Mousetrap Stuff

2467

THE UNIVERSITY OF CALGARY

2500 University Drive N.W., Calgary, Alberta, Canada T2N 1N4

Faculty of SCIENCE
Department of MATHEMATICS & STATISTICS

Facsimile: (403) 282-5150
Telephone (403) 220-5202

November 15, 1991

Sherwood Washburn,
Department of Mathematics and Computer Science,
Seton Hall University,
South Orange NJ 07079, U.S.A.

Dear Sherwood Washburn,

Thank you for your 91-10-25 letter and your note on Mousetrap. I'm not sure that I properly understand the rules. What might be of interest is to extend the array (which I calculated separately, according to my interpretation of the rules):

```
                        1
                     0     1
                  1     0     1
               2     2     0     2
            9     6     3     0     6
        44    31    19    11     0    15
    265   180   112    53    32     0    78
  1854  1267??  [last line doubtful, done by hand]
```

but, as you will see from the references below, this may be well known to those who well know it.

While I can see that you disagree with Cayley, it seems to me that you should also disagree with him about the perms 2,1,3,4 and 2,4,3,1, which, according to me, should throw out all four cards (in the orders 3,2,1,4 and 3,1,4,2 respectively).

Rouse Ball (& Coxeter), Math Recreations & Essays, 12th edition, U of Toronto Press, 1974, p. 336(–7), quotes the second Cayley paper, but gives (effectively) the row of the above array starting with 9, which seems to agree with neither Cayley's table nor yours, as you state them.

I haven't checked the Steen reference, *Quart. J. Math.*, **15**(1878) 230–241, but I wouldn't be surprised if it contained much of what you want.

There is another paper of Cayley: *Proc. Roy. Soc. Edin.*, 9(1878) 388–391. See also

Louis Comtet, Advanced Combinatorics, D. Reidel, 1974, pp. 180ff.; p. 199, Exx. 4, 5; p. 201, Ex. 13; p. 256, Ex. 7 — and chase up the references there to Appell, Carlitz and Tricomi. On p. 257, Ex. 8, "p. 000" should read "pp. 51, 242–243."

I think that this will give you more than you want!

It's not difficult to write down the permutation which throws out all the cards in order: 1, 12, 132, 1423, 13254, 142563, 1527436, 16245378, 142863795, 182973X564, ..., this is a bit reminiscent of the Josephus problem.

Best wishes,

Yours sincerely,

Richard K. Guy

Richard K. Guy,
Emeritus Professor of
Mathematics.

RKG/rkg

ϕc: Richard J. Nowakowski,
Department of Mathematics & Statistics & Computing Science,
Dalhousie University,
Halifax, N.S. B3H 3J5.
Neil J.A. Sloane,
A.T.&T. Bell Laboratories 2C-376,
600 Mountain Avenue,
Murray Hill NJ 07974, U.S.A.

If you count those perms where just 1 card is thrown out, you get (E&OE)

# of cards	card # →							Total
1	1							1
2	0	0						0
3	1	0	1					2
4	2	1	1	2				6
5	9	5	5	3	9	7		31
6	44	31	25	20	16	44		180
7	265	203	167	148	117	102	265	1267?
8	1854						1854	

Part of letter to RJN, giving more information.

November 18, 1991

Richard J. Nowakowski,
Department of Mathematics & Statistics & Computing Science,
Dalhousie University,
Halifax, N.S. B3H 3J5

Dear Richard,

Thankyou for a valuable (to me) visit. This is to cover several things, not all of which will get done for a day or two. But the first, letter to Sherwood Washburn about Mousetrap, is on its way. Unfortunately, as you can tell from the alterations on your copy, I've got hooked on this, and I suspect that you may be also. Before we spend too much time on it, would be a good idea to check 3 Sylvester refs and Steen (see enclosed letter) and also to chase up Appell, Carlitz and Tricomi — even to read Comtet more carefully than I've done!

