CL-Chemy Transforms Fibonacci-Type Sequences to Arrays

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Abstract

The *CL*, or *coefficient list* concept is a natural adjunct to many recursive formulas and algorithms. The basic idea is that coefficients may vary as iteration proceeds. This article explores the effect of CLs on a fascinating family of sequences; namely the close relatives of the eponymous numbers brought to our attention by Leonardo da Pisa (aka *Fibonacci*) circa 1200 CE.

Terms of the Fibonacci sequence will derive, by Binet's formula, from powers of the zeros of $x^2 - x - 1$. By a mathematical rule-of-thumb, patterns spawned by 2nd-degree equations have a 2-dimensional representation. The exposition that follows provides a context in which the familiar Fibonacci sequence is properly thought of as a single row/column of a (degenerate) 2D array.

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Openings to a New Dimension in the Exploration of the Remarkable φ Sequence

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A Derivation of the Limit Ratios of a Generalized Fibonacci Sequence

We begin with a generalized Fibonacci formula

$$(c)F_n + (b)F_{n+1} = F_{n+2} \qquad F_0 = 0,\ F_1 = 1 \tag{1.1}$$

Henceforth (1.1) will be called a *φ-sequence* or φ_1.

Theorem: For b,c in $\mathbb{N}$ ($\mathbb{N}$ = 1,2,3… ∞), the ratio of adjacent terms in φ_1 converges to a real number, x. I.e.,

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = x$$

Proof: In the limit: $x^2 = \frac{F_{n+2}}{F_n} = \frac{(c)F_n + (b)F_{n+1}}{F_n} = c + \frac{(b)F_{n+1}}{F_n} = bx + c$, and $x^2 - bx - c = 0$.

Since the discriminant $b^2 - 4ac$ is always positive (see the quadratic formula in (1.2)), x is real.

QED

E.g., if b = 2 and c = 3, then φ_1 = 0, 1, 2, 7, 20, 61, 182, 547, 1640, 4921, 14762…

To what ratio does this sequence converge? Let $x = r_\pm$. Then by the *quadratic formula*

$$r_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.2}$$

the roots of $x^2 - 2x - 3 = 0$ are $r_+ = 3$ and $r_- = -1$.

Then, in the limit, the ratio of F_n to F_{n-1} is (1.2), the quadratic formula itself.

Note that rearranging (1.1) to $F_n = \frac{F_{n+2} - (b)F_{n+1}}{c}$ generates terms leftward of F_0.

In the above example, terms to the left of F_0 are (moving to the left from zero) 0.333…, –0.222…, 0.259…, –0.247… …This part of the sequence converges to r_-.

For convenience, let $Q = x^2 - bx - c$. Now, by way of *Binet's formula*,

$$\frac{r_+^n - r_-^n}{r_+ - r_-} = F_n \tag{1.3}$$

the zeros (roots) of Q generate the nth term of φ_1.

E.g., take the sequence from above, 0, 1, 2, 7, 20, 61, 182, 547, 1640, 4921, 14762…. The roots of $x^2 - 2x - 3$ are $r_+ = 3$ and $r_- = -1$. Let's say $n = 7$. Then

$$F_7 = \frac{3^7 - (-1)^7}{3 - (-1)} = 547$$

Thus, the characteristics of a φ_1 sequence are encoded in the associated quadratic equation, Q. The patterns associated with a 2nd-degree equation are normally, naturally two-dimensional, yet (1.3) generates only a linear sequence. So these φ_1 sequences are missing a dimension, it seems…

Dynamic Coefficients

Indeed, this 2nd dimension has usually evaded observation. It will be opened to exploration by the use of *coefficient lists* (*CL*s). The idea behind such lists is that the coefficients of variables F_n and F_{n+1} are themselves a sequence, and terms in a list will apply sequentially as iteration proceeds.

A more general version of (1.1) employs two coefficient lists, β and γ.

$$(\gamma)F_n + (\beta)F_{n+1} = F_{n+2} \qquad F_0 = 0,\, F_1 = 1 \tag{1.4}$$

Where β and γ are defined as

$$\beta = [b_1, b_2 \ldots b_i] \; and \; \gamma = [c_1, c_2 \ldots c_j] \tag{1.5}$$

Upon the first iteration of (1.4), b_1 and c_1 apply; on the second, b_2 and c_2 and so forth…

Let the indices i and $j = \lambda$, so β and γ each contain λ terms, where $1 \le \lambda < \infty$. Then λ = the *period* of the sequence φ_λ. Again, as (1.4) is iterated and expanded recursively, the terms in β and γ apply sequentially, and, because $\lambda < \infty$, cyclically as iteration goes on. E.g., for $\lambda \ge 4$:

$$\varphi_4 = 0, 1, b_1, c_2 + b_1b_2, c_3b_1 + c_2b_3 + b_1b_2b_3, c_2c_4 + c_4b_1b_2 + c_3b_1b_4 + c_2b_3b_4 + b_1b_2b_3b_4 \ldots \tag{1.6}$$

Note that β and γ, as defined in (1.5) may have different lengths; i.e., i and j respectively. So, given $i \ne j$, we say that $\lambda = \mathrm{LCM}(i,j)$. This is for economy; else given, say, a φ_6 sequence such as β = [1,2,3], γ = [1,2], it would be necessary to write out β = [1,2,3,1,2,3] and γ = [1,2,1,2,1,2].

Now, for an example of numerical coefficient lists in action, take β [1,2,3] and γ [1] (where, for brevity, the = sign is implied). At the first iteration, the first (and only) term in γ applies to F_0, and the first term in β to F_1: (1)0 + (1)1 = 1 = F_2. On the second iteration, $c = 1$ and $b_2 = 2$ apply to F_1 and F_2 respectively: (1)1 + (2)1 = 3 = F_3. For the third iteration, (1)1 + (3)3 = 10 = F_4, and on the fourth iteration, the cycle starts over. This process generates the sequence below:

φ_3 = 0, 1, 1, 3, 10, 13, 36, 121, 157, 435, 1462, 1897, 5256, 17665, 22921, 63507, 213442…

Convergence to Multiple Limit Ratios

Where the example sequence on page 1, φ_1 = 0, 1, 2, 7, 20, 61, 182, 547, 1640, 4921… converges to the roots of a single equation, the positive (to the rightward of zero) section of φ_3 above converges simultaneously to three ratios. These are, in the limit, the roots $r_{1+} \approx 3.3609$, $r_{2+} \approx 1.2975$ and $r_{3+} \approx 2.7707$, i.e., the positive roots of the quadratic equations Q_1, Q_2 and Q_3. The next step is to fashion a procedure that generates Q_j coefficients in terms of elements in β and γ.

It will be seen that, in general, a sequence φ_λ converges to the roots of λ quadratics, Q_j. The approach to finding the coefficients of these equations will be an extension of the strategy used for φ_1. To this end, (1.4) is expanded below in an indeterminate form.

