A note on the Catalan transform of a sequence

Peter Bala, May 12, 2017

Difference tables can be used to calculate the binomial transform of a sequence. We show a similar method can be used to find the Catalan transform of a sequence.

The Catalan transformation

The Catalan transform is a sequence transform introduced by Barry [BA2005]. Recall the Catalan numbers $C(n)$ are given by

$$C(n) \quad = \quad \frac{1}{2n+1}\binom{2n}{n},$$

with ordinary generating function (o.g.f.) $c(x)$ given by the formula

$$\begin{aligned} c(x) \quad &= \quad \frac{1-\sqrt{1-4x}}{2x}. \\ &= \quad 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots . \end{aligned}$$

Definition. *Let $(a_n)_{n\geq 0}$ be a sequence with the generating function $A(x) = a_0 + a_1x + a_2x^2 + \cdots$. The Catalan transform of the sequence (a_n) is defined to be the sequence whose o.g.f. is $A(xc(x))$.*

We describe the function $A(xc(x))$ as the Catalan transform of the function $A(x)$. Calculation gives

$$\begin{aligned} A(xc(x)) \quad = \quad & a_0 + a_1x + (a_1 + a_2)\,x^2 + (2a_1 + 2a_2 + a_3)\,x^3 + \\ & (5a_1 + 5a_2 + 3a_3 + a_4)\,x^4 + \cdots . \end{aligned}$$

Barry shows the Catalan transform of a sequence corresponds to the matrix multiplication

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & 1 & 1 & & & \\ & 2 & 2 & 1 & & \\ & 5 & 5 & 3 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_1 + a_2 \\ 2a_1 + 2a_2 + a_3 \\ 5a_1 + 5a_2 + 3a_3 + a_4 \\ \vdots \end{pmatrix} \qquad (1)$$

where the lower triangular array on the left side of (1) is the Catalan convolution array A106566; this array is the element of the Bell subgroup of the Riordan group given by $(1, xc(x))$. See A033184 for another version of the array.

An alternative approach to finding the Catalan transformation of a sequence is as follows: given a sequence $(a_n)_{n\geq 0}$, construct a lower triangular array $(T(n,k))_{n,k\geq 0}$ by putting the sequence as column 0 of the array and completing the remaining columns of the array by means of the recurrence equation

$$T(n,k) \quad = \quad T(n,k-1) + T(n-1,k) \quad [1 \leq k \leq n]. \tag{2}$$

Then the leading diagonal of the resulting array is equal to the Catalan transform of the sequence a_n. The first few rows of the array are shown below.

$$\begin{pmatrix} a_0 & & & & \\ a_1 & a_1 & & & \\ a_2 & a_1+a_2 & a_1+a_2 & & \\ a_3 & a_1+a_2+a_3 & 2a_1+2a_2+a_3 & 2a_1+2a_2+a_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that in the above, if instead of the recurrence (2) we used the recurrence $T(n,k) = T(n-1,k-1) + T(n-1,k)$, then the leading diagonal of the resulting array would give the binomial transform $\sum_k \binom{n}{k} a_k$ of the original sequence a_n.

Problem 1. Find the first few terms in the expansion of $c(x)^3$.

The o.g.f. $x^3c(x)^3$ is the Catalan transform of the function x^3, which is the o.g.f. for the sequence $(0,0,0,1,0,0,...)$. Using the recurrence relations (2) to find the Catalan transform of the sequence $(0,0,0,1,0,0,...)$ results in the triangle

$$\begin{pmatrix} 0 & & & & & & & & \\ 0 & 0 & & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 1 & 1 & 1 & 1 & & & & & \\ 0 & 1 & 2 & 3 & 3 & & & & \\ 0 & 1 & 3 & 6 & 9 & 9 & & & \\ 0 & 1 & 4 & 10 & 19 & 28 & 28 & & \\ 0 & 1 & 5 & 15 & 34 & 62 & 90 & 90 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The coefficients in the expansion of $x^3c(x)^3$, and hence of $c(x)^3$, can be read off the leading diagonal of the triangle yielding

$$c(x)^3 = 1 + 3x + 9x^2 + 28x^3 + 90x^4 + \cdots .$$

$\square$

In order to prove that the above triangular process when applied to a formal power series $A(x)$ produces the Catalan transform of $A(x)$ as the leading diagonal of the triangle, it is sufficient to prove it for the case when $A(x) = x^n$, $n = 1, 2, \ldots$ is a monomial. The general case will then follow by the linearity of the process. The case of monomials is the content of Theorem 20 in [TE2011]. The lower triangular array that our approach produces when applied to find the Catalan transform of the monomial x^{k+1} is equivalent to Tedford's array $\overline{P}_k$. Tedfords's arrays are essentially our arrays read by shallow diagonals. For example, the array calculated in Problem 1 when read by shallow diagonals corresponds to $\overline{P}_2$ in [TE2011, Section 5, Fig.4].

Exponential generating functions

We can find the Catalan transform $A(xc(x))$ of the exponential generating function $A(x) = \sum_{n\geq 0} a_n \dfrac{x^n}{n!}$ by the above triangular method. However, if we wish to avoid working with rational numbers we need to modify the process as follows. After placing the sequence (a_n) in the first column of the array, the remainder of the array is now completed using the recurrence equations

$$T(n,k) \quad = \quad T(n,k-1) + nT(n-1,k) \quad [1 \leq k \leq n]. \tag{3}$$

The factor of n in the recurrence compensates for the $n!$ terms in the exponential generating function. As an example, the following table shows the steps in calculating the Catalan transform of the exponential function $\exp(x)$.

Row	a_n									
0	**1**									
1	**1**	$\leftarrow$x1$\rightarrow$	**1**							
			$\updownarrow$x2							
2	**1**	$\leftarrow$x1$\rightarrow$	**3**	$\leftarrow$x1$\rightarrow$	**3**					
			$\updownarrow$x3		$\updownarrow$x3					
3	**1**	$\leftarrow$x1$\rightarrow$	**10**	$\leftarrow$x1$\rightarrow$	**19**	$\leftarrow$x1$\rightarrow$	**19**			
			$\updownarrow$x4		$\updownarrow$x4		$\updownarrow$x4			
4	**1**	$\leftarrow$x1$\rightarrow$	**41**	$\leftarrow$x1$\rightarrow$	**117**	$\leftarrow$x1$\rightarrow$	**193**	$\leftarrow$x1$\rightarrow$	**193**	
$\vdots$										$\ddots$

Table 1. Catalan transform of $\exp(x)$.

From the leading diagonal of the table we find

$$\exp(xc(x)) \quad = \quad 1 + x + \frac{3x^2}{2!} + \frac{19x^3}{3!} + \frac{193x^4}{4!} + \cdots .$$

The coefficient sequence $[1, 1, 3, 19, 193, ...]$ is the sequence of values of the Bessel polynomials $y_n(x)$ at $x = 2$. See A001517.

References

[BA2005] P. Barry, *A Catalan Transform and Related Transformations of Integer Sequences*, Journal of Integer Sequences, Vol. 8 (2005), Article 05.4.4

[TE2011] S. J. Tedford, *Combinatorial interpretations of convolutions of the Catalan numbers*, Integers 11 (2011) #A3