COMMENTS ON A002080 AND RELATED SEQUENCES BASED ON THRESHOLD FUNCTIONS

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We identify an incorrect value in the OEIS sequence A002080 and also suggest changes to the descriptions of this and related sequences to make the terminology more consistent.

1. Summary of corrections

1.1. **A002080.** The 7th term is listed as 123565, but it should be 122921.

The value 123565 is not consistent with A002077 and A000609. The latter sequence is well-checked and surely correct. See Table 2.2 and calculations after it for more details.

1.2. **A002080.** This sequence is more accurately described as the number of "N-equivalence classes of self-dual threshold functions of n or fewer variables". Compare with

- A002078 for same use of "N-equivalence classes"
- A002077 for same use of "threshold"
- A000609 for same use of "or fewer"

1.3. **A002077.** This sequence is more accurately described as the number of "N-equivalence classes of self-dual threshold functions of exactly n variables". Compare with

- A000615 for same use of "exactly"
- A002079 for same use of "N-equivalence classes" (but note that "gates" is inconsistent).

1.4. **A002079.** This sequence is more consistently described as the number of "N-equivalence classes of threshold functions of exactly n variables". Compare with

- A002078 (and all others listed here) for same use of "functions"

1.5. **A001532.** This sequence is more accurately described as the number of "NP-equivalence classes of self-dual threshold functions of n or fewer variables". Compare with

- A000617 for same use of "NP-equivalence classes"

1.6. **A003184.** This sequence is more accurately described as the number of "NP-equivalence classes of self-dual threshold functions of exactly n variables". Compare with

- A000619 for same use of "NP-equivalence classes"
- A001532 (and other "self-dual" sequences here) for why it is more consistent to be n and not $n+1$ variables, i.e. with offset 1 not 0.

1.7. **Relationships.** The equations (2.2)–(2.8) below express relationships between these sequences that would be suitable to include in the respective comments sections of OEIS. Note that (2.5) is already expected, as the comments for A002078 observe that it appears to be the binomial mean transform of A000609.

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2. Background on threshold functions

A *threshold function* of n (or fewer) variables is a function $f\colon \{0,1\}^n \to \{0,1\}$ which can be computed as the truth value for the 'threshold condition'

$$\sum_{i=1}^{n} w_i x_i \geqslant w_0 \tag{2.1}$$

where $x_i \in \{0,1\}$ are the n (Boolean) variables and $w_i \in \mathbb{R}$ are prescribed weights. By perturbing the weights (without changing the function f), we can assume that equality in (2.1) does not happen. Thus threshold functions are in bijection with the chambers of an arrangement of 2^n linear hyperplanes in $\mathbb{R}^{n+1}$, given by setting the threshold condition to equality for each of the input values for $x \in \{0,1\}^n$. Methods exist for counting such chambers (e.g. [1, §6.2]) and the number of threshold functions is a well-checked sequence.

There is an action of $(S_2)^n$ on threshold functions giving rise to *N-equivalence*. It is given by complementing each variable x_i separately or changing the sign of the weights and modifying the threshold w_0 accordingly. The number of threshold functions in each N-equivalence class is 2^k when the function (in Boolean form) depends (explicitly) on exactly k variables. There is additionally an action of S_n by permuting the variables. Combined with N-equivalence this gives rise to *NP-equivalence*.

For threshold functions there is also a notion of being *positive*. This is often defined as being expressible (as a Boolean function) in disjunctive form without negation, but in practice it means that all the weights $w_i \geqslant 0$. The useful lemma is that there is one positive function in each N-equivalence class.

The OEIS [4] contains the following sequences, with values listed in Table 2.1.

$$\begin{aligned}
\mathrm{TF}(n) &= \#\{\text{threshold functions of } n \text{ or fewer variables}\}\\
\mathrm{TF}^x(n) &= \#\{\text{threshold functions of exactly } n \text{ variables}\}\\
\mathrm{TF}/N(n) &= \#\{\text{N-equivalence classes of threshold functions of } n \text{ or fewer variables}\}\\
\mathrm{TF}^x/N(n) &= \#\{\text{N-equivalence classes of threshold functions of exactly } n \text{ variables}\}\\
\mathrm{TF}/NP(n) &= \#\{\text{NP-equivalence classes of threshold functions of } n \text{ or fewer variables}\}\\
\mathrm{TF}^x/NP(n) &= \#\{\text{NP-equivalence classes of threshold functions of exactly } n \text{ variables}\}
\end{aligned}$$