As just explicated, a sequence φ_λ is generated by applying the terms of β and γ in order as iterations are performed. At the first iteration, the initial terms F_0 and F_1 are multiplied by c_1 and b_1 respectively; at the second, F_1 and F_2 are multiplied by c_2 and b_2; for the third iteration c_3 and b_3 apply and so on. After λ iterations, the cycle repeats, and for $\lambda \geq 2$ the sequence begins like this:

$$\varphi_{\lambda \geq 2} = F_0,\ \ F_1,\ \ (c_1)F_0 + (b_1)F_1 = F_2,\ \ (c_2)F_1 + (b_2)F_2 = F_3 = (b_2c_1)F_0 + (b_1b_2 + c_2)F_1\ldots$$

It so happens that there is sufficient information in any such single sequence to provide the coefficients for all of the associated Q_j. This approach, however, spawns huge, unwieldy equations for even relatively small λ. As an alternative, note that any of the φ_λ has λ–1 ‘siblings’, and such information as they hold is most readily accessed when they are all utilized in tandem to form an *array*. The key to generating the constituent sequences of an array is the cyclical permutation of elements in the CLs β and γ. E.g., take β' [b_2, b_3… b_λ, b_1] and γ' [c_2, c_3… c_λ, c_1], which generates the sequence φ_λ':

$$\varphi_\lambda' = F_0,\ \ F_1,\ \ (c_2)F_0 + (b_2)F_1 = F_2,\ \ (c_3)F_1 + (b_3)F_2 = F_3 = (b_3c_2)F_0 + (b_2\, b_3 + c_3)F_1\ldots$$

Then β'' [b_3, b_4… b_λ, b_1, b_2], γ'' [c_3, c_4… c_λ, c_1, c_2] generates the sequence φ_λ''; β''' [b_4… b_λ, b_1, b_2, b_3], γ''' [c_4… c_λ, c_1, c_2, c_3] generates φ_λ''' and so on.

Such φ_λ sequences aligned as an array will be represented as Φ_λ [b_1, b_2… b_i][c_1, c_2… c_j]. The familial φ_λ sequences such arrays comprise will be notated as S_j. On route to this, though, is a look at ways in which a single φ_λ sequence may be mined for the coefficients of all its associated Qs.

A Single-Sequence Method that Solves for Q_j’s Coefficients

Take for example a φ_2 sequence, β [b_1, b_2], γ [c_1, c_2]. For $\beta, \gamma \in \mathbb{N}$, the ratios of adjacent terms rightward of $F_0 = 0$ converge alternatively to r_{1+} and r_{2+}, the positive roots of the equations Q_1 and Q_2 respectively. To find the coefficients of Q_1, we take four consecutive terms of φ_2:

$$F_n,\, F_{n+1},\, F_{n+2} = (c_1)F_n + (b_1)F_{n+1},\, F_{n+3} = (b_2c_1)F_n + (b_1b_2 + c_2)F_{n+1}$$

Then in the limit as $n \to \infty$: $\dfrac{F_{n+1}}{F_n} = \ x_1 = \dfrac{F_{n+3}}{F_{n+2}}$ and $x_1 \cdot F_{n+2} = F_{n+3}$

Multiplication by x_1 gives $x_1^2 \cdot F_{n+2} = x_1 \cdot F_{n+3}$

Or, in terms of F_n and F_{n+1}: $(c_1)F_n \cdot x_1^2 + (b_1)F_{n+1} \cdot x_1^2 = (b_2c_1)F_n \cdot x_1 + (b_1b_2 + c_2)F_{n+1} \cdot x_1$

But $F_n = \dfrac{F_{n+1}}{x_1}$, so $(c_1)F_{n+1} \cdot x_1 + (b_1)F_{n+1} \cdot x_1^2 = (b_2c_1)F_{n+1} + (b_1b_2 + c_2)F_{n+1} \cdot x_1$

Then dividing by F_{n+1} and collecting terms;

$$Q_1 = b_1x_1^2 - (b_1b_2 - c_1 + c_2)x_1 - b_2c_2 = 0 \qquad (1.7)$$

For a numerical example, take Φ_2 [1,2][1]. Iterating per the CL definition gives the sequence:

S_1 = 0, 1, 1, 3, 4, 11, 15, 41, 56, 153, 209, 571, 780, 2131…

Solve F_{11}/F_{10} = 571/209 for the ratio 2.7321… Now, substituting terms from β[1,2], γ[1] into (1.7) gives $Q_1 = 1x_1^2 - (1 \cdot 2 - 1 + 1)x_1 - 2,$ which equals $x_1^2 - 2x_1 - 2$, and $r_{1+} \approx 2.7321$.

The coefficients of Q_2 may then be found by cyclically permuting the indices of the coefficients of Q_1 in (1.7). Another alternative is to start from the beginning, i.e., perform the above derivation of the coefficients with the terms in β and γ permuted, as shown below. Obviously this last method takes more work:

$$\begin{aligned}
&F_n \\
&F_{n+1} \\
&F_{n+2} = (c_2)F_n + (b_2)F_{n+1} \\
&F_{n+3} = (b_1c_2)F_n + (b_1b_2 + c_1)F_{n+1}
\end{aligned}$$

$Q_2 = b_2x_2^2 - (b_1b_2 - c_2 + c_1)x_2 - b_1c_2$ results from the methods outlined above. Substituting from β [1,2], γ[1] gives $2x_2^2 - 2x_2 - 1$, and $r_{2+} \approx F_{12}/F_{11}$ = 780/571 ≈ 1.3660.

In the above, an equation that solves for the roots of Q_1 derives from setting x_1 equal to two fractions expressed indeterminately in terms of φ_2. Cyclical permutation of Q_1's indices then provides the coefficients for Q_2.

Next, for any λ, a general method for expressing the coefficients of Q_1 in terms of the elements in β and γ is described in the following steps:

- Beginning at F_n, take λ + 2 consecutive terms of φ_λ: F_n, F_{n+1}… $F_{n+\lambda}$, $F_{n+\lambda+1}$
- Then set $\dfrac{F_{n+1}}{F_n} = x_1 = \dfrac{F_{n+\lambda+1}}{F_{n+\lambda}}$
- Clear the second fraction and express $F_{n+\lambda}$ and $F_{n+\lambda+1}$ in terms of F_n and F_{n+1}
- Multiply both sides by x_1. Now, because $F_n = \dfrac{F_{n+1}}{x_1}$, F_n can be transformed to F_{n+1}
- Dividing out F_{n+1} then leaves a 2nd degree equation in x_1, with coefficients that are expressed as terms from β and γ

A derivation of Q_1 for a φ_3 sequence $\beta\,[b_1, b_2, b_3]$, $\gamma\,[c_1, c_2, c_3]$ by this regime is illustrative. First, the sequence is expanded out to five terms:

$$
\begin{aligned}
&F_n \\
&F_{n+1} \\
&F_{n+2} = (c_1)F_n + (b_1)F_{n+1} \\
&F_{n+3} = (b_2c_1)F_n + (b_1b_2 + c_2)F_{n+1} \\
&F_{n+4} = (b_2b_3c_1 + c_1c_3)F_n + (b_1b_2b_3 + b_1c_3 + b_3c_2)F_{n+1}
\end{aligned}
$$

Next the fraction in $x_1 = \dfrac{F_{n+4}}{F_{n+3}}$ is cleared; $x_1 \cdot F_{n+3} = F_{n+4}$ is then expressed in terms of F_n and F_{n+1}, and both sides are multiplied by x_1:

$$(b_2c_1)F_n \cdot x_1^2 + (b_1b_2 + c_2)F_{n+1} \cdot x_1^2 = (b_2b_3c_1 + c_1c_3)F_n \cdot x_1 + (b_1b_2b_3 + b_1c_3 + b_3c_2)F_{n+1} \cdot x_1 \tag{1.8}$$

Substitute $F_n = \dfrac{F_{n+1}}{x_1}$ in the above to eliminate F_n, divide by F_{n+1} and collect the terms:

$$Q_1 = (b_1b_2 + c_2)x_1^2 - (b_1b_2b_3 + b_1c_3 - b_2c_1 + b_3c_2)x_1 - (b_2b_3c_1 + c_1c_3) = 0 \tag{1.9}$$

Then for $\beta[1,2,3]$ and $\gamma[1,2,3]$, $Q_1 = 4x_1^2 - 13x_1 - 9$. Cyclical permutation of the terms in β and γ, (or indices of Q_1 in (1.9)) gives $Q_2 = 9x_1^2 - 5x_1 - 8$ and $Q_3 = 4x_1^2 - 11x_1 - 12$. So, the sequence S_1 = 0, 1, 1, 4, 15, 19, 68, 261… converges to the roots $r_{1+} \approx 3.8364$, $r_{2+} \approx 1.2607$ and $r_{3+} \approx 3.5865$.