Having made no real progress on extending the table in the letter (except to note, of course, that the left entry on row n, for n cards, counting $n = 0, 1, \ldots,$ is $d(n)$, the number of derangements, discordant perms, subfactorial n, for which recursions are well known to those who well know them — sequence 766 in Sloane's first edition) I started to produce another array, for the numbers of perms from which just the single card, card $\#c$, $1 \le c \le n$ is thrown out. I think I've had a little more luck:

1

0 0

1 0 1

2 1 1 2

9 5 5 3 9

44 31 25 20 16 44

265 203 167 148 117 102 265

1854 1501 1854

(E&OE, of course) At first sight, not much new, except that $d(n-1)$ appears at both left and right (we're now starting with $n = 1$). That is, the number of perms which throw out card #1 (only) and card $\#n$. However, if I've got the answer right, I think that I can prove that card #2 (only) is thrown out by

$$d_2(n) = (n-3)d(n-2) + (n-4)d(n-3)$$

perms. It may be a little less opaque to write this as

$$d_2(n) = d(n-2) + (n-4)d'(n-2)$$

where $d'(n)$ is the number of derangements of n when one of the hats has lost its label, e.g., $d'(4)$ is the number of ways 1, 2, 3 and 4 can put on hats 1, 3, 4 and an unlabelled one, with noone having his right label (or hats 1, 2, 4, 5, say). If I've got it right, there's a fairly simple recursion:

$$d'(n+1) = d(n) + nd'(n)$$

so that $d'(n) = d(n) + d(n-1)$. This gives Seq 1166 in Sloane I:

1, 1, 1, 3, 11, 53, 309, 2119, 16687, 148329, ... for which the refs are Riordan, Intro to Combin Anal, p. 188, (where I discover my ignorance of rook polynomials — the array there is not the same as either of mine, but is presumably related — how?), *Math Gaz* **52**(1968) 381 (Max Rumney & EJF Primrose do some nice algebra and show connexion with d(n), but give no motivation or combin interpretation) and David, Kendall & Barton, Sym Fn & Allied Tables, p. 263 (not immediately available).

But the seq d_2 may be the first contribution to Sloane III:

0, 0, 1, 5, 31, 203, 1501, 12449, 114955, 1171799, 13082617, 158860349, 2085208951, 29427878435, 444413828821, 7151855533913, 122190894996451, 2209057440250799, ...

Best wishes to you and Fran from Louise and

Yours sincerely,

Richard K. Guy
Emeritus Professor of Mathematics

RKG/rkg

encl: copy of letter to Washburn,
copy of ,
copy of letter to ,
copy of letter to Andrew B.,
recent version of nitelite.

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2500 University Drive N.W., Calgary, Alberta, Canada T2N 1N4

Faculty of SCIENCE
Department of MATHEMATICS & STATISTICS

Facsimile: (403) 282-5150
Telephone (403) 220-5202

December 11, 1991

Sherwood Washburn,
Department of Mathematics and Computer Science,
Seton Hall University,
South Orange NJ 07079, U.S.A.

Dear Sherwood Washburn,

I continue my recent letter. Richard Nowakowski has run (our version of) Mousetrap on a machine. and corrects and extends the table I sent.

1
0 1
1 0 1
2 2 0 2
9 6 3 0 6
44 31 19 11 0 15
265 180 105 54 32 0 84
1854 1255 771 411 281 138 0 330
14833 9949 6052 3583 2057 1366 668 0 1812
133496 89162 55340 32135 19026 12685 6753 4305 0 9978

As I said in my earlier letter, I haven't checked the Steen reference, *Quart. J. Math.*, 15(1878) 230–241, but Richard Nowakowski notes that it is made by Sloane after Seq. 1186, The Game of Mousetrap:

1, 3, 13, 65, 403, 2885, 23515, 214805

which I haven't yet succeeded in relating to any of our calculations. It's interesting that Sloane gives only these terms, as if Steen didn't have any general formula.