The information appears to come from [3, Table I] (see also [5, Table V]).

n	0	1	2	3	4	5	6	7	OEIS
$\mathrm{TF}(n)$	2	4	14	104	1,882	94,572	15,028,134	8,378,070,864	A000609
$\mathrm{TF}^x(n)$	2	2	8	72	1,536	86,080	14,487,040	8,274,797,440	A000615
$\mathrm{TF}/N(n)$	2	3	6	20	150	3,287	244,158	66,291,591	A002078
$\mathrm{TF}^x/N(n)$	2	1	2	9	96	2,690	226,360	64,646,855	A002079
$\mathrm{TF}/NP(n)$	2	3	5	10	27	119	1,113	29,375	A000617
$\mathrm{TF}^x/NP(n)$	2	1	2	5	17	92	994	28,262	A000619

Table 2.1. Counting threshold functions.

The relationships between the first four entries in Table 2.1 are as follows.

$$\mathrm{TF}(n) = \sum_{k=0}^{n} \mathrm{TF}^x(k) \binom{n}{k} \tag{2.2}$$

for example,

$$1882 = 1536 * 1 + 72 * 4 + 8 * 6 + 2 * 4 + 2 * 1$$
$$94572 = 86080 * 1 + 1536 * 5 + 72 * 10 + 8 * 10 + 2 * 5 + 2 * 1$$

and

$$\mathrm{TF}/N(n) = \sum_{k=0}^{n} \mathrm{TF}^x/N(k)\binom{n}{k} \tag{2.3}$$

for example,

$$150 = 96 * 1 + 9 * 4 + 2 * 6 + 1 * 4 + 2 * 1$$
$$3287 = 2690 * 1 + 96 * 5 + 9 * 10 + 2 * 10 + 1 * 5 + 2 * 1$$

and

$$\mathrm{TF}(n) = \sum_{k=0}^{n} \mathrm{TF}^x/N(k)\binom{n}{k}2^k \tag{2.4}$$

for example,

$$1882 = 96 * 1 * 16 + 9 * 4 * 8 + 2 * 6 * 4 + 1 * 4 * 2 + 2 * 1 * 1$$
$$94572 = 2690 * 1 * 32 + 96 * 5 * 16 + 9 * 10 * 8 + 2 * 10 * 4 + 1 * 5 * 2 + 2 * 1 * 1$$

Note that all formulae can be recursively inverted, so that each of these four sequences is equivalent information. It is a straightforward consequence of (2.3) and (2.4) that also

$$\mathrm{TF}/N(n) = \frac{1}{2^n}\sum_{k=0}^{n} \mathrm{TF}(k)\binom{n}{k}, \tag{2.5}$$

which is to say that A002078 is the binomial mean transform of A000609, as already observed in the comments for A002078 on OEIS. For example,

$$150 = (1882 + 104 * 4 + 14 * 6 + 4 * 4 + 2)/16$$
$$3287 = (94572 + 1882 * 5 + 104 * 10 + 14 * 10 + 4 * 5 + 2)/32$$

The *dual* $\overline{f}$ of a threshold function f is given by complementing all inputs and the output, that is, $\overline{f}(x) = f\big(x^c\big)^c$. Thus f is *self-dual* when $\overline{f} = f$, i.e. $f\big(x^c\big) = f(x)^c$.

A standard way to introduce more symmetry into the problem of counting threshold functions is to use the bijection(s) between self-dual threshold functions of n variables and threshold functions of $n-1$ variables, obtained by setting one variable to be constant (and extending in the unique way to go back). Since N-equivalence and NP-equivalence restrict to self-dual threshold functions, the equation $\mathrm{TF}(n) = \mathrm{SD}(n+1)$ effectively introduces extra symmetry into the counting problem.