As a practical matter, this method is impossibly cumbersome for large λ: e.g., a $\lambda = 10$ equation has about 1600 β and γ terms in its coefficients. This difficulty however is to a great extent mitigated in the context of numerical arrays, in which burdensome symbolic expressions are reduced to more compact and manageable terms. The symbols-to-numbers transition is described below.

A General Quadratic in x_j Solves for All Limit Ratios in Φ_λ

To show how the process of deriving coefficients may be simplified, a set of φ_3 sequences will be aligned as an array. Where the symbolic version of the φ_3 sequence in (1.8) provided components for the coefficients of Q_1 in (1.9), now set $F_n = F_0$ to construct a more specific array.

Once the sequence S_1 is established, cycling the indices of the coefficients gives S_2 and S_3.

Table I

$\Phi_3\,[b_1, b_2, b_3][c_1, c_2, c_3]$

	S_1	S_2	S_3
F_0 =	F_0	F_0	F_0
F_1 =	F_1	F_1	F_1
F_2 =	$(c_1)F_0 + (b_1)F_1$	$(c_2)F_0 + (b_2)F_1$	…
F_3 =	$(b_2c_1)F_0 + (b_1b_2 + c_2)F_1$	$(b_3c_2)F_0 + (b_2b_3 + c_3)F_1$	…
F_4 =	$(b_2b_3c_1 + c_1c_3)F_0 + (b_1b_2b_3 + b_1c_3 + b_3c_2)F_1$	$(b_1b_3c_2 + c_1c_2)F_0 + (b_1b_2b_3 + b_1c_3 + b_2c_1)F_1$	…

Multiplying by $F_0 = 0$ and $F_1 = 1$ simplifies the situation considerably, but at the cost of causing some important information to disappear.

Table 2

	S_1	S_2	S_3
F_1	1	1	1
F_2	b_1	b_2	b_3
F_3	$b_1b_2 + c_2$	$b_2b_3 + c_3$	$b_1b_3 + c_1$
F_4	$b_1b_2b_3 + b_1c_3 + b_3c_2$	$b_1b_2b_3 + b_2c_1 + b_1c_3$	$b_1b_2b_3 + b_3c_2 + b_2c_1$

A comparison of the two tables shows that the zeroing out process has resulted in a substantial loss of information, the importance of which will be emphasized below. As a first step, recall that coefficients in (1.9) were found by setting F_4 over F_3, which can be written like so:

$$x_1 = \frac{(b_2b_3c_1 + c_1c_3)F_n + (b_1b_2b_3 + b_1c_3 + b_3c_2)F_{n+1}}{(b_2c_1)F_n + (b_1b_2 + c_2)F_{n+1}} \tag{1.10}$$

The information in (1.10) is that which was manipulated to produce the formula in (1.9). However, in numerical expressions of φ_λ, the initial terms as defined in (1.4) are always $F_0 = 0$ and $F_1 = 1$. Thus when $F_n = F_0$, the terms (b_2c_1) and $(b_2b_3c_1 + c_1c_3)$ in the equation above will vanish, and so (1.9) would now appear as below:

$$Q_1 = (b_1b_2 + c_2)x_1^2 - (b_1b_2b_3 + b_1c_3 + b_3c_2)x_1 = 0 \tag{1.11}$$

(1.11) solves for F_4/F_3, but not for the numbers to which S_1 in Φ_3 will eventually converge. Yet reference to table 2 shows that, serendipitously, terms zeroed from S_1 are available in the adjacent column S_2, although each lacks their common factor c_1. The strategy then is to construct a formula (expressed as an equation) that restores (1.11) to (1.9).

The self-similarity inherent in these arrays allows a generalized formula to apply to every column in Φ_λ. We'll see later that this generality extends to rows (periodically) as well. For now, we'll reference the array in table 2 and derive a general formula from that.

Some additional notation facilitates expression of the formula, so let the initial index of $F_{i,\,j}$ (where $F_{i,\,j}$ is an element of an array Φ_λ) represent its row and the second its column. Then let F_λ denote the 'baseline' of Φ_λ; that is, the row of number coincident with the period length λ. The coefficients a, b and c of Q_j are now expressed in terms from S_j and S_{j+1} in the following manner:

$$a = F_{\lambda\,,\,j} \;\; ; \quad b = (F_{\lambda+1\,,\,j} - c_j \cdot F_{\lambda-1\,,\,j+1}) \;\; ; \quad c = -\,c_j \cdot F_{\lambda\,,\,j+1} \tag{1.12}$$

Referencing (1.11) vis-à-vis (1.9), nothing was lost from the a coefficient, so in (1.12) it remains unchanged. The c coefficient of (1.9), missing entirely in (1.11), is retrieved from the next column (S_{j+1}) on the same row. Then still in that same column, but going back a row, we find the fragment that completes b. The new equation looks like this:

$$Q_j = F_{\lambda,j} \cdot x_j^2 - (F_{\lambda+1,j} - c_j \cdot F_{\lambda-1,j+1})x_j - c_j \cdot F_{\lambda,j+1} \tag{1.13}$$

For verification, (1.13) applied to S_1 and S_2 in table 2 returns the equation in (1.9):

$$Q_1 = (b_1b_2 + c_2)x_1^2 - (b_1b_2b_3 + b_1c_3 - b_2c_1 + b_3c_2)x_1 - (b_2b_3c_1 + c_1c_3)$$

Extraction of Q_j Coefficients from Numerical Arrays

If Φ_3 is generated in numerical form, though, then this clutter of symbols disappears. E.g., take the array Φ_3 [1,2,3][1,2,3] in table 3 below.

Table 3

Φ_3 [1,2,3][1,2,3]	S_1	S_2	S_3	S_1	S_2	S_3	S_1	S_2	…
F_0	0	0	0	0	0	0	0	0	
F_1	1	1	1	1	1	1	1	1	
F_2	1	2	3	1	2	3	1	2	
F_λ	4	9	4	4	9	4	4	9	
F_4	15	11	14	15	11	14	15	11	
F_5	19	40	54	19	40	54	19	40	
$F_{2\lambda}$	68	153	68	68	153	68	68	153	
F_7	261	193	244	261	193	244	261	193	
F_8	329	692	936	329	692	936	329	692	

The arrows connect the terms of Φ_3 that are needed to construct the coefficients of Q_1 according to (1.12). Obviously there is information in table 2 well beyond that required to find the coefficients of Q_1, Q_2 and Q_3. This is due partially to redundancy, as columns repeat in a cycle of λ. Since the second index of $F_{i,j}$ cannot exceed λ, we can forego this needless repetition. As for the rows, the equation in (1.13) requires that $F_{\lambda+1,\,j}$ be available, but nothing beyond it. So, in this λ_3 example, we need only nine numbers beyond the initial zeros and ones, as seen below:

$$Q_j = F_{\lambda\,,\,j} \cdot x_j^2 - (F_{\lambda+1,\,j} - c_j \cdot F_{\lambda-1,\,j+1})x_j - c_j \cdot F_{\lambda,\,j+1}$$

Φ_3 [1,2,3][1,2,3]	S_1	S_2	S_3
F_2	1	2	3
F_λ	4	9	4
F_4	15	11	14

$$Q_1 = 4x_1^2 - (15 - 1\cdot 2)x_1 - 1\cdot 9 = 4x_1^2 - 13x_1 - 9$$

$$Q_2 = 9x_2^2 - (11 - 2\cdot 3)x_2 - 2\cdot 4 = 9x_2^2 - 5x_2 - 8$$

$$Q_3 = 4x_3^2 - (14 - 3\cdot 1)x_3 - 3\cdot 4 = 4x_3^2 - 11x_3 - 12$$

Applying (1.13) to Φ_3 [1,2,3][1,2,3] above gives the equations Q_1, Q_2 and Q_3. The sequence 0, 1, 1, 4, 15, 19, 68, 261… therefore converges to the positive roots $r_{1+} \approx 3.8364$, $r_{2+} \approx 1.2607$ and $r_{3+} \approx 3.5865$, as do sequences S_2 and S_3. These Q_j are the same equations that were stated earlier, just below the equation in (1.9), verifying that the single sequence and numerical array methods provide identical results.

