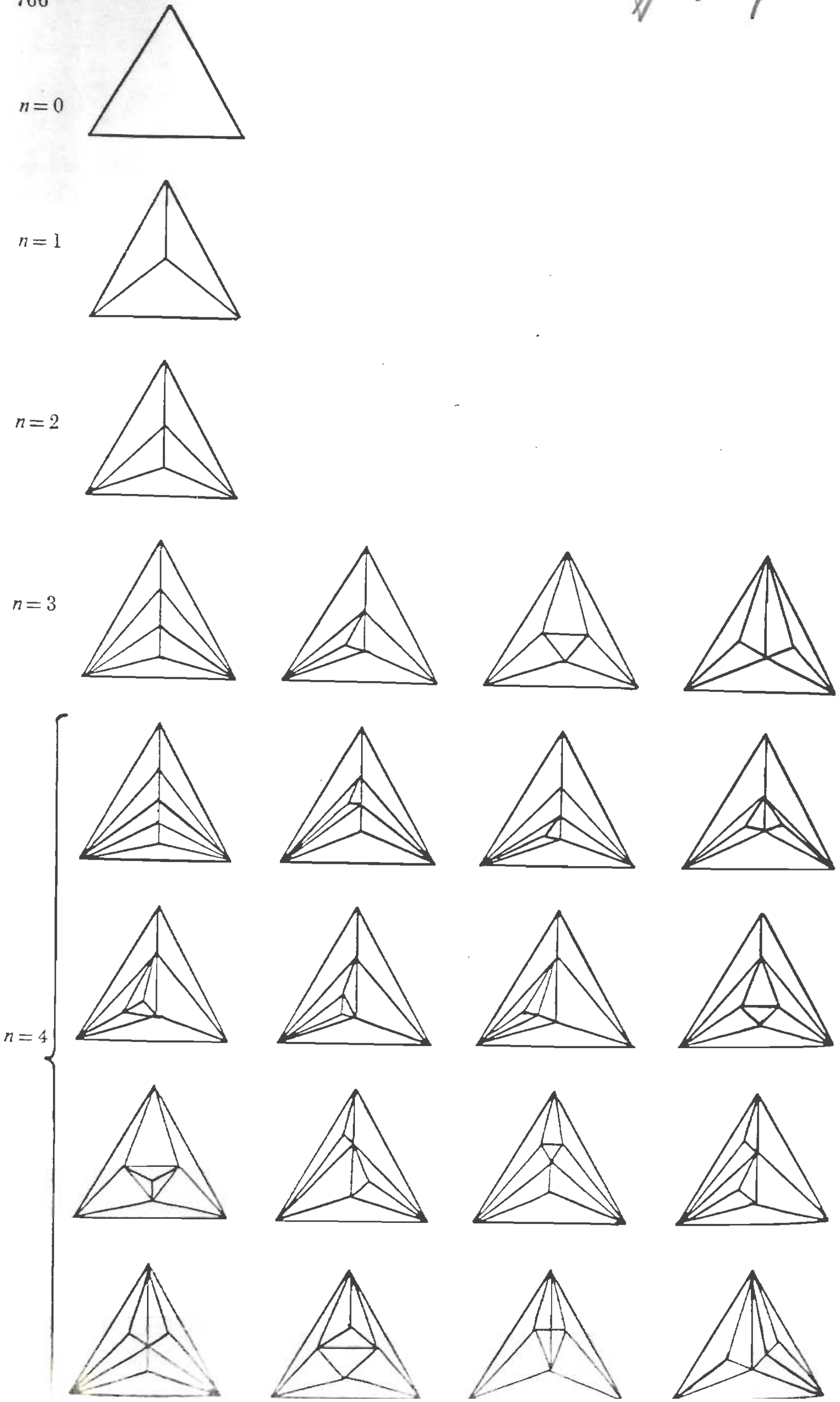

Then, by Lagrange's theorem,

(14.3) $$N = 1+xN^3.$$

Thus $M-N \ll x(M-N)(M^2+MN+N^2)$. It follows by induction on the coefficients in M and N that $M \ll N$. Hence

$$J_0 \ll J_0^2 \ll \sum_{n=0}^{\infty} \frac{(3n+3)!}{(n+1)!\,(2n+3)!} x^n$$

and

(14.4) $$K_{n,0} \leqslant \frac{(3n+3)!}{(n+1)!\,(2n+3)!} \quad (n = 0, 1, \ldots).$$

By Stirling's theorem,

(14.5) $$K_{n,0} \lesssim \frac{1}{4}\left(\frac{27}{4}\right)^{n+1} n^{-3/2} \sqrt{\left(\frac{3}{\pi}\right)} \quad \text{as} \quad n \to \infty.$$

Since $27/4 < 256/27$, $K_{n,0}/D_{n,0} \to 0$ as $n \to \infty$. Thus $R_{n,0} \sim D_{n,0}/6$ as $n \to \infty$; i.e. almost all triangulations of type $[n, 0]$ are asymmetrical.

IV. STRONG TRIANGULATIONS

15. Derivation of Tutte's function $\psi(x, y)$ from $D(x, y)$

The interior edges which join external vertices of a triangulation uniquely decompose the triangulation into strong triangulations. Thus we can think of any rooted triangulation as being obtained from a unique rooted strong triangulation by adjoining to every external edge except the root a unique rooted triangulation, possibly degenerate. It follows that

(15.1) $$D(x,y) = [\bar{D}(x,y)]^2 \psi(x, y\bar{D}(x,y))$$

or

(15.2) $$z^2 \psi(x,z) = z - y,$$

where y is the solution of

(15.3) $$y = z\bar{D}(x,y)^{-1}$$

which vanishes when $z = 0$. Substituting (15.3) in (12.5) we obtain

$$y^2 + [z^2 - (1 + xD(x,0))z - x]y + xz = 0,$$

whence

(15.4) $$y = -z^2 + [1 + xD(x,0)]z + x \pm \tfrac{1}{2}(z - uv)[(v^2 - z)^2 - 4uvz]^{\frac{1}{2}},$$

where u and v are defined as in § 4. Selecting the negative sign in (15.4)

2709

we obtain, after expanding by the binomial theorem,

$$y = z - z^2 - z^2 \sum_{m=0}^{\infty} \sum_{s=0}^{m} \frac{(m+s)!}{s!\,(s+1)!\,(m-s)!} u^{s+1} v^{-(2m+s+1)} z^m \tag{15.5}$$

$$+ z^2 \sum_{m=0}^{\infty} \sum_{s=0}^{m+1} \frac{(m+s+1)!}{s!\,(s+1)!\,(m-s+1)!} u^{s+2} v^{-(2m+s+2)} z^m.$$

By substituting in (15.2) and applying (4.6), it is possible to obtain expressions for the numbers $\psi_{n,m}$.

In the same way, the numbers of rooted strong triangulations of types $[n, m; r]$ and $[n, m]^-$ could be derived from the functions ${}_rE(x,y)$, $K(x,y)$, and $L(x, y)$. The author first computed the functions ${}_3E(x,0)$ and $J(x,0)$ from equations developed for the purpose of enumerating the various types of strong triangulations; the development of those equations was, however, considerably more complicated than that followed in this paper.

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