In Sloane's 2nd edition (not out yet!) there's a sequence, provisionaly labelled A6347, which looks close to things we want. It satisfies the recurrence $a(n) = (n+1)a(n-1) + (-1)^n$:

0, 1, 3, 16, 95, 666, 5327, 47944, 479439, ...

as well as Seq. 1221, discordant permutations, from the first edition:

1, (1, 0,) 3, 16, 95, 672, 5397, 48704.

Again, only a finite number of terms. The reference is to *Scripta Math.*, 19(1953) 118, which I haven't pursued.

The next sequence (1222) *is* relevant:

(0, 0,) 1, 3, 16, 96, 675, 5413, 48800, 488592, ...

There is a reference to Ahrens which I should look up. Also to a paper of Max Wyman & Leo Moser, *Canad. Math. J.*, **10**(1958) 468–480 and to Riordan, Introduction to Combinatorial Analysis, p. 198. The W & M paper gives nice asymptotic formulas which enable the ménage numbers to be calculated exactly (i.e., the error is only a small fraction of an integer). They don't actually sum the ménage numbers, but it is the sums which give the numbers of perms which throw out just card $\#(n-1)$ from n cards (according to the rules I'm using).

Here's my array for card $\#c$ (only) thrown out from n cards:

1
0 0
1 0 1
2 1 1 2
9 5 5 3 9
44 31 25 20 16 44
265 203 167 142 117 96 265
1854 1501 1267 1075 932 791 675 1854
14833 12449 10745 9311 8241 5413 14833

which has slightly fewer errors than before, but has been calculated by hand. The two outside diagonals on each side are in Sloane's Handbook. I have a formula for the second from the left (which I believe I gave in a letter to Nowakowski and sent to you). I can give an algorithm for calculating $d_3(n)$, the number of perms which eject just card #3. Start from the following array:

1
0 1
1 1 2
2 3 4 6
9 11 14 18 24
44 53 64 78 96 120
265 309 362 426 504 600 720
1854 2119 2428 2790 3216 3720 4320 5040
14833 16687 18806 21234 24024 27240 30960 35280 40320

This is just a difference table for $n!$, and the first 5 diagonals on the right appear in Sloane with a reference to *Crelle*, **198**(1957) 61. The first 2 diags on the left are what I've called $d(n)$ and $d'(n)$ and these and only these appear in Sloane. Think of the entries as the numbers of perms of n which are discordant with a given perm in r places. The two diags just mentioned are $r = n$ and $r = n - 1$. The general formula for an entry is

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}(n-i)!$$

Then the number of perms which eject just card #3 is

$$d_3(n) = 5d'(n-4) + (13n-68)d''(n-4) + (n-6)(7n-39)d'''(n-4) + (n-6)^2(n-7)d^{iv}(n-4)$$

where the ds are entries in successive diagonals, reading from the left, in the above array. So the sequence for d_3 reads:

0, 0, 0, 1, 1, 5, 25, 167, 1267, 10745, 101005, 1044395, 11795863, 144605933, 1913265985, ... [E&OE, as usual]

In theory one could express d_4 in terms of $d'(n-5)$, etc. as a sum of five terms with polynomial coefficients of degrees 0, 1, 2, 3, 4. And so on, but there may be simpler expressions and they may be in the literature, which I still haven't searched.

Best wishes,

Yours sincerely,

Richard K. Guy

Richard K. Guy,
Emeritus Professor of Mathematics.

RKG/rkg

ϕc: Richard J. Nowakowski,
Department of Mathematics & Statistics & Computing Science,
Dalhousie University,
Halifax, N.S. B3H 3J5.
Neil J.A. Sloane,
A.T.&T. Bell Laboratories 2C-376,
600 Mountain Avenue,
Murray Hill NJ 07974, U.S.A.