Note that equations generated in this way share a single discriminant (D), where $D = b^2 - 4ac$. In the case just above, $D = 13^2 - 4 \cdot -9 = 5^2 - 9 \cdot -8 = 11^2 - 4 \cdot -12 = 313$. This will be of interest later on.

Writer's note:

Steps involved in generalizing φ-sequences to arrays have, so far, been mostly empirical; picking out patterns by inspection and trying to describe them in terms of algorithms and equations. Up to now, it's been obvious and easy, and a rigorous approach to the material would probably not require much more effort or time. But it's about to become more complicated, and it may present some challenges to describe what follows in a formal style of theorems and proofs. Or maybe not. This article will continue with the 'heuristic' approach to identifying patterns. Anyone who discovers something of interest here is, of course, welcome to try to tighten it up...

Generalizations of Q_j

The equation for Q_j in the form of (1.13) is a useful tool for delving into patterns that emerge as Fibonacci-type sequences with $\lambda > 1$ are expanded into two dimensions. Yet there are expressions of (1.13) that have greater generality and power. These versions have qualities that are requisite for certain applications, such as the 2D version of Binet's formula that appears in (1.17). Derivations of these generalized formulas are presented below.

First up is to enhance our notation. Let $Q = ax^2 + bx + c = 0$ and we seek to define, for k an integer, the symbol Q^k. To this end, consider $r_\pm$, the roots of Q, in their quadratic formula representation.

$$r_+ = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } r_- = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Then, since $Q = x^2 - (r_+ + r_-)x + (r_+ \cdot r_-)$, we say $Q^{-1} = x^2 - (r_+^{-1} + r_-^{-1})x + (r_+^{-1} \cdot r_-^{-1})$

Thus if $r_+^{-1} = \dfrac{2a}{-b+\sqrt{b^2-4ac}}$ and $r_-^{-1} = \dfrac{2a}{-b-\sqrt{b^2-4ac}}$

then $r_+^{-1} + r_-^{-1} = \dfrac{-4ab}{4ac}$ and $r_+^{-1} \cdot r_-^{-1} = \dfrac{4a^2}{4ac}$. Hence

$$Q^{-1} = x^2 - (\frac{-4ab}{4ac})x + \frac{4a^2}{4ac} = 4acx^2 + 4abx + 4a^2 = cx^2 + bx + a$$

Also, $Q \cdot Q^{-1} = Q^0 = x^2 - (r_+ \cdot r_+^{-1} + r_- \cdot r_-^{-1})x + (r_+ \cdot r_+^{-1} \cdot r_- \cdot r_-^{-1}) = x^2 - 2x + 1$

In general, $Q^k = x^2 - (r_+^k + r_-^k)x + (r_+^k \cdot r_-^k)$. We'll walk through Q^2 for an example.

$$r_+^2 = \frac{b^2 - 2ac - b\sqrt{b^2-4ac}}{2a^2} \quad \text{and} \quad r_-^2 = \frac{b^2 - 2ac + b\sqrt{b^2-4ac}}{2a^2}\text{, so}$$

$$Q^2 = x^2 - (r_+^2 + r_-^2)x + (r_+^2 \cdot r_-^2) = x^2 - \frac{(b^2-2ac)}{a^2}x + \frac{c^2}{a^2} = a^2x^2 - (b^2 - 2ac)x + c^2$$

For instance, if $Q = x^2 - 2x - 3$, then $Q^2 = x^2 - 10x + 9$.

Note then, that for any k the radical vanishes, and the coefficients of Q^k will be integers. Now consider two distinct equations, Q and Q'. We will soon see, in numerous examples, that it is possible for the equation below to also have integer coefficients.

$$Q'' = Q \cdot Q' = x^2 - (r_+ \cdot r_+' + r_- \cdot r_-')x + (r_+ \cdot r_+' \cdot r_- \cdot r_-')$$

A seemingly necessary, but apparently not sufficient condition on this is that the discriminants of Q and Q' are of equal value (or one is an n^2 multiple of the other, $n = 1,2,3\ldots$). That said, take the formula in (1.1) and set $b = c = 1$. Iteration in both directions yields the familiar Fibonacci sequence.

…–21, 13, –8, 5, –3, 2, –1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21…

This is of course, *the* Fibonacci sequence, (call it ϕ), which converges, leftward and rightward to the roots of $Q_\phi = x^2 - x - 1 = 0$. The positive root of this equation (or its inverse) is often called the divine proportion or the golden mean: $r_{\phi+} = 1.618034\ldots$, $r_{\phi-} = -0.618034\ldots$ The relationship of the roots of Q_ϕ to its b and c coefficients is $r_{\phi+} + r_{\phi-} = -b$ and $r_{\phi+} \cdot r_{\phi-} = c$.

Consequently, equations Q_ϕ^k with integral coefficients are formed on the roots $r_{\phi\pm}^k$ by

$$Q_\phi^k = x^2 - (r_{\phi+}^k + r_{\phi-}^k)x \ + \ r_{\phi+}^k \cdot r_{\phi-}^k$$

But $r_{\phi+}^n + r_{\phi-}^n$ is also a formula for L_n, which is the n^{th} term of the Lucas sequence, where

$$(c)L_n + (b)L_{n+1} = L_{n+2}; \quad L_0 = 2, L_1 = 1$$

Therefore, for $b = c = 1$, the Lucas numbers 2, 1, 3, 4, 7, 11, 18, 29, 47… are the b coefficients of successive Q_ϕ^k, $k \geq 0$. But the Lucas sequence can also be stated in terms of Fibonacci numbers as $L_n = F_{n-1} + F_{n+1}$. With that in mind, the array Φ_ϕ [1][1] is constructed on ϕ below.

Note: the patterns that associated adjacent table 3 elements to form the coefficients of Q_j in (1.13) are, as indicated by the arrows, extended throughout this array.

Table 4

Φ_ϕ [1][1]	S_1	S_2	S_3	S_4	S_5	S_6	…
…							
F_{-4}	-3	-3	-3	-3	-3	-3	
F_{-3}	2	2	2	2	2	2	
F_{-2}	-1	-1	-1	-1	-1	-1	
F_{-1}	1	1	1	1	1	1	
F_0	0	0	0	0	0	0	
F_λ	1	1	1	1	1	1	
F_2	1 ↓	1	1	1	1	1	
F_3	2 ↓	2	2	2	2	2	
F_4	3 ↓	3	3	3	3	3	
F_5	5 ↓	5	5	5	5	5	
F_6	8 ↓	8	8	8	8	8	
F_7	13 ↓	13	13	13	13	13	

The equalities in (1.12) are now modified to form coefficients for Q_ϕ^k from terms in Φ_ϕ:

$$a = F_{1,1};\ b = -(F_{1+k,1} + F_{1-k,1+k} \cdot (-1)^k\ );\ c = F_{1,1+k} \cdot (-1)^k$$

$$Q_\phi^k = F_{1,1}x^2 - (F_{1+k,1} + F_{1-k,1+k} \cdot (-1)^k\ )x + F_{1,1+k} \cdot (-1)^k \tag{1.14}$$

According to the patterns in Φ_ϕ, the a coefficient is fixed. As k increases, the first term of b moves downward through the first column as the second term moves up and rightward on a diagonal, while

c slides across the columns on the first row. From (1.14), which embodies these patterns, the coefficients of Q_ϕ^k for the first few values of k are given below.