The OEIS [4] contains the following sequences, with values listed in Table 2.2, also taken from [3, Table I].

$$\mathrm{SD}(n) = \#\{\text{self-dual threshold function of } n \text{ or fewer variables}\} = \mathrm{TF}(n-1)$$
$$\mathrm{SD}/N(n) = \#\{\text{N-equiv. classes of self-dual threshold function of } n \text{ or fewer variables}\}$$
$$\mathrm{SD}^x/N(n) = \#\{\text{N-equiv. classes of self-dual threshold function of exactly } n \text{ variables}\}$$
$$\mathrm{SD}/NP(n) = \#\{\text{NP-equiv. classes of self-dual threshold function of } n \text{ or fewer variables}\}$$
$$\mathrm{SD}^x/NP(n) = \#\{\text{NP-equiv. classes of self-dual threshold function of exactly } n \text{ variables}\}$$

The relationships between the entries in Table 2.2 are similar to those in Table 2.1.

$$\mathrm{SD}/N(n) = \sum_{k=1}^{n} \mathrm{SD}^x/N(k)\binom{n}{k} \tag{2.6}$$

n	1	2	3	4	5	6	7	8	OEIS
$\mathrm{SD}(n)$	2	4	14	104	1,882	94,572	15,028,134	8,378,070,864	A000609[1]
$\mathrm{SD}/N(n)$	1	2	4	12	81	1,684	122,921	33,207,256	A002080
$\mathrm{SD}^x/N(n)$	1	0	1	4	46	1,322	112,519	32,267,168	A002077
$\mathrm{SD}/NP(n)$	1	1	2	3	7	21	135	2,470	A001532
$\mathrm{SD}^x/NP(n)$	1	0	1	1	4	14	114	2,335	A003184

TABLE 2.2. Counting self-dual threshold functions.

for example,

$$\begin{aligned} 1684 &= 1322 * 1 + 46 * 6 + 4 * 15 + 1 * 20 + 0 * 15 + 1 * 6 \\ 122921 &= 112519 * 1 + 1322 * 7 + 46 * 21 + 4 * 35 + 1 * 35 + 0 * 21 + 1 * 7 \\ 33207256 &= 32267168 * 1 + 112519 * 8 + 1322 * 28 + 46 * 56 + 4 * 70 + 1 * 56 + 0 * 28 + 1 * 8 \end{aligned}$$

and

$$\mathrm{SD}(n) = \sum_{k=1}^{n} \mathrm{SD}^x/N(k) \binom{n}{k} 2^k \tag{2.7}$$

for example,

$$\begin{aligned} 94572 &= 1322 * 1 * 64 + 46 * 6 * 32 + 4 * 15 * 16 + 1 * 20 * 8 + 1 * 6 * 2 \\ 15028134 &= 112519 * 1 * 128 + 1322 * 7 * 64 + 46 * 21 * 32 + 4 * 35 * 16 \\ &\quad + 1 * 35 * 8 + 1 * 7 * 2 \end{aligned}$$

As before, both formulae can be recursively inverted, so that each of the three sequences is equivalent information. As for (2.5), it follows from (2.6) and (2.7) that

$$\mathrm{SD}/N(n) = \frac{1}{2^n} \sum_{k=1}^{n} \mathrm{SD}(k) \binom{n}{k} \tag{2.8}$$

that is, A002080 is the binomial mean transform of (A000609 offset by 1), as we can set $\mathrm{SD}(0) = 0$ to make this strictly analogous to (2.5). For example,

$$\begin{aligned} 1684 &= (94572 + 1882 * 6 + 104 * 15 + 14 * 20 + 4 * 15 + 2 * 6)/64 \\ 122921 &= (15028134 + 94572 * 7 + 1882 * 21 + 104 * 35 + 14 * 35 + 4 * 21 + 2 * 7)/128 \\ 33207256 &= (8378070864 + 15028134 * 8 + 94572 * 28 + 1882 * 56 + 104 * 70 \\ &\quad + 14 * 56 + 4 * 28 + 2 * 8)/256 \end{aligned}$$

Note: the value $\mathrm{SD}/N(7) = 123565$ was given in [3, Table I] and transferred to OEIS, but the calculations above show this must be an error, because it is not consistent with the value $\mathrm{SD}^x/N(7) = 112519$ and the reliable value $\mathrm{SD}(7) = \mathrm{TF}(6) = 15028134$.

REFERENCES

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