$$
\begin{aligned}
Q_\phi^0 &= 1x^2 - (1+1(-1)^0)x + 1(-1)^0 = x^2 - 2x + 1 \\
Q_\phi^1 &= 1x^2 - (1+0(-1)^1)x + 1(-1)^1 = x^2 - x - 1 \\
Q_\phi^2 &= 1x^2 - (2+1(-1)^2)x + 1(-1)^2 = x^2 - 3x + 1 \\
Q_\phi^3 &= 1x^2 - (3-1(-1)^3)x + 1(-1)^3 = x^2 - 4x - 1 \\
Q_\phi^4 &= 1x^2 - (5+2(-1)^4)x + 1(-1)^4 = x^2 - 7x + 1 \\
Q_\phi^5 &= 1x^2 - (8-3(-1)^5)x + 1(-1)^5 = x^2 - 11x - 1
\end{aligned}
$$

Further Generalizations of Q_J

This shows that the array Φ_ϕ contains the information necessary to provide the coefficients for the equations with roots that are powers of $r_{\phi+}$. This extended pattern also applies to higher order arrays, but to navigate those realms requires modifying the notation again. To this end, the symbol for the exponent is now enhanced.

By convention, B^k represents B to the power of k. For k a positive integer, this is a simple and convenient product notation; e.g., $B^3 = B \cdot B \cdot B$. We will say in this case that the base B is *monotonic*. Now an underline to $\underline{k}$ in $B^{\underline{k}}$ will be taken to signify that B is *polytonic*, i.e., it comprises two or more elements (not necessarily all distinct), say, $B = [t_1, t_2 \ldots t_\lambda]$.

A property of B is that any term may be the initial, as in $t_j^{\underline{k}}$. For example, take $B = \beta = [b_1, b_2, b_3]$; then $b_2^{\underline{5}} = b_2 \cdot b_3 \cdot b_1 \cdot b_2 \cdot b_3$. (A potential problem with the underline arises in this example, as the superscript and subscript in $b_2^{\underline{5}}$ may seem to combine to form a fraction. Bear in mind that fractional exponents will not appear in this context; but if they were necessary, the somewhat clumsy but still intelligible $b_j^{\underline{y/z}}$ could be used.)

For further examples, if $\beta\,[1,2,3]$ then, $b_2^{\underline{5}} = 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 = 36$. For $\Phi_3\,[1,2,3][2]$ then $c_2^{\underline{7}} = 2^7$. It could be said that in general any base is polytonic, but when each $t_i = t_j$, then $\underline{k} = k$. Note that for a base B of λ elements, the product $t_j^{\underline{\lambda}}$ is the same regardless of which element is chosen as the initial; i.e., $t_1^{\underline{\lambda}} = t_2^{\underline{\lambda}} = t_3^{\underline{\lambda}} \ldots = t_\lambda^{\underline{\lambda}}$.

Problem: Define the exponent $\underline{k}$ for negative integer values

Having derived (1.14) empirically from table 4, we go on to devise a more general expression for a period-λ pattern, $\Phi_\lambda\,[b_1, b_2 \ldots b_\lambda][1]$. Beyond underlining exponents, it's necessary to ensure that the coefficients are taken from the proper rows. This entails that F's initial subscript be equal to the period λ; thus every coefficient in the new formula will preface as F_λ, as in, say, $F_{\lambda-k,j+k}$.

With this established, the pattern used in Φ_ϕ (table 4) applies now to the array $\Phi_3\,[1,2,3][1]$, in table 5 below:

Table 5

Φ_3 [1,2,3][1]	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
…								
F_{-4}	-11	-10	-9	-11	-10	-9	-11	-10
$F_{-\lambda}$	3	7	4	3	7	4	3	7
F_{-2}	-2	-3	-1	-2	-3	-1	-2	-3
F_{-1}	1	1	1	1	1	1	1	1
F_0	0	0	0	0	0	0	0	0
F_1	1	1	1	1	1	1	1	1
F_2	1	2	3	1	2	3	1	2
F_λ	3	7	4	3	7	4	3	7
F_4	10	9	11	10	9	11	10	9
F_5	13	25	37	13	25	37	13	25
$F_{2\lambda}$	36	84	48	36	84	48	36	84
F_7	121	109	133	121	109	133	121	109
F_8	157	302	447	157	302	447	157	302

Define Q_j^k as the equation that has its *a* coefficient in the column S_j, and the roots of which are the product $r_j \cdot r_{j+1} \cdot \ldots \cdot r_{j+k-1} = r_j^k$ for r_{j+} and r_{j-}.

The equation in (1.14) is generalized so as to derive the coefficients of Q_j^k from Φ_λ:

$$Q_j^k = F_{\lambda,j} \cdot x^2 - (F_{\lambda+k,j} + F_{\lambda-k,j+k} \cdot (-1)^k)x + F_{\lambda,j+k} \cdot (-1)^k \tag{1.15}$$

The formula in (1.15) is now applied to the Φ_3 array above, and the coefficients of Q_1^k for a few sequential values of *k* are shown below.

$$\begin{aligned}
Q_1^0 &= 3x_1^2 - (3+3(-1)^0)x_1 + 3(-1)^0 &&= x_1^2 - 2x_1 + 1 \\
Q_1^1 &= 3x_1^2 - (10+2(-1)^1)x_1 + 7(-1)^1 &&= 3x_1^2 - 8x_1 - 7 \\
Q_1^2 &= 3x_1^2 - (13+1(-1)^2)x_1 + 4(-1)^2 &&= 3x_1^2 - 14x_1 + 4 \\
Q_1^\lambda &= 3x_1^2 - (36+0(-1)^3)x_1 + 3(-1)^3 &&= x_1^2 - 12x_1 - 1 \\
Q_1^4 &= 3x_1^2 - (121+1(-1)^4)x_1 + 7(-1)^4 &&= 3x_1^2 - 122x_1 + 7 \\
Q_1^5 &= 3x_1^2 - (157-1(-1)^5)x_1 + 4(-1)^5 &&= 3x_1^2 - 158x_1 - 4
\end{aligned}$$

Moving over to the column S_2 gives the coefficients for Q_2^k.

$$\begin{aligned}
Q_2^0 &= 7x_2^2 - (7+7(-1)^0)x_2 + 7(-1)^0 &&= x_2^2 - 2x_2 + 1 \\
Q_2^1 &= 7x_2^2 - (9+3(-1)^1)x_2 + 4(-1)^1 &&= 7x_2^2 - 6x_2 - 4 \\
Q_2^2 &= 7x_2^2 - (25+1(-1)^2)x_2 + 3(-1)^2 &&= 7x_2^2 - 26x_2 + 3 \\
Q_2^\lambda &= 7x_2^2 - (84+0(-1)^3)x_2 + 7(-1)^3 &&= x_2^2 - 12x_2 - 1 \\
Q_2^4 &= 7x_2^2 - (109+1(-1)^4)x_2 + 4(-1)^4 &&= 7x_2^2 - 110x_2 + 4 \\
Q_2^5 &= 7x_2^2 - (302-2(-1)^5)x_2 + 3(-1)^5 &&= 7x_2^2 - 304x_2 - 3
\end{aligned}$$

A shift to S_3 gives the coefficients for Q_3^k.

$$\begin{aligned}
Q_3^0 &= 4x_3^2-(4+4(-1)^0)x_3+4(-1)^0 &&= x_3^2-2x_3+1\\
Q_3^1 &= 4x_3^2-(11+1(-1)^1)x_3+3(-1)^1 &&= 4x_3^2-10x_3-3\\
Q_3^2 &= 4x_3^2-(37+1(-1)^2)x_3+7(-1)^2 &&= 4x_3^2-38x_3+7\\
Q_3^\lambda &= 4x_3^2-(48+0(-1)^3)x_3+4(-1)^3 &&= x_3^2-12x_3-1\\
Q_3^4 &= 4x_3^2-(133+1(-1)^4)x_3+3(-1)^4 &&= 4x_3^2-134x_3+3\\
Q_3^5 &= 4x_3^2-(447-3(-1)^5)x_3+7(-1)^5 &&= 4x_3^2-450x_3-7
\end{aligned}$$

To reiterate, where the roots of Q_ϕ^k were powers of $r_{\phi\,\pm}$, the roots of Q_j^k here are products of some or all of three limit ratios in Φ_3 [1,2,3][1]. More examples will help to clarify this idea.

Given S_j in Φ_λ, then $\dfrac{F_{n\lambda+1,\,j}}{F_{n\lambda,\,j}} \to r_j$; $\dfrac{F_{n\lambda+2,\,j}}{F_{n\lambda+1,\,j}} \to r_{j+1}$; $\dfrac{F_{n\lambda+\lambda,\,j}}{F_{n\lambda+\lambda-1,\,j}} \to r_{j+1}$ as $n \to \infty$.

In Φ_3 [1,2,3][1], S_1 = 0, 1, 1, 3, 10, 13, 36, 121, 157, 435, 1462, 1897, 5256, 17665…

One expression of $\dfrac{F_{2\lambda+1,1}}{F_{2\lambda,1}}$ is $\dfrac{17665}{5256}$, which ≈ r_{1+} of $Q_1 = 3x_1^2-8x_1-7 \approx 3.36092$

$$\frac{F_{2\lambda+2,1}}{F_{2\lambda+1,1}} = \frac{1897}{1462} \approx r_{2+} \text{ of } Q_2 = 7x_2^2-6x_2-4 \approx 1.29754$$

$$\frac{F_{2\lambda+3,1}}{F_{2\lambda+2,1}} = \frac{5256}{1897} \approx r_{3+} \text{ of } Q_3 = 4x_3^2-10x_3-3 \approx 2.77069$$

And $r_{1+} \cdot r_{2+} \cdot r_{3+} = r_{j+}^{\lambda} \approx 12.08276\ldots$, a root of $Q_j^\lambda = x^2-12x-1$.

A look at some discriminants of equations from the array Φ_3 [1,2,3][1]:

Q_j^k	$D =$	Q_j^k	$D =$
$Q_1^2 = 3x^2-14x+4$	148	$Q_2^4 = 7x_2^2-110x_2+4$	148 * 9^2
$Q_1^3 = x_1^2-12x_1-1$	148	$Q_2^5 = 7x_2^2-304x_2-3$	148 * 25^2
$Q_1^4 = 3x^2-122x+7$	148 * 10^2	$Q_3^2 = 4x_3^2-38x_3+7$	148 * 3^2
$Q_1^5 = 3x_1^2-158x_1-4$	148 * 13^2	$Q_3^3 = x_3^2-12x_3-1$	148
$Q_2^2 = 7x_2^2-26x_2+3$	148 * 2^2	$Q_3^4 = 4x^2-134x+3$	148 * 25^2
$Q_2^3 = x_2^2-12x_2-1$	148	$Q_3^5 = 4x^2-450x-7$	148 * 37^2

The basic structure of these Φ_λ arrays is established; yet they have intriguing properties that remain to be explored. More on this is to follow in other articles. At this point, after all the effort to forge our way into this new territory, suppose we just look around for a while, to try to get a feel for the lay of the land.

Fibonacci Identities and Binet's Formula in 2D

First up, we'll examine some of the *Fibonacci identities*. These are formulas that equate various terms of the ϕ-sequence in diverse and often unexpected ways. What happens to such an identity in Φ_λ? Let's check out some examples…

(i) $F_n^2 + F_{n+1}^2 = F_{2n+1} \;\rightarrow\; F_{n\lambda,\,j} \cdot F_{n\lambda,\,j+1} \cdot c_j + F_{n\lambda+1,\,j}^2 = F_{2n\lambda+1,\,j}$ (where $\rightarrow$ is 'maps to')

(ii) $F_n \cdot (F_{n+1} + F_{n-1}) = F_{2n} \rightarrow F_{n\lambda,\,j} \cdot \;(F_{n\lambda+1,\,j} + F_{n\lambda-1,\,j+1} \cdot \; c_j) = F_{2n\lambda,\,j}$

(iii) $F_n \cdot F_{n+1} - F_{n-1} \cdot F_{n-2} = F_{2n-1} \rightarrow \dfrac{F_{n\lambda,\,j} \cdot F_{n\lambda+1,\,j+1} - F_{n\lambda-1,\,j+1} \cdot F_{n\lambda-2,\,j+2} \cdot c_j^2}{b_j} = F_{2n\lambda-1,\,j+1}$

These formulas for $F_{n>0}$ are verified using the array in table 6 below.

Table 6

Φ_3 [1,1,2,3][1,2]	S_1	S_2	S_3	S_4
F_0	0	0	0	0
	1	1	1	1
	1	1	2	3
	3	3	8	4
F_λ	7	11	10	10
	27	14	26	24
	34	36	62	92
	88	86	238	116
$F_{2\lambda}$	210	330	300	300
	806	416	776	716
	1016	1076	1852	2748
	2628	2568	7108	3464
$F_{3\lambda}$	6272	9856	8960	8960
	24072	12424	23176	21384
	30344	32136	55312	82072
	78488	76696	212288	103456
$F_{4\lambda}$	187320	294360	267600	267600
	718936	371056	692176	638656

Beginning with (i): If $F_{n\lambda\,,\,j}$ = $F_{\lambda\,,1}$; then $7 \cdot 11 \cdot 1 + 27^2 = 806 = F_{2\lambda+1,1}$
$F_{\lambda\,,2}$; then $11 \cdot 10 \cdot 2 + 14^2 = 416 = F_{2\lambda+1,2}$
$F_{2\lambda\,,3}$; then $300 \cdot 300 \cdot 1 + 776^2 = 692176 = F_{4\lambda+1,3}$
$F_{2\lambda\,,4}$; then $300 \cdot 210 \cdot 2 + 716^2 = 638656 = F_{4\lambda+1,4}$

For identity (ii): If $F_{n\lambda\,,\,j}$ = $F_{\lambda\,,1}$; then $7 \cdot 27 + 7 \cdot 3 \cdot 1 = 210 = F_{2\lambda\,,1}$
$F_{\lambda\,,2}$; then $11 \cdot 14 + 11 \cdot 8 \cdot 2 = 330 = F_{2\lambda\,,2}$
$F_{2\lambda\,,3}$; then $300 \cdot 776 + 300 \cdot 116 \cdot 1 = 267600 = F_{4\lambda\,,3}$
$F_{2\lambda\,,4}$; then $300 \cdot 716 + 300 \cdot 88 \cdot 2 = 267600 = F_{4\lambda\,,4}$

Then, for (iii): If $F_{n\lambda,j}$ = $F_{\lambda,1}$; then $(7 \cdot 14 - 3 \cdot 2 \cdot (1 \cdot 2))/1 = 86 = F_{2\lambda-1,2}$
$F_{\lambda,2}$; then $(11 \cdot 26 - 8 \cdot 3 \cdot (2 \cdot 1))/1 = 238 = F_{2\lambda-1,3}$
$F_{2\lambda,4}$; then $(300 \cdot 806 - 88 \cdot 36 \cdot (2 \cdot 1))/3 = 78488 = F_{4\lambda-1,1}$

(What happens if these Φ_λ formulas are applied over $F_{n<0}$? A worksheet is provided on page 20 for those who wish to investigate.)

There is another way to verify these formulas that is rather interesting. Take this version of the sequence φ_1:

$$(-1)F_n + (2)F_{n+1} = F_{n+2} \qquad F_0 = y,\ F_1 = y+1 \tag{1.16}$$

For y an integer, working in both directions generates $\mathbb{Z}$, the ring of integers; i.e.:

$$\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \ldots$$

If we construct the array $\Phi_{(z)}$ [2][–1] on (1.16) sequences, then the generalized identities i, ii, and iii above reduce to ordinary algebra:

(i) $\rightarrow (-1)y^2 + (y+1)^2 = 2y + 1$
(ii) $\rightarrow y((y+1) + (-1)(y-1)) = 2y$
(iii) $\rightarrow (y(y+1) - (-1)^2(y-1)(y-2))/2 = 2y - 1$

Finally, consider a version of Binet's φ_1 formula (1.3) generalized for φ_λ:

$$F_{\lambda,j} \cdot \frac{r_{j+}^{n} - r_{j-}^{n}}{r_{j+}^{\lambda} - r_{j-}^{\lambda}} = F_{n,j} \tag{1.17}$$

Recall that $F_{\lambda,j}$ is the a coefficient of Q_j^n. We'll verify this formula with an example. Suppose that $F_{7,1}$ (i.e., the 7th term of S_1) in Φ_3 [1,2,3][1,2,3] is to be determined. Then, since $7 = 2\lambda + 1$, only the roots of Q_1 and Q_j^λ are needed. The truncated array

Φ_3 [1,2,3][1,2,3]

…	S_1	S_2	S_3
	1	2	3
F_λ	4	9	4
	15	11	14

supplies the numbers to plug into the generalized formula

$$Q_j^k = F_{\lambda,j} \cdot x_j^2 - (F_{\lambda+k,j} + F_{\lambda-k,j+k} \cdot c_j^k \cdot (-1)^k)\, x_j + F_{\lambda,j+k} \cdot c_j^k \cdot (-1)^k = 0 \tag{1.18}$$

Whence $Q_1 = 4x_1^2 - 13x_1 - 9$. We can also use this Φ_3 fragment to find roots of Q_j^λ by the formula

$$Q_j^\lambda = x_j^2 - (F_{\lambda+1,j} + c_j \cdot F_{\lambda-1,j+1})x_j + c_j^\lambda \cdot (-1)^\lambda = 0 \tag{1.19}$$

from which $Q_j^\lambda = x^2 - 17x - 6$.

The equations in (1.18) and (1.19) above have roots: $r_{1+} = 3.8365$; $r_{1-} = -.5865$; $r_{j+}^{\lambda} = 17.3459$; $r_{j-}^{\lambda} = -.3459$. Then $r_{1+} \cdot r_{j+}^{2\lambda} - r_{1-} \cdot r_{j-}^{2\lambda} = 1154.3202 + .0702 = 1154.3904$, and $r_{j+}^{\lambda} - r_{j-}^{\lambda} = 17.6918$. Then 1154.3904/17.6918 = 65.25, and 65.25 · 4 = <u>261</u>. S_1 = 0, 1, 1, 4, 15, 19, 68, <u>261</u>… voila!

In closing, here's an array with a monotonic β.

Table 7

Φ_3 [1][1,2,3]	S_1	S_2	S_3	S_4	S_5	S_6	S_7	…
…								
F_{-4}	$-4/18$	$-6/6$	$-5/12$	$-4/18$	$-6/6$	$-5/12$	$-4/18$	
$F_{-\lambda}$	$3/6$	$4/6$	$2/6$	$3/6$	$4/6$	$2/6$	$3/6$	
F_{-2}	$-1/6$	$-1/3$	$-1/2$	$-1/6$	$-1/3$	$-1/2$	$-1/6$	
F_{-1}	$1/3$	$1/1$	$1/2$	$1/3$	$1/1$	$1/2$	$1/3$	
F_0	0	0	0	0	0	0	0	
F_1	1	1	1	1	1	1	1	
F_2	1	1	1	1	1	1	1	
F_λ	3	4	2	3	4	2	3	
F_4	6	5	4	6	5	4	6	
F_5	9	13	10	9	13	10	9	
$F_{2\lambda}$	21	28	14	21	28	14	21	
F_7	48	41	34	48	41	34	48	

Recall the array Φ_ϕ in table 4, where each column is the ϕ-sequence, 0,1,1,2,3…. The formula for the second part of the b coefficient of Q_ϕ^k in (1.14) is $F_{1-k,1+k} \cdot (-1)^k$. Indices $1-k,1+k$ entail moving across the array in a diagonal direction; but these same numbers are available in every column, including the first, so in this case, diagonal motion isn't necessary. We could find those same numbers by $1-k,1$; i.e., moving straight 'down'.

But in arrays of higher order, adjacent columns (sequences) differ. Thus the indices $\lambda-k,\lambda+k$ from Q_j^k in (1.18) will take a term from each sequence in turn. Although this diagonal motion seems, for $\lambda > 1$, a necessity, it turns out that if the second element in the b coefficient is given a different form, there is apparently a way to construct it from the first column alone. To show how this idea has developed so far, the terms connected by arrows in table 7 above are equated by the formula:

$$F_{\lambda-k,\,j+k} \cdot c_j^k \cdot (-1)^k = F_{-\lambda+k,\,j} \cdot c_j^\lambda \cdot (-1)^{\lambda-1}$$

Thus for <u>k</u> ≥ 0, if this equivalence is correct, the coefficients of (1.18) can also be stated as:

$$Q_j^k = F_{\lambda,\,j} \cdot x_j^2 - (F_{\lambda+k,\,j} - F_{-\lambda+k,\,j} \cdot c_j^\lambda \cdot (-1)^\lambda)x_j + F_{\lambda,\,j+k} \cdot c_j^k \cdot (-1)^k$$

Problem(s): Provide a formal proof for any of the formulas/equations/identities stated above.

Addenda

Recall the formula in (1.16) which, for y = an integer generates the ring of integers. This sequence converges to $F_{n+1}/F_n = 1$ in both directions. It would seem that no integer sequence could converge more slowly than …–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5… But other sequences, found in arrays such as Φ_4 [1, –2,3, –4][1] and Φ_4 [–1,2, –3,4][1], do converge at the same rate. The discriminant D = 0 (D_0) seems likely be the key here.

How many D_0 arrays are there? An infinite number perhaps? What sort of patterns will emerge in β and γ if all of the D_0 arrays are themselves considered as a set?

The empirical formula below gives, in test cases, coefficients for the equation with roots $r^2_{j\pm}$:

$$Q_j^2 = F^2_{\lambda,\,j} \cdot x_j^2 - (F^2_{\lambda+1,\,j} + F^2_{\lambda-1,\,j+1} - 2c_j^{\lambda} \cdot (-1)^{\lambda}) \cdot x_j + F^2_{\lambda,\,j+1} \cdot c_j^2$$

Should this be valid for all arrays, perhaps a pattern is incipient here, and an algorithm can be found to generate coefficients for Q_j^k when k is an ordinary exponent.

A 3rd degree sequence, $(c)F_n + (b)F_{n+1} + (a)F_{n+2} = F_{n+3}$, converges to a root (or roots) of a cubic equation. Is there something analogous to the arrays Φ_λ for use in 3D?

For all of the aesthetic expressions of the Fibonacci sequence and the golden mean that abound in geometry, art and nature, are expressions of any $\Phi_{\lambda>1}$ sequences and/or ratios to be found in these realms?

Using $\mathbb{N}$ = 1,2,3… ∞ as a coefficient of the sequence in (1.4), then $F_n + (\mathbb{N})F_{n+1}$ = OEIS A058307

0, 1, 3, 13, 68, 421, 3015, 24541, 223884, 2263381, 25121075, 303716281, 3973432728…

$(\mathbb{N})F_n + (\mathbb{N})F_{n+1}$ = A002467

0, 1, 1, 4, 15, 76, 455, 3186, 25487, 229384, 2293839, 25232230, 302786759, 3936227868…

$(\mathbb{N})F_n + F_{n+1}$ = A000932

0, 1, 1, 3, 6, 18, 48, 156, 492, 1740, 6168, 23568, 91416, 374232, 1562640, 6801888…

References:

N.J.A. Sloane's The Online Encyclopedia of Integer Sequences
http://oeis.org/

Ron Knott's Fibonacci Numbers and the Golden Section
http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/

Appendix

A survey of the arrays that are the theme of this paper is not complete without mentioning continued fractions, as a close correlation exists between the two. In what follows, certain periodic continued fractions (θ_λ) can be seen as a φ_λ-sequence turned inside out.

Recall that at the beginning of the paper, the formula

$$(c)F_n + (b)F_{n+1} = F_{n+2} \;; \qquad F_0 = 0,\; F_1 = 1$$

was the basis for a derivation of $Q = x^2 - bx - c$.

By substitution, Q expands to an infinite, cyclical continued fraction of period length λ_1. I.e., since $x^2 - bx - c = 0$, then $x = b + \dfrac{c}{x}$; hence, continuous substitution of $b + \dfrac{c}{x}$ for x on the right-hand side yields the continued fraction $x = b + \cfrac{c}{b + \cfrac{c}{b + \cfrac{c}{b\ldots}}}$

To restore this fraction to a quadratic form, let the iterative algorithm

$$c/x + b \to x \tag{2.1}$$

represent its solution. Let I designate the number of iterations of (2.1). Then;

$$\lim_{I\to\infty} (c/x + b \to x) = (c/x + b = x)$$

and $x^2 - bx - c = 0$.

Thus for λ_1 and b,c positive integers, both the infinite continued fraction and the ratio of adjacent terms of a φ_1 series converge to the same number.

For $\lambda > 1$, we said that in the limit as $n \to \infty$, $\dfrac{F_{n+1}}{F_n} = x_1 = \dfrac{F_{n+\lambda+1}}{F_{n+\lambda}}$. By use of the procedures that were then devised, such fractions reduce to λ quadratics. So, for S_1 and S_2 in Φ_2, we find

$$Q_1 = b_1 x_1^2 - (b_1 b_2 - c_1 + c_2)x_1 - b_2 c_1 = 0$$

$$Q_2 = b_2 x_2^2 - (b_1 b_2 + c_1 - c_2)x_2 - b_1 c_2 = 0$$

Now for $\lambda > 1$, θ_λ can be 'factored' into λ fractions of the form:

$$c_j/x_j + b_j \to x_{j+1} \tag{2.2}$$

Thus for λ_2, (2.2) has two expressions; ${c_1}/{x_1} + b_1 \to x_2$ and ${c_2}/{x_2} + b_2 \to x_1$

Then by substitution and taking limits: $x_1 = b_2 + \dfrac{c_2}{b_1 + {c_1}/{x_1}}$; $x_2 = b_1 + \dfrac{c_1}{b_2 + {c_2}/{x_2}}$

Clearing fractions and collecting terms:

$$Q_1 = b_1 x_1^2 - (b_1 b_2 - c_1 + c_2) x_1 - b_2 c_1 = 0$$

$$Q_2 = b_2 x_2^2 - (b_1 b_2 + c_1 - c_2) x_2 - b_1 c_2 = 0$$

These are the same equations as given by Φ_2. Note that although an infinite periodic continued fraction θ_λ is usually thought of as converging to one solution, in this context θ_λ converges to λ solutions, and all are of equal interest. Henceforth, let Θ_λ represent the set of these solutions and the associated quadratic equations. Now we have seen so far that for $0 < \lambda \le 2$, Θ_λ and Φ_λ converge to the same values. For $\lambda > 2$, a slight complication appears. To see why, take the φ_3 formula

$$Q_1 = (b_1 b_2 + c_2) x_1^2 - (b_1 b_2 b_3 + b_1 c_3 - b_2 c_1 + b_3 c_2) x_1 - (b_2 b_3 c_1 + c_1 c_3) = 0$$

Having derived this equation from S_1 in Φ_3 [b_1, b_2, b_3][c_1, c_2, c_3], we can expand it into a continued fraction θ_3. First, arrange the terms of Q_1 so that each is positive and then divide by x_1 for

$$(b_1 b_2 + c_2) x_1 + b_2 c_1 = \frac{b_2 b_3 c_1 + c_1 c_3}{x_1} + b_1 b_2 b_3 + b_1 c_3 + b_3 c_2$$

Some rearranging and tweaking of these terms allows for division by $b_1 + {c_1}/{x_1}$:

$$\left(\frac{b_1 b_2 x_1 + \dfrac{b_2 c_1 x_1}{x_1} + c_2 x_1 = b_1 b_2 b_3 + \dfrac{b_2 b_3 c_1}{x_1} + b_1 c_3 + \dfrac{c_1 c_3}{x_1} + b_3 c_2}{b_1 + \dfrac{c_1}{x_1}} \right) = \left(b_2 x_1 + \frac{c_2 x_1}{b_1 + \dfrac{c_1}{x_1}} = b_2 b_3 + \frac{b_3 c_2}{b_1 + \dfrac{c_1}{x_1}} + c_3 \right)$$

Dividing the last equation by $b_2 + \dfrac{c_2}{b_1 + \dfrac{c_1}{x_1}}$ yields $x_1 = b_3 + \dfrac{c_3}{b_2 + \dfrac{c_2}{b_1 + {c_1}/{x_1}}}$

This last equation in turn can be expanded indefinitely. The aforementioned complication is one of a notational nature, for it appears that Φ_3 [b_1, b_2, b_3][c_1, c_2, c_3] is associated with the fraction set Θ_3 [b_3, b_2, b_1][c_3, c_2, c_1]. But since θ_λ is solved ‘upwards’, the notation could be adjusted to say that Φ_3 [b_1, b_2, b_3][c_1, c_2, c_3] $\leftrightarrow$ Θ_3 [b_1, b_2, b_3][c_1, c_2, c_3]

Hence, finding Q_1 for any λ merely requires clearing $x_1 = b_\lambda + \cfrac{c_\lambda}{b_{\lambda-1} + \cfrac{c_{\lambda-1}}{b_{\lambda-2} + \cfrac{c_{\lambda-2}}{+\cfrac{\cdots}{b_1 + c_1/x_1}}}}$

WORKSHEET

For page 15

	S_1	S_2	S_3	S_4
	1247.375	2808.34375	724.71875	2703.8125
$F_{-4\lambda}$	-731.71875	-1149.84375	-1045.3125	-1045.3125
	299.59375	1658.5	404.125	613.1875
	-432.125	-641.1875	-237.0625	-251.0625
	167.0625	376.125	97.0625	362.125
$F_{-3\lambda}$	-98	-154	-140	-140
	40.125	222.125	54.125	82.125
	-57.875	-85.875	-31.75	-33.625
	22.375	50.375	13	48.5
$F_{-2\lambda}$	-13.125	-20.625	-18.75	-18.75
	5.375	29.75	7.25	11
	-7.75	-11.5	-4.25	-4.5
	3	6.75	1.75	6.5
$F_{-\lambda}$	-1.75	-2.75	-2.5	-2.5
	.75	4	1	1.5
	-1	-1.5	-.5	-.5
	.5	1	.5	1
Φ_4 [1,1,2,3][1,2]	0	0	0	0
	1	1	1	1
	1	1	2	3
	3	3	8	4
F_{λ}	7	11	10	10
	27	14	26	24
	34	36	62	92
	88	86	238	116
$F_{2\lambda}$	210	330	300	300
	806	416	776	716
	1016	1076	1852	2748
	2628	2568	7108	3464
$F_{3\lambda}$	6272	9856	8960	8960
	24072	12424	23176	21384
	30344	32136	55312	82072
	78488	76696	212288	103456
$F_{4\lambda}$	187320	294360	267600	267600
	718936	371056	692176	638656